Descent and peak polynomials

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Introduction

Roots

Coefficients

Conjectures and other work
The cast of characters
SB = Sara Billey
KB = Krzysztof Burdzy
FCV = Francis Castro-Velez
ADL = Alexander Diaz-Lopez
MF = Matthew Fahrbach
PH = Pamela Harris
EI = Erik Insko
MO = Mohamed Omar
RO = Rosa Orellana
JP = José Pastrana
DPL = Darleen Perez-Lavin
BES = Bruce E Sagan
AT = Alan Talmage
RZ = Rita Zevallos
\[[n] := \{1, 2, \ldots, n\},
\mathcal{S}_n := \text{symmetric group on } [n],
\]

\[l_0 := l \cup \{0\} \text{ for } l \text{ a finite set of positive integers,}\]

\[m := \max l_0.\]

Permutation \(\pi = \pi_1 \ldots \pi_n \in \mathcal{S}_n\) has \textit{descent set}

\[\text{Des } \pi = \{i \mid \pi_i > \pi_{i+1}\} \subseteq [n-1].\]

Given \(l\) and \(n > m\), define

\[D(l; n) = \{\pi \in \mathcal{S}_n \mid \text{Des } \pi = l\} \quad \text{and} \quad d(l; n) = \#D(l; n).\]

\textbf{Ex.} \(D(\{1, 2\}; 5) = \{32145, 42135, 52134, 43125, 53124, 54123\}.\)

\textbf{Theorem (MacMahon, 1916)}

\textit{We have }\(d(l; n)\) \textit{is a polynomial in }n\textit{, called the descent polynomial.}\n
\textbf{Proof.} Let \(l = \{i < j < \ldots\}\). Use inclusion-exclusion on \(\pi \in \mathcal{S}_n\) of the form \(\pi = \pi_1 < \cdots < \pi_i \pi_{i+1} < \cdots < \pi_j \cdots.\)

\textbf{Corollary (ADL-PH-EI-BES, 2016)}

\textit{If }\(l \neq \emptyset\) \textit{and }\(l^- = l - \{m\}\) \textit{then }\(d(l; n) = \binom{n}{m}d(l^-; m) - d(l^-; n).\)

\textit{So }\deg d(l; n) = m.\)
\[ [\ell, n] := [\ell, \ell + 1, \ldots, n]. \]

Permutation \( \pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n \) has \textit{peak set}

\[ \text{Peak} \, \pi = \{ i \mid \pi_{i-1} < \pi_i > \pi_{i+1} \} \subseteq [2, n-1]. \]

Note that if \( \text{Peak} \, \pi = I \) then \( I \) can not contain two consecutive integers and call such \( I \) \textit{admissible}. If \( n > m \) then define

\[ P(I; n) = \{ \pi \in \mathfrak{S}_n \mid \text{Peak} \, \pi = I \}. \]

**Ex.** \( P(\{2\}; 4) = \{1324, 1423, 1432, 2314, 2413, 2431, 3412, 3421\}. \)

**Theorem (SB-KB-BES, 2013)**

If \( I \neq \emptyset \) is admissible then \( \#P(I; n) = p(I; n)2^{n-\#I-1} \) where \( p(I; n) \) is a polynomial in \( n \) of degree \( m-1 \) called the \textit{peak polynomial}.

**Proof.** Use inclusion-exclusion on \( \pi \in \mathfrak{S}_n \) such that

\[ \text{Peak}(\pi_1 \ldots \pi_{m-1}) = I - \{m\} \quad \text{and} \quad \text{Peak}(\pi_m \ldots \pi_n) = \emptyset \]

and then induct. \( \square \)
The peak polynomial is not always real rooted. But it does have some interesting integral roots.

**Theorem (SB-MF-AT, 2016)**

Let $I = \{i_1 < \cdots < i_s\}$.

(i) If $i_{r+1} - i_r$ is odd for some $r$ then

$$p(l; 0) = p(l; 1) = \cdots = p(l; i_r) = 0.$$ 

(ii) If $i \in I$ then

$$p(l; i) = 0.$$
In some ways the descent polynomial behaves similarly.

**Theorem (ADL-PH-EI-BES, 2016)**

If \( i \in I \) then \( d(I; i) = 0 \).

**Proof.**

\[
d(I; n) = \binom{n}{m} d(I^-; m) - d(I^-; n)
\]

where \( I^- = I - \{m\} \). If \( i < m \) then, using induction,

\[
d(I; i) = \binom{i}{m} d(I^-; m) - d(I^-; i) = 0 \cdot d(I^-; m) - 0 = 0.
\]

If \( i = m \) then

\[
d(I; m) = \binom{m}{m} d(I^-; m) - d(I^-; m) = 0
\]

as desired. \( \square \)
**Ex.** Let \( I = \{1, 2\} \). Then

\[ D(I; n) = \{ \pi = \pi_1 > \pi_2 > \pi_3 < \pi_4 < \cdots < \pi_n \} . \]

So \( \pi_3 = 1 \). And picking any two elements of \([2, n]\) for \( \pi_1, \pi_2 \) determines \( \pi \). Thus

\[ d(I; n) = \binom{n-1}{2} = \frac{n^2 - 3n + 2}{2} \]

has negative, nonintegral coefficients.

The next peak polynomial result was conjectured by SB-KB-BES.

**Theorem (ADL-PH-EI-MO, 2016)**

*The coefficients in the expansion*

\[ p(I; n) = \sum_{k \geq 0} a_k(I) \binom{n-m}{k} \]

*are nonnegative integers.*

**Proof.** Use a new recursion for \( p(I; n) \) based on where \( n + 1 \) can be placed in passing from \( \mathcal{S}_n \) to \( \mathcal{S}_{n+1} \). \( \square \)
For descent polynomials, these coefficients have a combinatorial interpretation.

**Theorem (ADL-PH-EI-BES, 2016)**

*Define $b_k(I)$ as the coefficients in the expansion

$$d(I; n) = \sum_{k \geq 0} b_k(I) \binom{n - m}{k}.$$*

*Then $b_k(I)$ is the number of $\pi \in D(I; n)$ with

$$\{\pi_1 \ldots, \pi_m\} \cap [m + 1, n] = [m + 1, m + k]. \tag{1}$$*

**Proof.** Partition $D(I; n)$ into subsets $D_k(I; n)$ which contain those permutations in $D(I; n)$ such that $|\{\pi_1 \ldots, \pi_m\} \cap [m + 1, n]| = k$. Then show

$$|D_k(I; n)| = b_k(I) \binom{n - m}{k},$$

where $b_k(I)$ is given by equation (1).
More on roots (including complex).

Conjecture (SB-MF-AT for $p$, ADL-PH-EI-BES for $d$, 2016)

If $d(I; z) = 0$, or if $I$ is admissible and $p(I; z) = 0$ then

$$|z| \leq m \quad \text{and} \quad \Re(z) \geq -3.$$  

For $d(I; z)$ this conjecture has been checked for all $I$ with $m \leq 12$.

**Ex.** Roots of $d(I; z)$ for $I = \{4, 6\}$.
More on coefficients.

Problem

Find a combinatorial interpretation of the $a_k(I)$ in

$$p(I; n) = \sum_{k \geq 0} a_k(I) \binom{n-m}{k}.$$ 

Sequence $a_0, a_1, \ldots$ is log concave if, for all $k$, $a_{k-1}a_{k+1} \leq a_k^2$.

Conjecture (ADL-PH-EI-BES, 2016)

The sequence $b_0(I), b_1(I), \ldots$ is log concave where the $b_k(I)$ are defined by

$$d(I; n) = \sum_{k \geq 0} b_k(I) \binom{n-m}{k}.$$ 

Note that the stronger condition of the generating function for $b_0(I), b_1(I), \ldots$ being real rooted does not always hold.

Proposition (ADL-PH-EI-BES, 2016)

If $I = [\ell, m]$ then $b_0(I), b_1(I), \ldots$ is log concave.
Other Coxeter groups.
The symmetric group is the Coxeter group of type $A$. There are analogous results for types $B$ and $D$ which have been demonstrated by FCV-ADL-RO-JP-RZ (2013) and ADL-PH-EI-DPL (2016) for $p(l; n)$, and by ADL-PH-EI-BES (2016) for $d(l; n)$. For example, we view $\beta = \beta_1 \ldots \beta_n \in B_n$ as a signed permutation and extend $\beta$ to $\beta = \beta_0 \beta_1 \ldots \beta_n$ where $\beta_0 = 0$. Translating the usual definition of descent set for a Coxeter system into this setting gives

$$\text{Des } \beta = \{i \geq 0 \mid \beta_i > \beta_{i+1}\}.$$ 

Given a finite set $I$ of nonnegative integers, define

$$D_B(I; n) = \{\beta \in B_n \mid \text{Des } \beta = I\} \quad \text{and} \quad d_B(I; n) = \#D_B(I; n).$$ 

Using Inclusion-Exclusion, one obtains the following.

Proposition (ADL-PH-EI-BES, 2016)

If $I \neq \emptyset$ and $I^- = I - \{m\}$ then

$$d_B(l; n) = \binom{n}{m} 2^{n-m} d_B(I^-; m) - d_B(I^-; n).$$
THANKS FOR LISTENING!