Distance-preserving graphs

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Basic definitions

Simplicial vertices

Products
Let $G = (V, E)$ be a graph and let $d$ or $d_G$ denote its distance function. A subgraph $H \subseteq G$ is isometric, written $H \leq G$, if for every $u, v \in V(H)$ we have

$$d_H(u, v) = d_G(u, v)$$

**Ex.** Consider

$G = C_5 = \begin{array}{c}
\text{v} \\
\text{u} \\
\text{w} \\
\text{x} \\
\text{y}
\end{array}$

and

$H = \begin{array}{c}
\text{v} \\
\text{u} \\
\text{w}
\end{array}$

$H' = \begin{array}{c}
\text{v} \\
\text{u} \\
\text{w} \\
\text{x} \\
\text{y}
\end{array}$

Then $H \leq G$. But $H' \nsubseteq G$ since $d_{H'}(u, w) = 3$ and $d_G(u, w) = 2$. 
Call a connected graph $G$ **distance preserving** (dp) if it has an isometric subgraph with $k$ vertices for all $k$ with $1 \leq k \leq |V(G)|$.

**Ex.** From the previous example, $C_5$ is not dp since it has no isometric subgraph with 4 vertices. On the other hand, trees are dp: If $T$ is a tree and $v$ is a leaf then $T - v$ is an isometric subgraph of $T$. So by repeatedly removing leaves, one can find isometric subgraphs of $T$ with any number of vertices.

Roughly, cycles cause obstructions to being dp.

**Conjecture (Nussbaum-Esfahanian)**

Almost all connected graphs are dp. That is, if $d_n$ and $c_n$ are the number of dp graphs and connected graphs on $n$ vertices, respectively, then

$$\lim_{n \to \infty} \frac{d_n}{c_n} = 1.$$  

We will provide various techniques for constructing larger dp graphs from smaller ones.
The *neighborhood* of a vertex $v$ of $G = (V, E)$ is

$$N(v) = \{ w \mid vw \in E \}.$$ 

Call $v$ *simplicial* if $N(v)$ is the vertex set of a clique (complete subgraph) of $G$.

**Ex.** Consider $G = u v w x y z$. Then $u$ is simplicial since $N(u) = \{ v, w, x \}$, the vertices of a triangle. But $y$ is not simplicial since $N(y) = \{ w, z \}$ and $wz \notin E$. 
Theorem (Z)

*Let* $v$ *be simplicial in* $G$. *Then* $G' = G - v$ *is isometric in* $G$.

**Proof.** Consider $x, y \in V(G')$. It suffices to show that no $x$–$y$ geodesic ($x$–$y$ path of minimum length) in $G$ goes through $v$. Suppose, towards a contradiction, that there is such a geodesic

$$P : x = v_0, v_1, \ldots, v_s = v, \ldots, v_t = y.$$

Since $v$ is simplicial $v_{s-1}v_{s+1} \in E(G)$. So $P - v$ is a shorter path from $x$ to $y$, a contradiction. 

A graph $G$ is *chordal* if every cycle $C \subseteq G$ of length at least 4 has an edge of $G$ joining two vertices not adjacent along $C$.

**Corollary**

*Chordal graphs, and hence trees, are dp*

**Proof.** If $G$ is chordal, then it has a simplicial vertex $v$ with $G - v$ chordal. The result now follows by induction.
Let $G, H$ be graphs. Products of $G$ and $H$ have vertex set $V(G) \times V(H)$. Their *(Cartesian) product*, $G \Box H$, has edge set

$$E(G \Box H) = \{(a, x)(b, y) \mid x = y \text{ and } ab \in E(G), \text{ or } a = b \text{ and } xy \in E(H)\}.$$  

Their *lexicographic product*, $G[H]$, has edge set

$$E(G[H]) = \{(a, x)(b, y) \mid ab \in E(G), \text{ or } a = b \text{ and } xy \in E(H)\}.$$  

**Ex.** Consider

\[
\begin{align*}
G &= a \quad b \quad c \\
H &= x \quad y
\end{align*}
\]

Then

\[
\begin{align*}
G \Box H &= (a, y) \quad (b, y) \quad (c, y) \\
&\quad (a, x) \quad (b, x) \quad (c, x) \\
G[H] &= (a, y) \quad (b, y) \quad (c, y) \\
&\quad (a, x) \quad (b, x) \quad (c, x)
\end{align*}
\]
Theorem (HSZ)

Let $G$ be dp with at least two vertices. Then $G[H]$ is dp for any graph $H$.

Call $G$ **sequentially dp** if its vertex set can be ordered $v_1, v_2, \ldots, v_n$ so that the subgraphs

$$G, G - \{v_1\}, G - \{v_1, v_2\}, \ldots$$

are all isometric in $G$

**Ex.** Trees are sequentially dp by the same argument as before. Clearly $G$ sequentially dp implies $G$ dp. The converse is false.

Theorem (HSZ)

The product $G\boxtimes H$ is sequentially dp if and only if $G$ and $H$ are sequentially dp.

Conjecture (HSZ)

If $G$ and $H$ are dp then so is $G\boxtimes H$.

Note that the converse of this conjecture is false.
THANKS FOR LISTENING!