Pattern-avoiding polytopes and Bruhat orders I

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Introduction to polytopes

Pattern-avoiding Birkhoff polytopes and weak Bruhat order

The dimension of $B_n(132, 312)$
A polytope is the convex hull of (smallest convex body containing) a set of points $v_1, \ldots, v_k \in \mathbb{R}^n$, written

$$P = \text{conv}\{v_1, \ldots, v_k\}.$$ 

All our polytopes will be integral, meaning $v_1, \ldots, v_k \in \mathbb{Z}^n$.

(1) Dimension. The affine span of $P$, $\text{aff } P$, is the smallest affine subspace containing $P$. The dimension of $P$ is

$$\dim P = \dim \text{aff } P.$$ 

Ex. If $v_1 = (2, 0)$ and $v_2 = (0, 2)$ then $P_1 = \text{conv}\{v_1, v_2\}$ is

So $\dim P_1 = 1.$
(2) Volume. The \textit{(relative) volume} of polytope $P$ is

$$\text{vol } P = \text{volume with respect to the lattice } \mathbb{Z}^n \cap \text{aff } P.$$ 

A \textit{simplex} is $\Sigma = \text{conv}\{v_1, \ldots, v_{k+1}\}$ with $\dim \Sigma = k$. Call $\Sigma$ \textit{unimodular} if $\text{vol } \Sigma$ is minimum with respect to $\mathbb{Z}^n \cap \text{aff } \Sigma$. A unimodular simplex has volume $\text{vol } \Sigma = 1/(\text{dim } \Sigma)!$. The \textit{normalized volume} of polytope $P$ is

$$\text{Vol } P = (\text{dim } P)! \text{vol } P.$$

\textbf{Ex.} Let $P_1$ be as before and $P_2 = \text{conv}\{(0, 0), (1, 0), (0, 1)\}$.

So $\text{vol } P_1 = 2$, and $\text{vol } P_2 = 1/2$. Both $P_i$ are simplices with $P_2$ unimodular and $P_1$ not. Also $\text{Vol } P_1 = 2$ and $\text{Vol } P_2 = 1$. 
(3) $h^*$-polynomials. The $m$th dilate of polytope $P$ is

$$mP = \{mv \mid v \in P\}.$$  

The *Ehrhart polynomial* of $P$ is

$$\mathcal{L}_P(m) = |mP \cap \mathbb{Z}^n|.$$  

**Theorem (Ehrhart-Stanley)**

*If $P$ is integral then $\mathcal{L}_P(m)$ is a polynomial in $m$ and for some $d$

$$
\sum_{m \geq 0} \mathcal{L}_P(m)t^m = \frac{\sum_{j=0}^{d} h_j^* t^j}{(1 - t)^{\dim P + 1}}
$$

where $\sum_j h_j^* t^j \in \mathbb{Z}_{\geq 0}[t]$ is called the $h^*$-polynomial of $P$, $h^*(P; t)$.  

**Ex.** Let $P = \text{conv}\{(0,0), (1,0), (0,1), (1,1)\}$.

So $\mathcal{L}_P(m) = (m + 1)^2$. 

![Diagram of P and 2P with counters]
Let $\mathcal{S}_n$ be the $n$th symmetric group. If $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{S}_n$ and $\pi = \pi_1 \ldots \pi_k \in \mathcal{S}_k$ then $\sigma$ contains the pattern $\pi$ if there is a subsequence of $\sigma$ order isomorphic to $\pi$. Otherwise $\sigma$ avoids $\pi$.  

**Ex.** $\sigma = 2415376$ contains $\pi = 312$ because of the subsequence $413$ but avoids $\pi = 321$ since it has no subsequence $s_i > s_j > s_k$.

For any set of permutations $\Pi$, let

$$\text{Av}_n(\Pi) = \{ \sigma \in \mathcal{S}_n \mid \sigma \text{ avoids every } \pi \in \Pi \}.$$ 

If $M_\sigma$ is the permutation matrix of $\sigma$ then the Birkhoff polytope is

$$B_n = \text{conv}\{ M_\sigma \mid \sigma \in \mathcal{S}_n \} \subseteq \mathbb{R}^{n \times n}.$$ 

(1) dim $B_n = (n - 1)^2$,  
(2) vol $B_n$ has only been calculated for $n \leq 10$,  
(3) $h^*(B_n; t)$ is symmetric and unimodal. 

Define the $\Pi$-avoiding Birkhoff polytope by

$$B_n(\Pi) = \text{conv}\{ M_\sigma \mid \sigma \in \text{Av}_n(\Pi) \} \subseteq B_n.$$ 

Here we study $B_n(132, 312)$; other $\Pi$ are in our paper.
Let $Q_n(132, 312)$ be $\text{Av}_n(132, 312)$ partially ordered by weak Bruhat order, that is, we have a cover $\pi \preceq \sigma$ if for some $i$,

$$\sigma = \pi(i, i + 1) \text{ where } \pi_i < \pi_{i+1}.$$ 

Let $M(n)$ be the poset of shifted Young diagrams contained in $(n, \ldots, 2, 1)$ ordered by inclusion.

**Proposition**

*For all $n$ we have*

$$Q_n(132, 312) \cong M(n - 1).$$

**Proof sketch.** The map $\phi : Q_n(132, 312) \rightarrow M(n - 1)$ given by

$$\phi(\sigma) = \text{Des } \sigma$$

is an isomorphism where $\text{Des } \sigma$ is the descent set of $\sigma$. \qed
Ex.

$Q_4(132, 312)$ 3421

3241

3214 2341

3214 2314

2314 2341

2134

1234

$M(3)$
Let $\Delta(Q_n(132, 312))$ be the order complex of all chains $\Gamma$ in $Q_n(132, 312)$. Since $Q_n(132, 312) \cong M(n - 1)$ which is a distributive lattice, $\Delta(Q_n(132, 312))$ is shellable. Consider the map $f : \Delta(Q_n(132, 312)) \rightarrow B_n(132, 312)$ defined by

$$f(\sigma_1 < \cdots < \sigma_k) = \text{conv}\{M_{\sigma_1}, \ldots, M_{\sigma_k}\}.$$

**Proposition**

$$\mathcal{T}_n(132, 312) = \{f(\Gamma) \mid \Gamma \in \Delta(Q_n(132, 312))\}$$

is a set of unimodular simplices in $B_n(132, 312)$.

**Proof sketch.** Induct on maximal chains using the shelling order. □

From the previous result, for $\Gamma$ a maximal chain in $Q_n(132, 312)$,

$$\dim B_n(132, 312) \geq \dim \Gamma = |(n - 1, \ldots, 2, 1)| = \binom{n}{2}.$$

**Theorem**

$$\dim B_n(132, 312) = \binom{n}{2}. \quad \square$$
THANKS FOR LISTENING!

AND PLEASE STAY FOR THE NEXT TALK!