Factoring rook polynomials

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Basics

The Factorization Theorem

An application

$m$-level rook placements

Comments and open questions
Consider tiling the first quadrant of the plane with unit squares:

\[
Q = 
\begin{array}{cccc}
\vdots \\
(3, 4) \\
\vdots \\
\end{array}
\]

Let \((c, d)\) be the square in column \(c\) and row \(d\). A board is a finite set of squares \(B \subseteq Q\).

**Ex.** Let \(B_n\) be the \(n \times n\) chess board. For example,

\[
B_3 = 
\begin{array}{cccc}
\vdots \\
\end{array}
\]  
\[
P = 
\begin{array}{cc}
\begin{array}{cc}
\begin{array}{cc}
\end{array}
\end{array}
\begin{array}{cc}
\begin{array}{cc}
\end{array}
\end{array}
\end{array}
\]  
\[
P = 
\begin{array}{cc}
\begin{array}{cc}
\begin{array}{cc}
\end{array}
\end{array}
\begin{array}{cc}
\begin{array}{cc}
\end{array}
\end{array}
\end{array}
\]

A placement \(P\) of rooks on \(B\) is *attacking* if there is a pair of rooks in the same row or column. Otherwise it is *nonattacking*. 
Define the *rook numbers* of $B$ to be

$$r_k(B) = \text{number of ways of placing } k \text{ nonattacking rooks on } B.$$ 

For any board $B$ we have $r_0(B) = 1$ and $r_1(B) = |B|$ (cardinality).

**Ex.** We have

$$r_n(B_n) = \left( \# \text{ of ways to place a rook in column 1} \right) \cdot \left( \# \text{ of ways to then place a rook in column 2} \right) \cdots \left( \# \text{ of ways to then place a rook in column 2} \right) \cdots \left( \# \text{ of ways to then place a rook in column 2} \right) \cdots = n \cdot (n-1) \cdots = n!$$

There is a bijection between placements $P$ counted by $r_n(B_n)$ and permutations $\pi$ in the symmetric group $\mathcal{S}_n$ where $(c, d) \in P$ if and only if $\pi(c) = d$.

**Ex.** Let

$$D_n = B_n - \{(1,1), (2,2), \ldots, (n,n)\}.$$ 

Then

$$r_n(D_n) = \# \text{ of permutations } \pi \in \mathcal{S}_n \text{ with } \pi(c) \neq c \text{ for all } c = \text{ the } n\text{th derangement number}.$$
A *partition* is a weakly increasing sequence \((b_1, \ldots, b_n)\) of nonnegative integers. A *Ferrers board* is \(B = (b_1, \ldots, b_n)\) consisting of the lowest \(b_j\) squares in column \(j\) of \(Q\) for all \(j\). If \(x\) is a variable and \(n \geq 0\) then the corresponding *falling factorial* is

\[
x \downarrow_n = x(x - 1) \cdots (x - n + 1).
\]

**Theorem (Factorization Theorem: Goldman-Joichi-White)**

For any Ferrers board \(B = (b_1, \ldots, b_n)\) we have

\[
\sum_{k=0}^{n} r_k(B)x \downarrow_{n-k} = \prod_{j=1}^{n}(x + b_j - j + 1).
\]

**Ex.**

\[
B = (1, 1, 3) = \begin{array}{ccc}
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\end{array}
\]

\(r_0(B) = 1,\ r_1(B) = 5,\ r_2(B) = 4,\ r_3(B) = 0.\)

\[
\sum_{k=0}^{3} r_k(B)x \downarrow_{3-k} = 1 \cdot x \downarrow_3 + 5 \cdot x \downarrow_2 + 4 \cdot x \downarrow_1 = x^3 + 2x^2 + x
\]

\[
= (x + 1)x(x + 1) = (x + b_1)(x + b_2 - 1)(x + b_3 - 2).
\]
\[
\sum_{k=0}^{n} r_k(b_1, \ldots, b_n) \times \downarrow_{n-k} = \prod_{j=1}^{n} (x + b_j - j + 1).
\] (1)

**Proof.** It suffices to prove (1) for \( x \) a positive integer. Consider

\[
B_x = \begin{array}{c}
B \\
\downarrow \\
R \\
\end{array}
\]

Claim: both sides of (1) equal \( r_n(B_x) \). Placing rooks left to right

\[
\begin{align*}
\text{r}_n(B_x) &= \prod_{j=1}^{n} (\text{# of unattacked squares in column } j) \\
&= (x + b_1)(x + b_2 - 1) \ldots = \text{RHS of (1)}. \\
\text{r}_n(B_x) &= \sum_{k=0}^{n} (\text{# of ways to put } k \text{ rooks on } B \text{ and } n - k \text{ on } R) \\
&= \sum_{k=0}^{n} r_k(B) \cdot x(x - 1) \ldots (x - n + k + 1) = \text{LHS of (1)}.
\end{align*}
\]

\( \square \)
Call boards $B$ and $B'$ *rook equivalent*, $B \equiv B'$, if $r_k(B) = r_k(B')$ for all $k \geq 0$. Note that $B \equiv B'$ implies

$$\#B = r_1(B) = r_1(B') = \#B'.$$

**Ex.**

\[
B = (1, 1, 3) = \begin{array}{|c|c|c|}
\hline
& & \\
\hline
1 & 1 & \\
\hline
\end{array} \quad B' = (2, 3) = \begin{array}{|c|c|}
\hline
& \\
\hline
& \\
\hline
\end{array}
\]

For $B, B'$: $r_0 = 1, \ r_1 = 5, \ r_2 = 4, \ r_k = 0$ for $k \geq 3$ so $B \equiv B'$.

A Ferrers board $B = (b_1, \ldots, b_n)$ is *increasing* if $b_1 < \cdots < b_n$. In the example above, $B'$ is increasing but $B$ is not.

**Theorem (Foata-Schützenberger)**

*Every Ferrers board is rook equivalent to a unique increasing board.*
The root vector of \( B = (b_1, \ldots, b_n) \) is

\[
\zeta(B) = (0 - b_1, 1 - b_2, \ldots, n - 1 - b_n) = (0, 1, \ldots, n - 1) - (b_1, b_2, \ldots, b_n)
\]

The entries of \( \zeta(B) \) are exactly the zeros of \( \sum_k r_k(B) \times \downarrow_{n-k} \).

So if \( B = (b_1, \ldots, b_n) \) and \( B' = (b'_1, \ldots, b'_n) \) then

\[
B \equiv B' \iff \zeta(B) \text{ is a rearrangement of } \zeta(B').
\]

**Ex.** \( B = (1, 1, 3) \) so \( \zeta(B) = (0, 1, 2) - (1, 1, 3) = (-1, 0, -1) \).

\( B' = (0, 2, 3) \) so \( \zeta(B') = (0, 1, 2) - (0, 2, 3) = (0, -1, -1) \) \( \therefore B \equiv B' \).

Every Ferrers board \( B \) is rook equivalent to a unique increasing board.

**Proof sketch.** Pad \( B \) with zeros so that \( \zeta = \zeta(B) \) starts with 0 and has all entries \( \geq 0 \). Let \( m = \max \zeta(B) \). Rearrange \( \zeta \) to form

\[
\zeta' = (0, 1, 2, \ldots, m, \zeta'_{m+1}, \ldots, \zeta'_n)
\]

where \( \zeta'_{m+1} \geq \cdots \geq \zeta'_n \). Then \( \exists \) increasing \( B' \) with \( \zeta(B') = \zeta' \).

**Ex.** \( B = (0, 1, 1, 3) \) so \( \zeta(B) = (0, 1, 2, 3) - (0, 1, 1, 3) = (0, 0, 1, 0) \).

Now \( \zeta' = (0, 1, 0, 0) \) so \( B' = (0, 1, 2, 3) - (0, 1, 0, 0) = (0, 0, 2, 3) \).
Fix a positive integer $m$. Partition the rows of $Q$ into levels where the $ith$ level consists of rows $(i - 1)m + 1, (i - 1)m + 2, \ldots, im$.

**Ex.** If $m = 2$ then

An $m$-level rook placement on $B$ is a set $P$ of rooks with no two in the same level or column. A 1-level rook placement is just an ordinary placement. The $m$-level rook numbers of $B$ are

$$r_{k,m}(B) = \text{number of } m\text{-level rook placements on } B \text{ with } k \text{ rooks.}$$

**Ex.** If $m = k = 2$ and

$$B = (1, 2, 3) = \begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{array}$$

$$\therefore r_{2,2}(B) = 3 :$$
The $m$-level rook placements are related to $C_m \wr S_n$ where $C_m$ is the order $m$ cyclic group and $S_n$ is the $n$th symmetric group, e.g.,

$$r_{n,m}(mn, \ldots, mn) = (mn)(mn-m) \cdots (m) = m^n n! = \#(C_m \wr S_n).$$

Define the $m$-falling factorials by

$$x \downarrow_{n,m} = x(x - m)(x - 2m) \cdots (x - (n - 1)m).$$

A singleton board is $B = (b_1, \ldots, b_n)$ with at most one $b_j$ in each of the open intervals $(0, m), (m, 2m), (2m, 3m), \ldots$.

**Theorem (Briggs-Remmel)**

**If $B$ is a singleton board then**

$$\sum_{k=0}^{n} r_{k,m}(B) x \downarrow_{n-k,m} = \prod_{j=1}^{n} (x + b_j - (j - 1)m).$$

**Ex.** $[17]_3 = 15$ since $15 \leq 17 < 18$. 

Given an integer $m$, define the mod $m$ floor function by

$$\lfloor n \rfloor_m = \text{largest multiple of } m \text{ which is less than or equal to } n.$$
Define a *zone*, \( z = z(B) \), of a Ferrers board \( B = (b_1, \ldots, b_n) \) to be a maximal subsequence \((b_i, \ldots, b_j)\) with

\[
\lfloor b_i \rfloor_m = \cdots = \lfloor b_j \rfloor_m.
\]

Given a zone \( z = (b_i, \ldots, b_j) \) define its *remainder* to be

\[
\rho(z) = \sum_{t=i}^{j} (b_t - \lfloor b_t \rfloor_m).
\]

**Ex.** If \( m = 3 \) then \( B = (1, 1, 2, 3, 5, 7) \) has zones

\( \therefore z = (1, 1, 2), \ z' = (3, 5), \ z'' = (7) \).

Also \( \rho(z) = 1 + 1 + 2 = 4, \ \rho(z') = 0 + 2 = 2, \ \rho(z'') = 1. \)

**Theorem (Barrese-Loehr-Remmel-S)**

*Let \( B = (b_1, \ldots, b_n) \) be any Ferrers board. Then*

\[
\sum_{k=0}^{n} r_{k,m}(B) x_{\downarrow n-k,m} = \prod_{j=1}^{n} (x + \lfloor b_j \rfloor_m - (j - 1)m + \epsilon_j)
\]

*where*

\[
\epsilon_j = \begin{cases} 
\rho(z) & \text{if } b_j \text{ is the last column in zone } z, \\
0 & \text{else.}
\end{cases}
\]
\[
\sum_{k=0}^{n} r_{k,m}(B) \downarrow_{n-k,m} = \prod_{j=1}^{n} \begin{cases} 
x + \lfloor b_j \rfloor m - (j - 1)m + \rho(z) & \text{if } b_j \text{ last in } z, \\
x + \lfloor b_j \rfloor m - (j - 1)m & \text{else.}
\end{cases}
\]

**Ex.** Recall that if \( m = 3 \) and \( B = (1, 1, 2, 3, 5, 7) \) then we have zones \( z = (1, 1, 2), \ z' = (3, 5), \ z'' = (7), \) and remainders \( \rho(z) = 1 + 1 + 2 = 4, \ \rho(z') = 0 + 2 = 2, \ \rho(z'') = 1. \) Thus

\[
\sum_{k=0}^{n} r_{k,m}(B) \downarrow_{n-k,m} = (x + 0 - 0 + 0)(x + 0 - 3 + 0)(x + 0 - 6 + 4)
\cdot (x + 3 - 9 + 0)(x + 3 - 12 + 2)(x + 6 - 15 + 1).
\]

BLRS implies Goldman-Joichi-White: If \( m = 1 \) then it is clear that the LHS of both equations are the same. Also \( \lfloor b_j \rfloor_1 = b_j \) for all \( j. \) So \( \rho(z) = 0 \) for all \( z. \) Thus the RHS’s also agree.

BLRS imples Briggs-Remmel: Clearly the LHS’s are the same. If \( B \) is singleton, then \( \lfloor b_j \rfloor_m = b_j \) for every \( b_j \) in a zone except possibly the last. For the last \( b_j, \lfloor b_j \rfloor_m + \rho(z) = \lfloor b_j \rfloor_m + \rho(b_j) = b_j. \) So RHS’s agree factor by factor.
1. *m*-level rook equivalence. Say $B, B'$ are *m*-level rook equivalent if $r_{k,m}(B) = r_{k,m}(B')$ for all $k$. Call $B = (b_1, \ldots, b_n)$ *m*-increasing if $b_1 > 0$ and $b_j \geq b_{j-1} + m$ for $j \geq 2$. Note that $B$ is 1-increasing if and only if $B$ is increasing.

**Theorem (BLRS)**

*Every Ferrers board is m-level rook equivalent to a unique m-increasing board.*
2. **A $p, q$-analogue.** Permutation $\pi = a_1 \ldots a_n \in S_n$ has **inversion set** and **inversion number**

$$\text{Inv} \, \pi = \{(i, j) \mid i < j \text{ and } a_i > a_j\}, \quad \text{and} \quad \text{inv} \, \pi = \# \text{Inv} \, \pi.$$

If $B$ is a board then the **hook** of $(i, j) \in B$, $H_{i,j}$, is all cells directly south or directly east of $(i, j)$. If $P$ is a rook placement on $B$ then the **Rothe diagram** of $P$ is the skew diagram

$$R(P) = B \setminus \bigcup_{(i,j) \in P} H_{i,j}$$

If $P_\pi$ is the permutation matrix of $\pi$ then $\text{inv} \, \pi = \# R(P_\pi)$.

BLRS have a generalization of the factor theorem with two parameters $p, q$ keeping track of inversions and non-inversions.

**Ex.** $\pi = 4132 \implies \text{Inv} \, \pi = \{(1, 2), (1, 3), (1, 4), (3, 4)\}$, $\text{inv} \, \pi = 4.$
3. Counting equivalence classes. Write $\zeta \geq 0$ if $\zeta$ is a nonnegative sequence. In this case, the *multiplicity vector* of $\zeta$ is

$$
n(\zeta) = (n_0, n_1, \ldots) \text{ where } n_i = \text{the number of } i\text{'s in } \zeta.
$$

**Theorem (Goldman-Joichi-White)**

If Ferrers board $B$ has $\zeta = \zeta(B) \geq 0$ and $n(\zeta) = (n_0, n_1, \ldots)$ then

$$
\text{# of Ferrers boards equivalent to } B = \prod_{i \geq 0} \left( \binom{n_i + n_{i+1} - 1}{n_i - 1} \right).
$$

The *$m$-root vector* of $B = (b_1, \ldots, b_n)$ is

$$
\zeta_m(B) = (0 - b_1, m - b_2, 2m - b_3, \ldots, (n - 1)m - b_n).
$$

**Theorem (BLRS)**

Let $B$ be singleton with $\zeta = \zeta_m(B) \geq 0$ and $n(\zeta) = (n_0, n_1, \ldots)$.

$$
\text{# of singleton boards equivalent to } B = \prod_{i \geq 0} \left( \binom{n_{im} + \cdots + n_{im+m} - 1}{n_{im} - 1, n_{im+1}, \ldots, n_{im+m}} \right).
$$

It would be interesting to find a result holding for all Ferrers $B$. 
4. File placements. A file placement $F$ on $B$ is a placement of rooks with no two in the same column. Fix $m \geq 1$ and let the $m$-weight of $F$ be

$$\text{wt}_m F = 1\downarrow_{y_1,m} \cdot 1\downarrow_{y_2,m} \cdots$$

where $y_i$ is the number of rooks of $F$ in row $i \geq 1$. Let

$$f_{k,m}(B) = \sum_F \text{wt}_m F$$

where the sum is over all file placements $F$ of $k$ rooks on $B$.

Ex. $F = \begin{array}{ccc}
\text{\large \text{\textbullet}} & \text{\large \text{\textbullet}} \\
\text{\large \text{\textbullet}} & \text{\large \text{\textbullet}} & \text{\large \text{\textbullet}} \\
\text{\large \text{\textbullet}} & \text{\large \text{\textbullet}} & \text{\large \text{\textbullet}}
\end{array}$

$F$ has $y_1 = 3$, $y_2 = 0$, $y_3 = 2$. If $m = 4$ then

$$\text{wt}_4 F = 1\downarrow_{3,4} \cdot 1\downarrow_{0,4} \cdot 1\downarrow_{2,4}$$

$$= (1)(-3)(-7) \cdot (1)(-3) = -63.$$

Theorem (BLRS)

For any Ferrers board $B = (b_1, \ldots, b_n)$

$$\sum_{k=0}^{n} f_{k,m}(B)x\downarrow_{n-k,m} = \prod_{j=1}^{n}(x + b_j - (j - 1)m).$$
5. **Higher $q,t$-Catalan numbers.** The $m$-triangular board is

$$\Delta_{n,m} = (0, m, 2m, \ldots, (n-1)m).$$

If $B = (b_1, \ldots, b_n) \subseteq \Delta_{n,m}$ then $\zeta_m(B) = (z_1, \ldots, z_n)$ gives the heights of the columns of $\Delta_{n,m} \setminus B$. Define $\text{area}_m(B) = \#B$ and

$$\text{dinv}_m(B) = \sum_{k=0}^{m-1} \#\{i < j : 0 \leq z_i - z_j + k \leq m\}.$$

The **higher $q,t$-Catalan numbers** are

$$C_{n,m}(q, t) = \sum_{B \subseteq \Delta_{n,m}} q^{\text{dinv}_m(B)} t^{\text{area}_m(\Delta_{n,m} \setminus B)}.$$

We also have

$$C_{n,m}(q, t) = \sum_{B \subseteq \Delta_{n,m}} q^{\text{area}_m(\Delta_{n,m} \setminus B)} t^{\text{bounce}_m(B)}$$

for another statistic $\text{bounce}_m(B)$. Using the $C_{n,m}(q, t)$, BLRS derives a formula for the number of boards $m$-weight equivalent to a given board as a product of binomial coefficients.
6. Hyperplane arrangements. Given \( \pi \in S_n \) the corresponding inversion arrangement is the set of hyperplanes in \( \mathbb{R}^n \)

\[
A(\pi) = \{ x_i = x_j \mid (i, j) \in \text{Inv } \pi \}.
\]

If \( \pi = a_1 \ldots a_n \) then its non-inversion board is

\[
B(\pi) = \{ (i, j) \mid i < j \text{ and } a_i < a_j \} \subseteq B_n.
\]

**Theorem (Hultman, Lewis-Morales)**

For all \( \pi \in S_n \), the number of regions of the arrangement \( A(\pi) \) equals the rook number \( r_n(B_n \setminus B(\pi)) \).

Barrese, Hultman and S are looking for a type B analogue.

**Ex.** If \( \pi = 213 \) then \( \text{Inv } \pi = \{(1, 2)\} \) and \( A(\pi) = \{ x_1 = x_2 \} \).

So the non-inversions of \( \pi \) are \( (1, 3), (2, 3) \) and

\[
B(\pi) = \begin{array}{ccc}
\text{R} & \text{R} & \text{R} \\
\text{R} & \text{R} & \text{R} \\
\text{R} & \text{R} & \text{R} \\
\text{R} & \text{R} & \text{R} \\
\text{R} & \text{R} & \text{R} \\
\end{array}
\]

Kenneth Barrese, Nicholas Loehr, Jeffrey B. Remmel, and Bruce E. Sagan, \textit{m}-level rook placements, preprint, 2013.


THANKS FOR LISTENING!