Introduction to Möbius Inversion over Posets

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Outline

Example A: Combinatorics.

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Theorem (PIE)

Let S be a finite set and $S_1, \ldots, S_n \subseteq S$.

$$\left| S - \bigcup_{i=1}^{n} S_i \right| = |S| - \sum_{1 \le i \le n} |S_i| + \sum_{1 \le i < j \le n} |S_i \cap S_j| - \cdots + (-1)^n \left| \bigcap_{i=1}^{n} S_i \right|. \qquad \square$$

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The Fundamental Theorem of the Difference Calculus or FTDC is as follows.

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Theorem (Number Theory MIT)

Let $f, g : \mathbb{Z}_{>0} \to \mathbb{R}$ satisfy

$$f(n) = \sum_{d|n} g(d)$$

for all $n \in \mathbb{Z}_{>0}$. Then

$$g(n) = \sum_{d|n} \mu(n/d) f(d). \qquad \square$$

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2. It makes the number-theoretic definition transparent.
3. It can be used to solve combinatorial problems.

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If $x, y \in P$ then *x is covered by y* or *y covers x*, written $x \lessdot y$, if $x < y$ and there is no z with $x < z < y$. The *Hasse diagram* of P is the (directed) graph with vertices P and an edge from x up to y if $x \lessdot y$.

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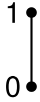

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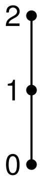

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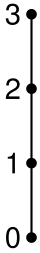

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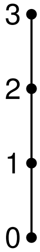

Note that C_n looks like a chain. zo mj

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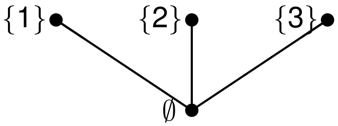

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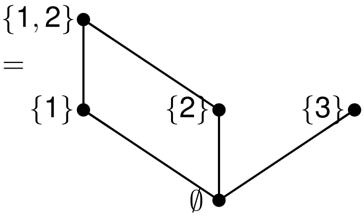

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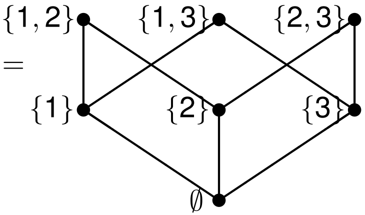

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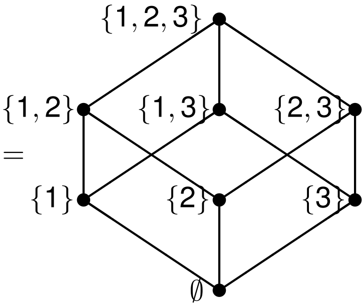

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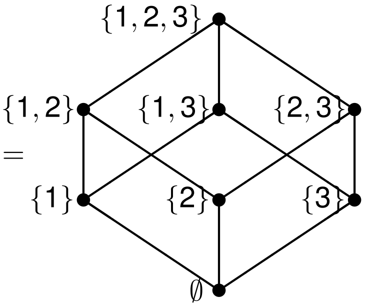

Note that B_3 looks like a cube.

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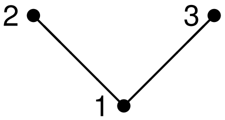

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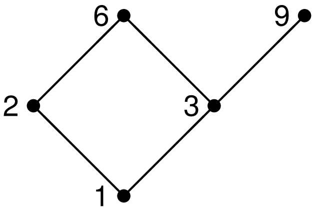

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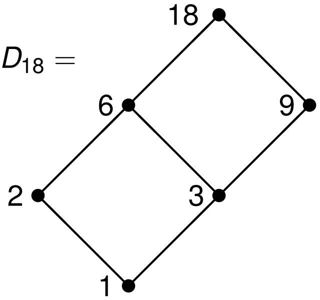

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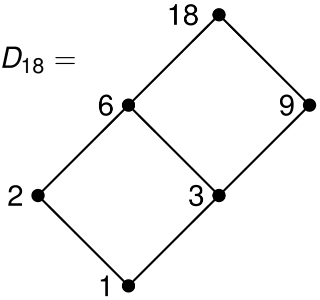

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Note that $[x, y]$ is a poset in its own right and it has a $\hat{0}$ and $\hat{1}$:

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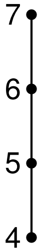

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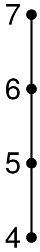

This interval looks like C_3.

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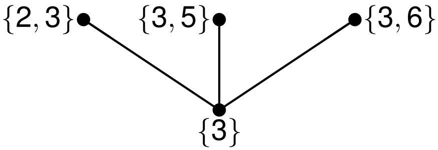

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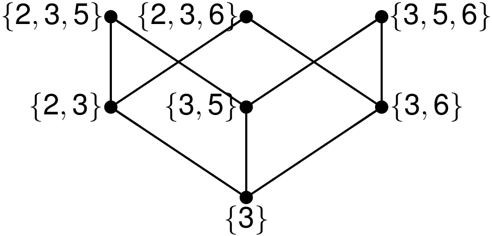

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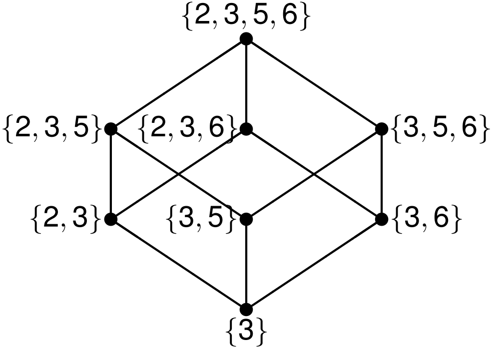

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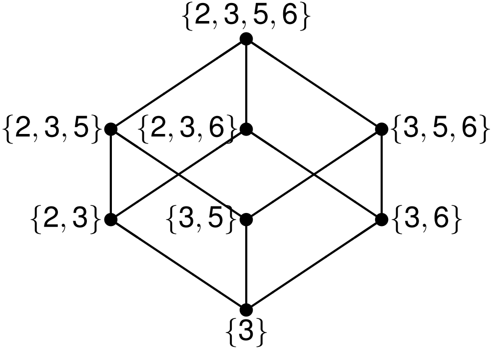

Note that this interval looks like B_3.

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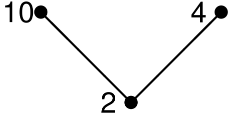

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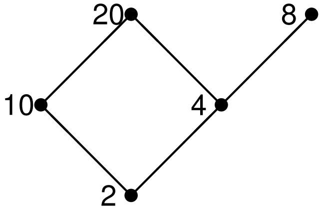

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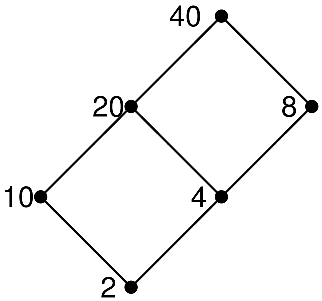

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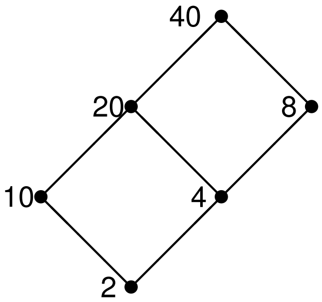

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The *product* of P, Q is $P \times Q = \{(a, x) \mid a \in P,\ x \in Q\}$ with

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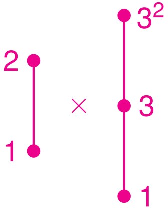

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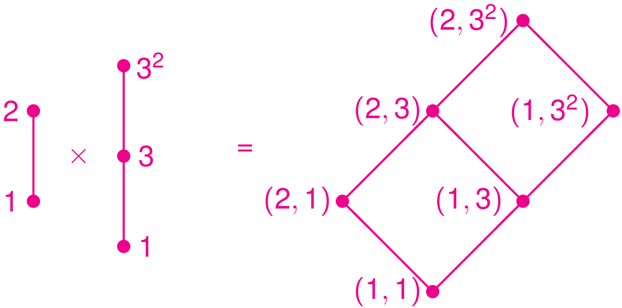

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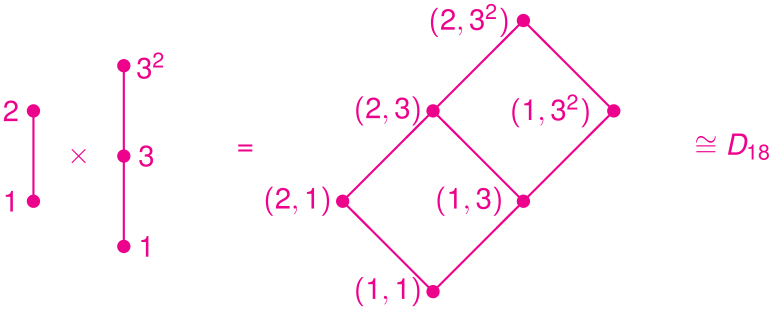

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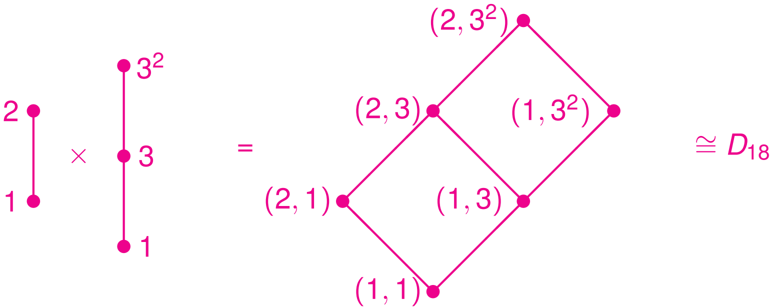

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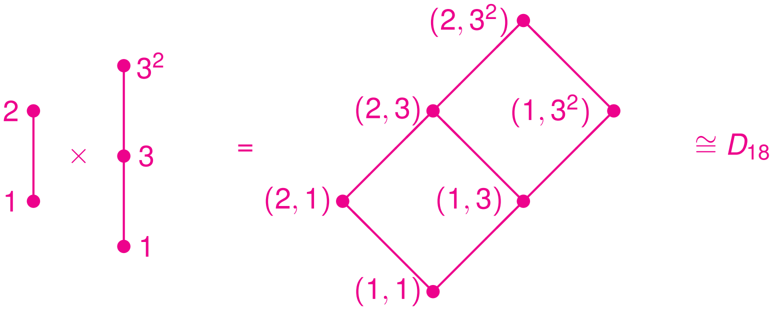

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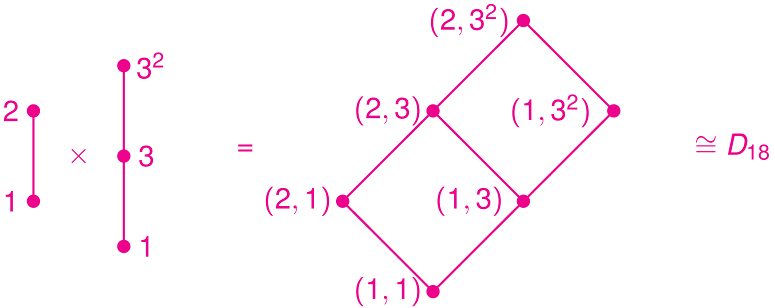

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Proof for C_n**.** Define $f : B_n \to C_1^n$ by $f(S) = (x_1, \ldots, x_n)$ where $x_i = 1$ if $x_i \in S$ and $x_i = 0$ else. Then f is an isomorphism. $\square$

Outline

If P has a $\hat{0}$ then its *Möbius function*, $\mu : P \to \mathbf{Z}$, is defined recursively by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0}, \\ -\sum_{y<x} \mu(y) & \text{if } x > \hat{0}. \end{cases}$$

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Equivalently

$$\sum_{y \le x} \mu(y) = \delta_{x,\hat{0}}$$

where $\delta_{x,\hat{0}}$ is the Kronecker delta,

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \ne y. \end{cases}$$

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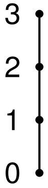

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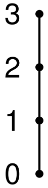
$C_3 =$ 

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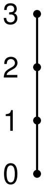

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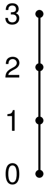
$C_3 =$ 

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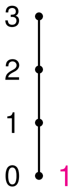

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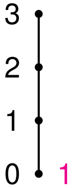

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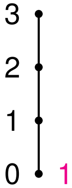
$C_3 =$ 

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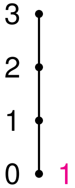
$C_3 =$ 

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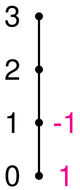

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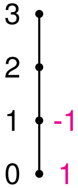

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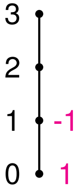

$\mu(0) = \mu(\hat{0}) = 1.$

$0 = \mu(1) + \mu(0) = \mu(1) + 1 \implies \mu(1) = -1.$

$0 = \mu(2) + \mu(1) + \mu(0) = \mu(2) - 1 + 1$

$$\sum_{y \le x} \mu(y) = \delta_{x,\hat{0}}.$$

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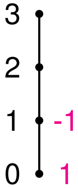

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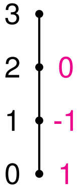
$C_3 =$ 

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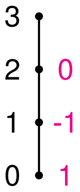
$C_3 =$ 

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Similarly $\mu(3) = 0$.

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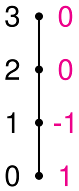
Example: The Chain.

$C_3 =$ 

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Similarly $\mu(3) = 0$.

$$\sum_{y \le x} \mu(y) = \delta_{x,\hat{0}}.$$

Example: The Chain.

$$C_3 = \begin{array}{ccc} 3 & \bullet & 0 \\ 2 & \bullet & 0 \\ 1 & \bullet & -1 \\ 0 & \bullet & 1 \end{array}$$

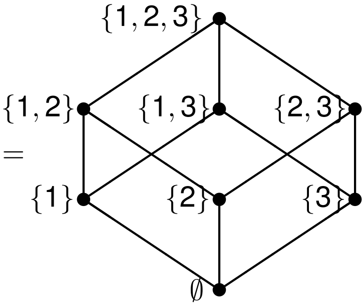
$\mu(0) = \mu(\hat{0}) = 1.$
$0 = \mu(1) + \mu(0) = \mu(1) + 1 \implies \mu(1) = -1.$
$0 = \mu(2) + \mu(1) + \mu(0) = \mu(2) - 1 + 1 \implies \mu(2) = 0.$
Similarly $\mu(3) = 0$.

Proposition

In C_n we have $\mu(i) = \begin{cases} 1 & \textit{if } i = 0 \\ -1 & \textit{if } i = 1, \\ 0 & \textit{else.} \end{cases}$ $\square$

Example: The Boolean Algebra.

$B_3 =$



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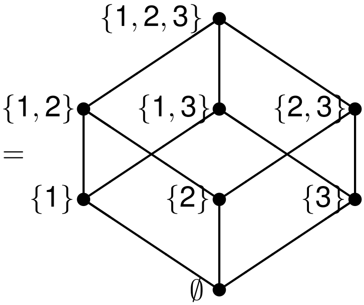

$\mu(\emptyset)$

Example: The Boolean Algebra.

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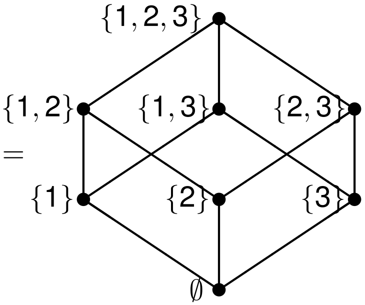
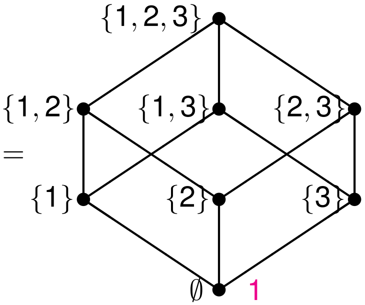


$\mu(\emptyset) = \mu(\hat{0}) = 1,$

Example: The Boolean Algebra.

$B_3 =$



$$\mu(\emptyset) = \mu(\hat{0}) = 1,$$

Example: The Boolean Algebra.

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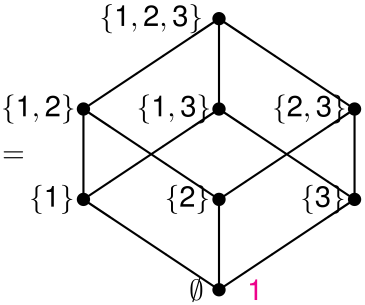
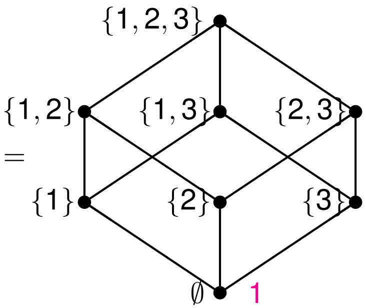


$\mu(\emptyset) = \mu(\hat{0}) = 1,$
$0 = \mu(\{1\}) + \mu(\emptyset)$

Example: The Boolean Algebra.

$B_3 =$



$$\mu(\emptyset) = \mu(\hat{0}) = 1,$$
$$0 = \mu(\{1\}) + \mu(\emptyset) \implies \mu(\{1\}) = -1.$$

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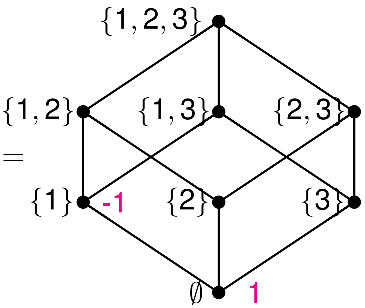
$B_3 =$



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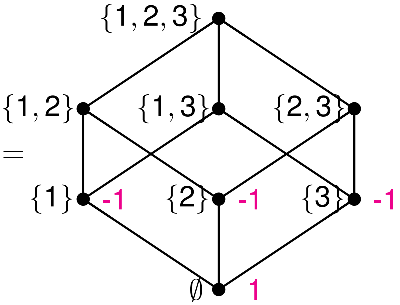
$B_3 =$ 

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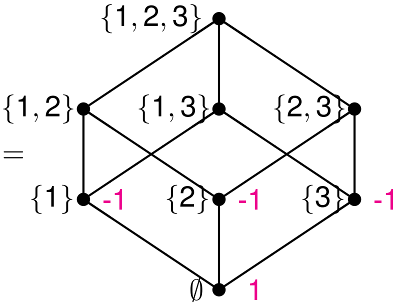

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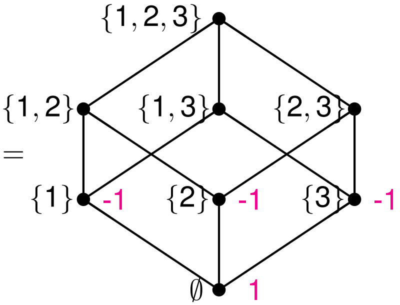
$B_3 =$



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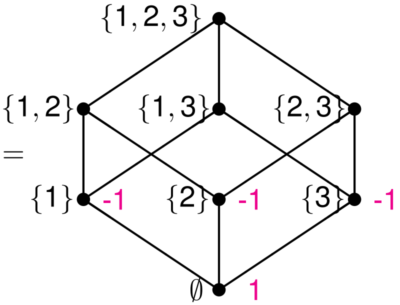
Example: The Boolean Algebra.

$B_3 =$



$$
\begin{aligned}
&\mu(\emptyset) = \mu(\hat{0}) = 1,\\
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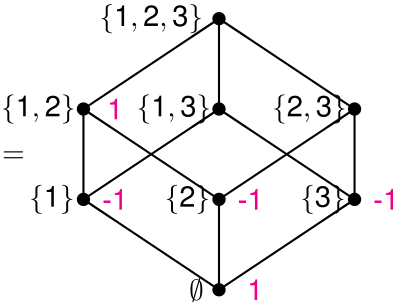
&\qquad\qquad\qquad \implies \mu(\{1,2\}) = 1,
\end{aligned}
$$

Example: The Boolean Algebra.

$B_3 =$



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$$\implies \mu(\{1,2\}) = 1,$$

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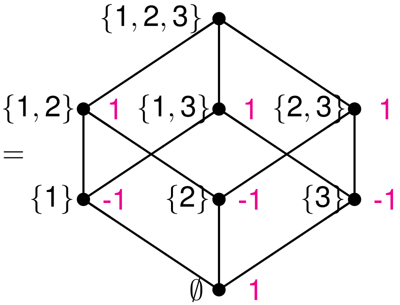

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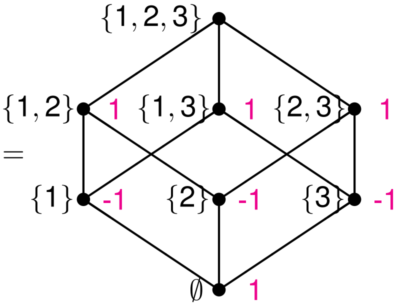
Example: The Boolean Algebra.

$B_3 =$



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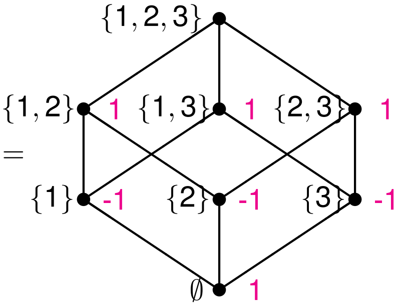
&\qquad\qquad\qquad\qquad \implies \mu(\{1,2\}) = 1, \\
&0 = \mu(\{1,2,3\}) + 1 + 1 + 1 - 1 - 1 - 1 + 1
\end{aligned}$$

Example: The Boolean Algebra.

$B_3 =$



$$
\begin{aligned}
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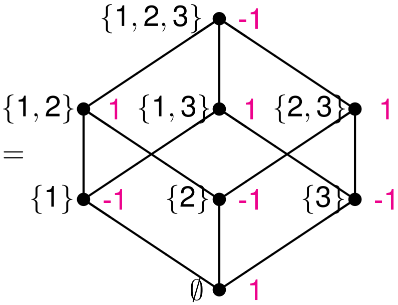
\end{aligned}
$$

Example: The Boolean Algebra.

$B_3 =$



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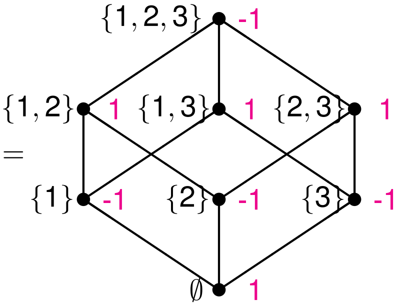
$B_3 =$ 

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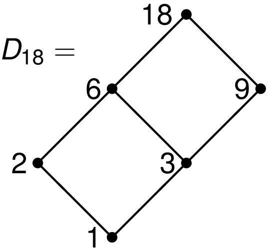
$$\implies \mu(\{1,2\}) = 1,$$
$$0 = \mu(\{1,2,3\}) + 1 + 1 + 1 - 1 - 1 - 1 + 1 \implies \mu(\{1,2,3\}) = -1$$

Conjecture

In B_n we have $\mu(S) = (-1)^{|S|}$.

Example: The Divisor Lattice.

$D_{18} =$



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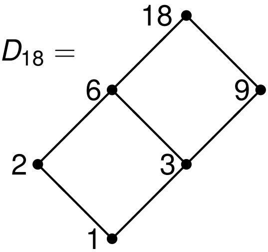
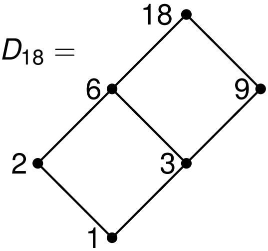


$\mu(1)$

Example: The Divisor Lattice.

$D_{18} =$



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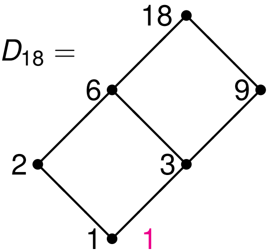

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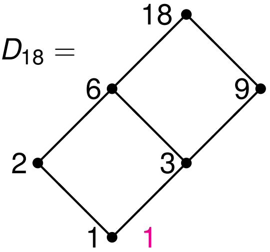

$\mu(1) = \mu(\hat{0}) = 1.$

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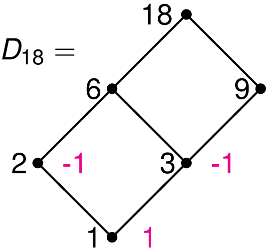
Example: The Divisor Lattice.

$D_{18} =$



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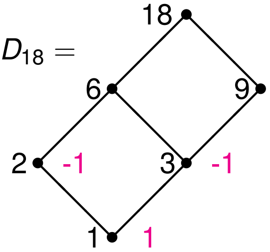


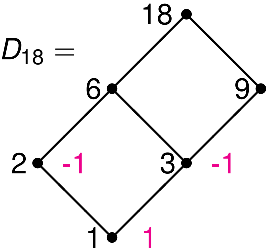
$\mu(1) = \mu(\hat{0}) = 1.$

$\mu(2) = \mu(3) = -1.$

$0 = \mu(6) + \mu(2) + \mu(3) + \mu(1)$

Example: The Divisor Lattice.

$D_{18} =$



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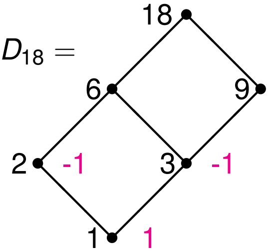
$0 = \mu(6) + \mu(2) + \mu(3) + \mu(1) = \mu(6) - 1 - 1 + 1$

Example: The Divisor Lattice.

$D_{18} =$



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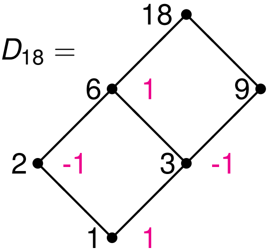
$D_{18} =$ 

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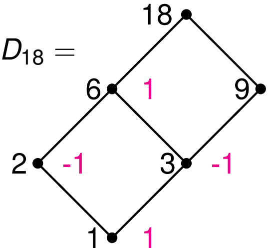
$D_{18} =$ 

$\mu(1) = \mu(\hat{0}) = 1.$
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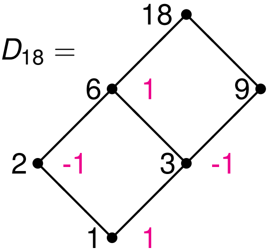
$0 = \mu(6) + \mu(2) + \mu(3) + \mu(1) = \mu(6) - 1 - 1 + 1 \implies \mu(6) = 1.$
$0 = \mu(9) + \mu(3) + \mu(1)$

Example: The Divisor Lattice.

$D_{18} =$



$\mu(1) = \mu(\hat{0}) = 1.$

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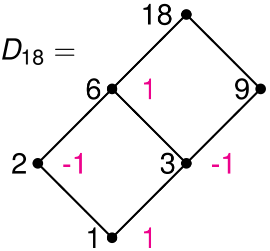
$0 = \mu(6) + \mu(2) + \mu(3) + \mu(1) = \mu(6) - 1 - 1 + 1 \implies \mu(6) = 1.$

$0 = \mu(9) + \mu(3) + \mu(1) = \mu(9) - 1 + 1$

Example: The Divisor Lattice.

$D_{18} =$



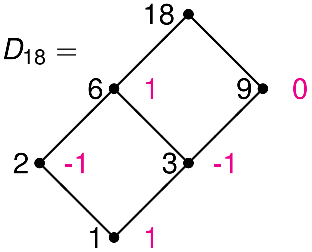
$\mu(1) = \mu(\hat{0}) = 1.$

$\mu(2) = \mu(3) = -1.$

$0 = \mu(6) + \mu(2) + \mu(3) + \mu(1) = \mu(6) - 1 - 1 + 1 \implies \mu(6) = 1.$

$0 = \mu(9) + \mu(3) + \mu(1) = \mu(9) - 1 + 1 \implies \mu(9) = 0.$

Example: The Divisor Lattice.



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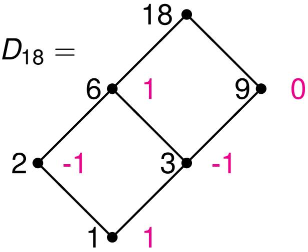
Example: The Divisor Lattice.

$D_{18} =$



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$\mu(2) = \mu(3) = -1.$

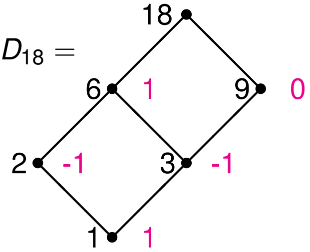
$0 = \mu(6) + \mu(2) + \mu(3) + \mu(1) = \mu(6) - 1 - 1 + 1 \implies \mu(6) = 1.$

$0 = \mu(9) + \mu(3) + \mu(1) = \mu(9) - 1 + 1 \implies \mu(9) = 0.$

$0 = \mu(18) + 1 + 0 - 1 - 1 + 1$

Example: The Divisor Lattice.

$D_{18} =$



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$0 = \mu(18) + 1 + 0 - 1 - 1 + 1 \implies \mu(18) = 0.$

Example: The Divisor Lattice.

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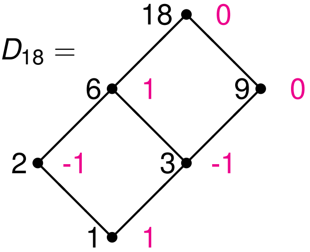

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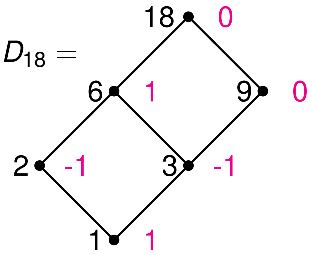

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Conjecture

If $d \in D_n$ has prime factorization $d = p_1^{n_1} \cdots p_k^{n_k}$ then

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Notes. 1. The two parts of the MIT are actually “if and only if.”
2. One can apply the MIT to C_n, B_n, and D_n to obtain the FTDC, PIE, and MIT from Number Theory, respectively.

Outline

1. (a) Prove that if $S \subseteq T$ in B_n then $[S, T] \cong B_{|T-S|}$.
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   (b) Prove the PIE by Möbius inversion over B_n.
   (c) Prove the Number Th'y MIT by Möbius inversion over D_n.

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THANKS FOR LISTENING!