Partially Ordered Sets and their Möbius Functions IV: Factoring the Characteristic Polynomial

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June 2, 2014

Outline

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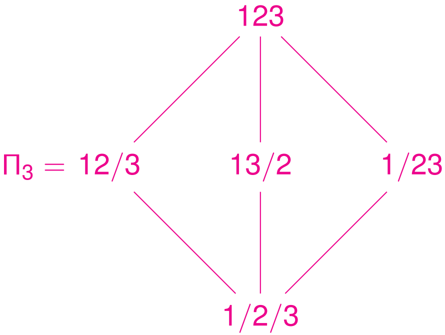

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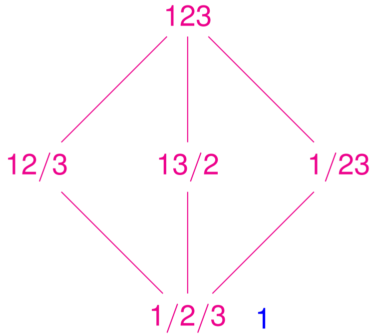

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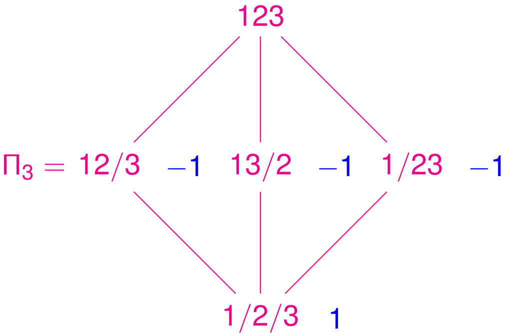

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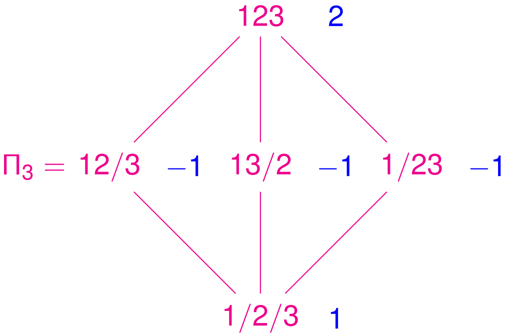

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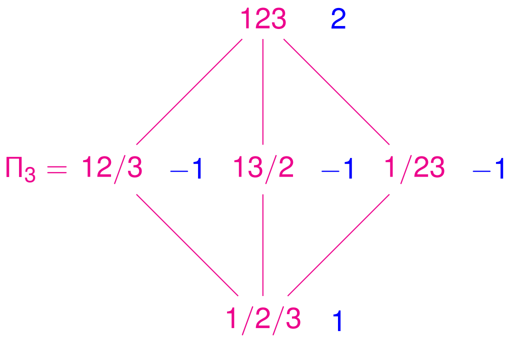

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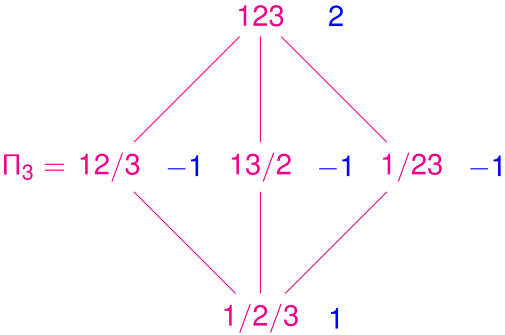

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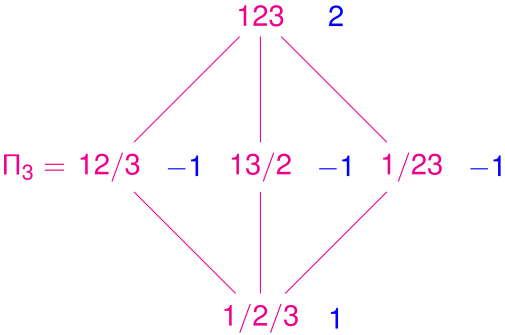

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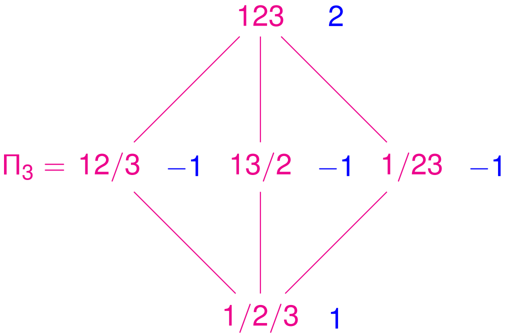

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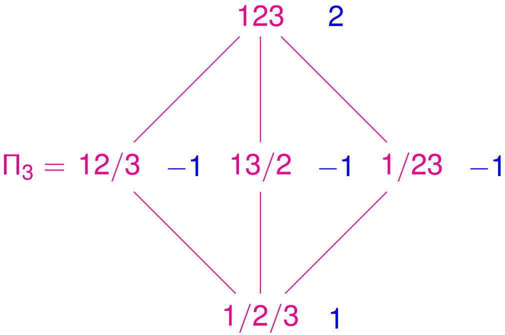

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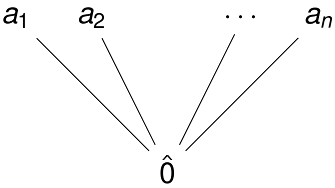
$CL_n =$ 

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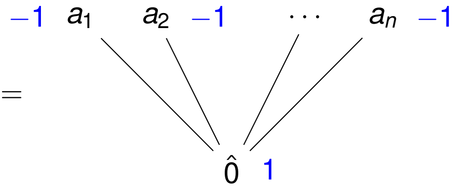
$CL_n =$ 

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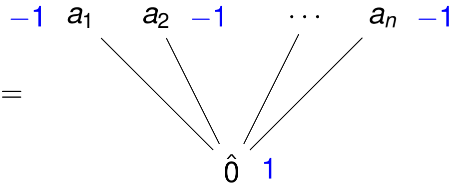

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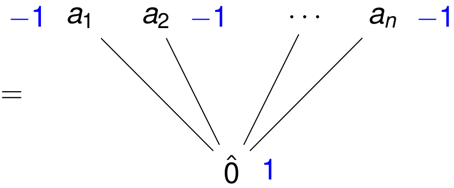

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So the characteristic polynomial of CL_n can give us any positive integer root as n varies.

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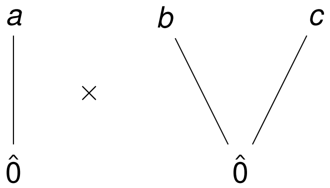

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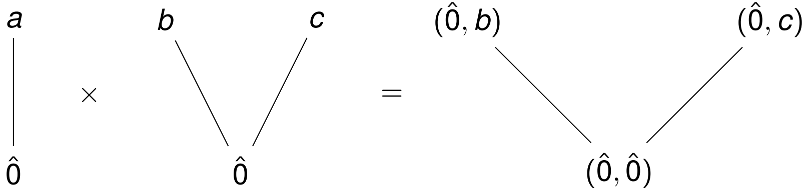

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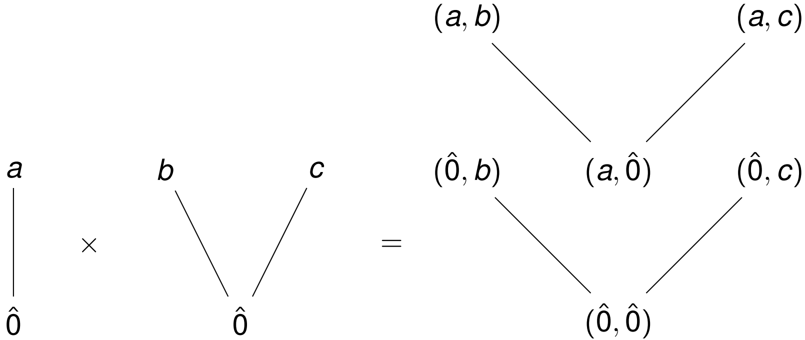

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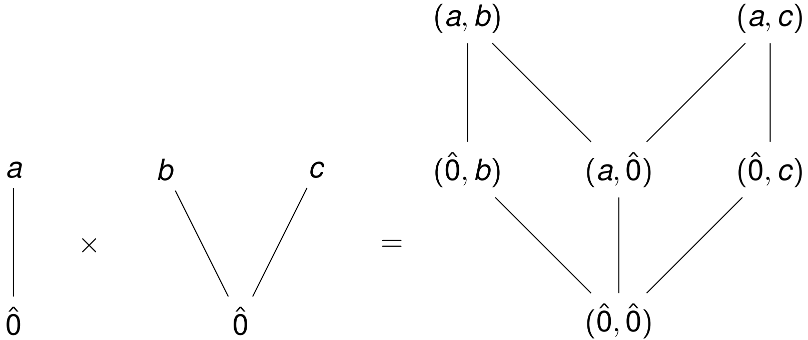

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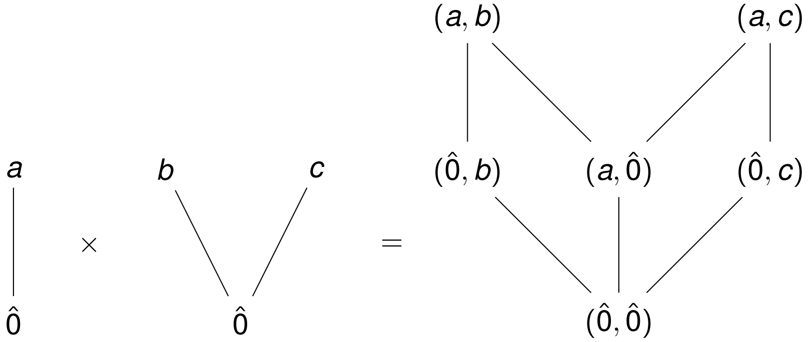

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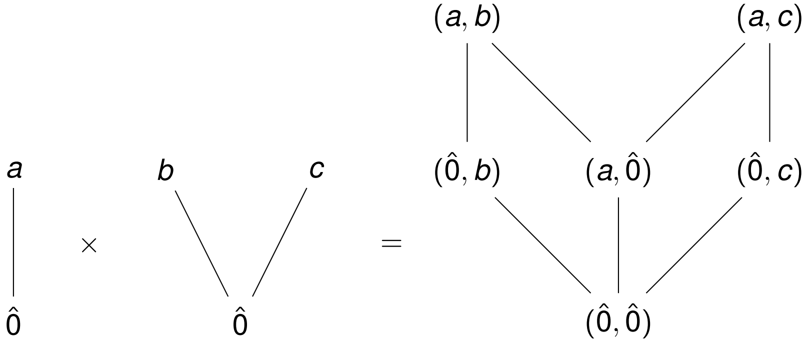

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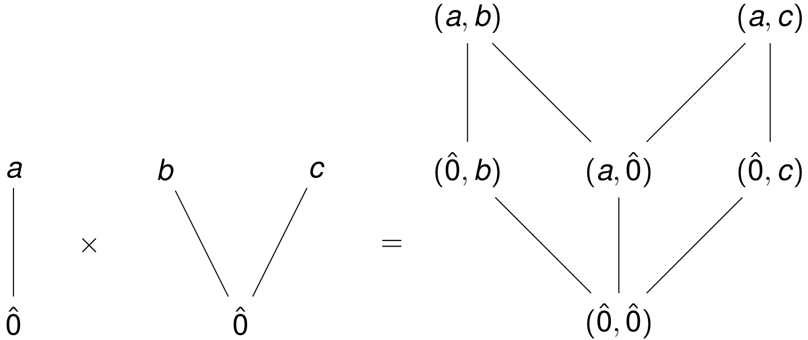

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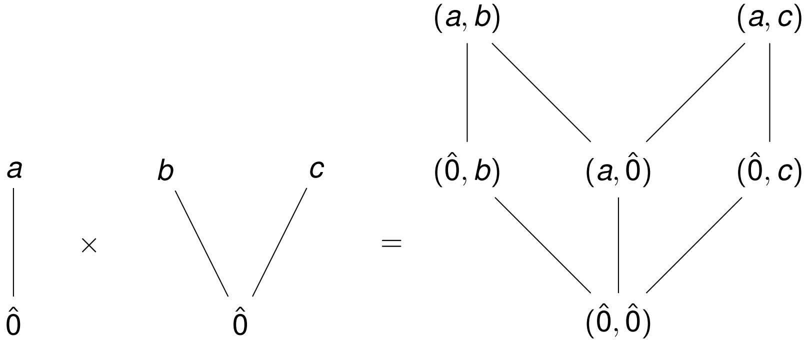

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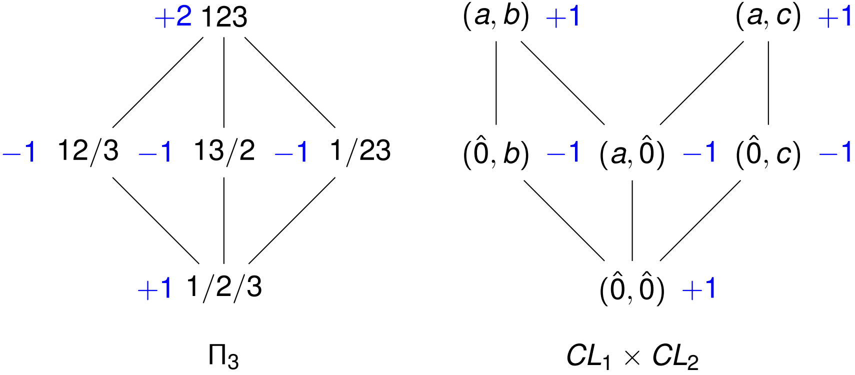

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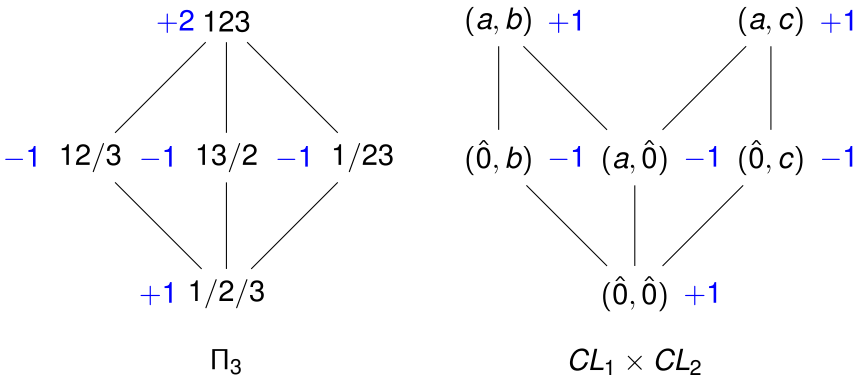

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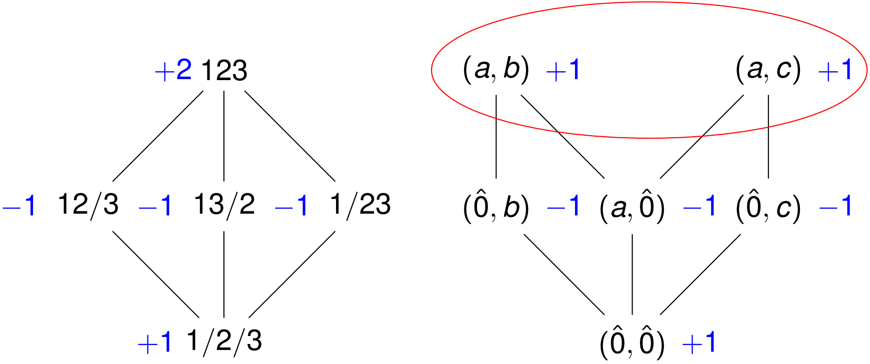

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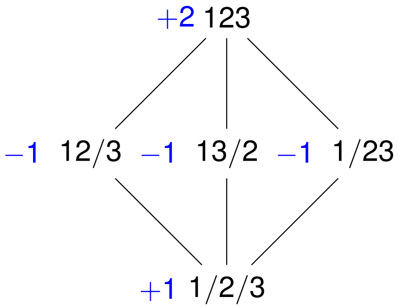

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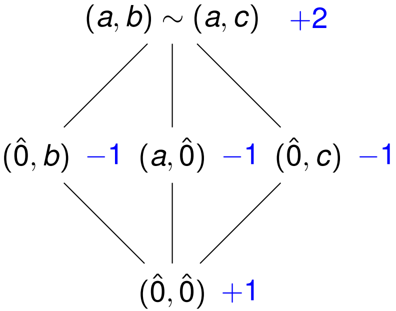

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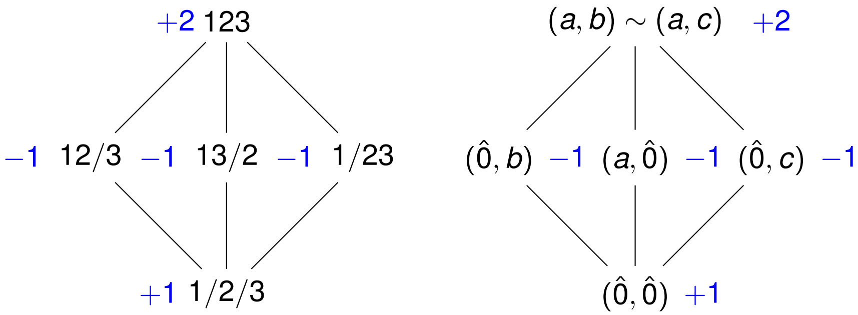

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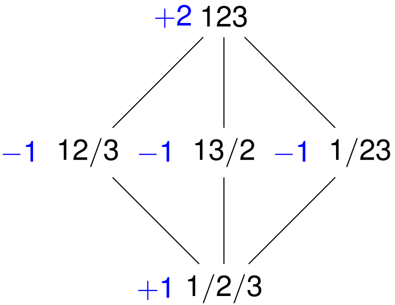

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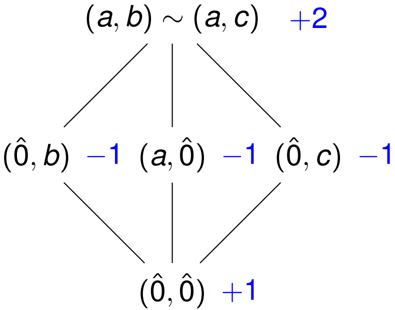

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Outline

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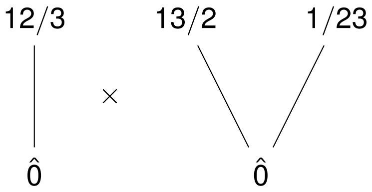

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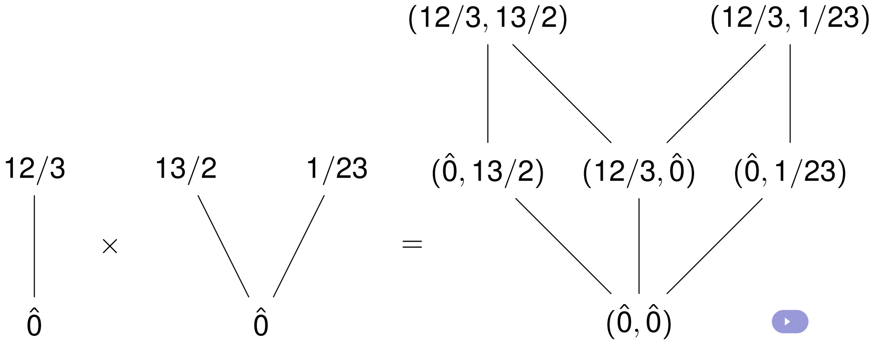

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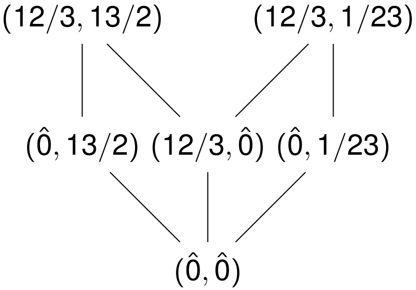

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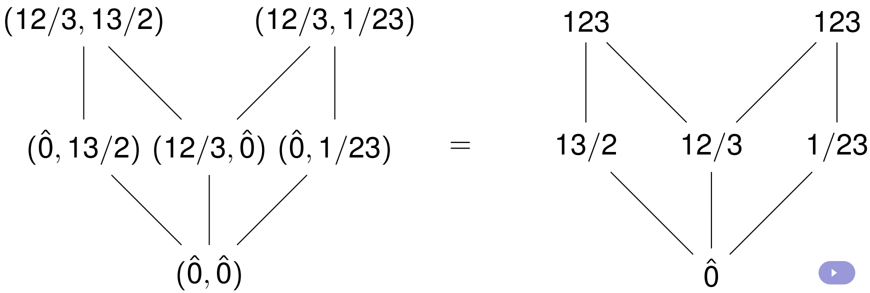

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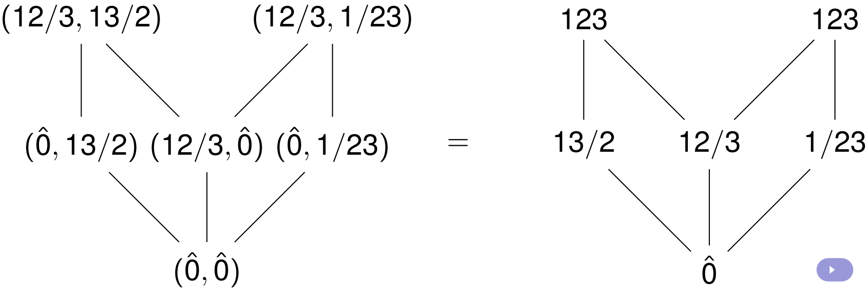

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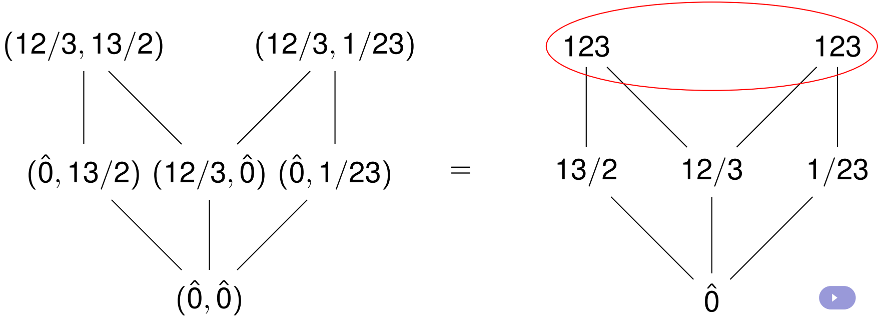

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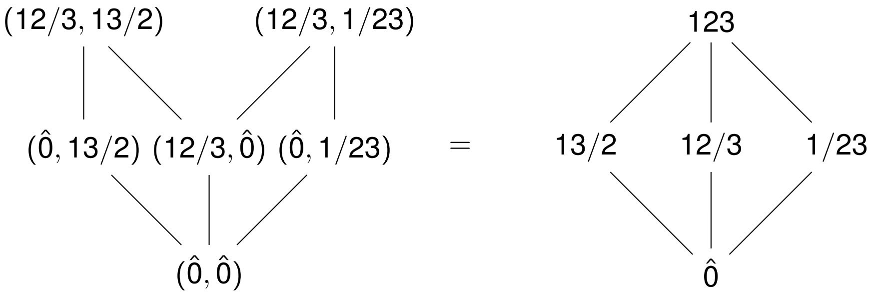

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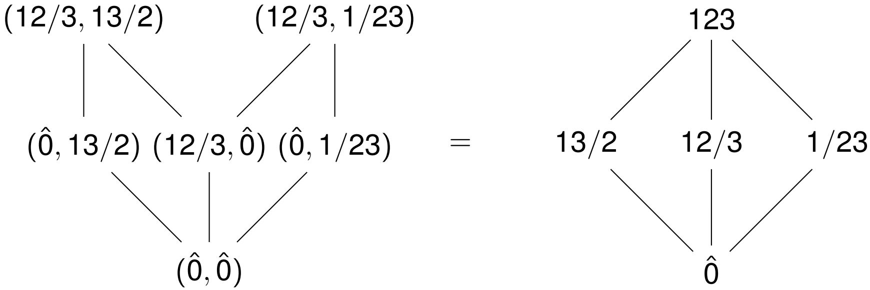

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Outline

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$x \in \Pi_3$	$\mathbf{t} \in \mathcal{T}_x^a$	$\operatorname{supp} \mathbf{t}$	$\rho(x)$
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Let L be a lattice, $(A_1, \ldots, A_n) \vdash \mathcal{A}(L)$ and $Q = \prod_i CL_{A_i}$. Suppose that for all $x \in L$ and all $\mathbf{t} \in \mathcal{T}_x^a$ we have

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Then the standard equivalence relation is homogeneous, $Q/\sim$ is ranked, and

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Condition (1) is used to prove that the map $(Q/\sim) \to L$ by $\mathcal{T}_x^a \mapsto x$ is surjective.

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Proof. If $i < j$ let $\{i, j\}$ be the atom of Π_n having this set as its unique non-singleton block.

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Outline

How do we find an appropriate atom partition?

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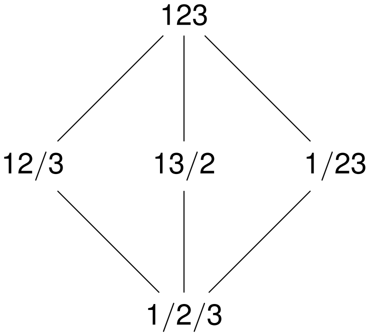

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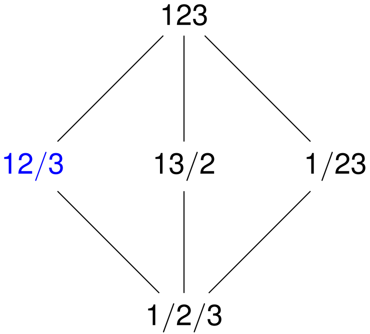

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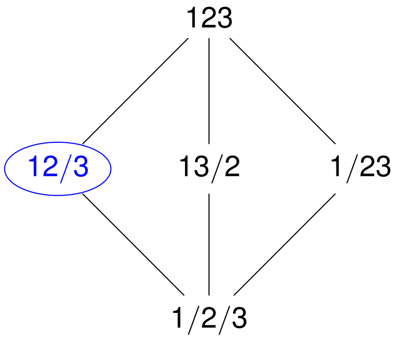

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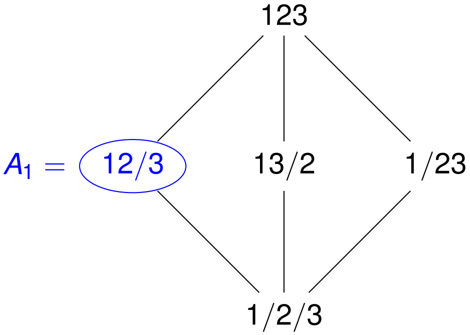

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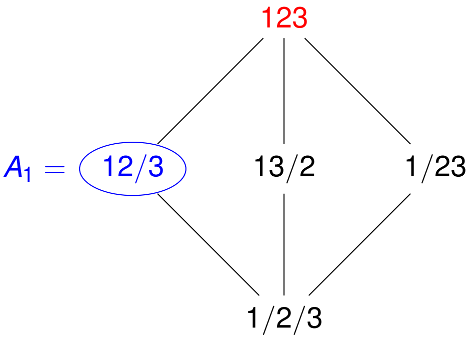

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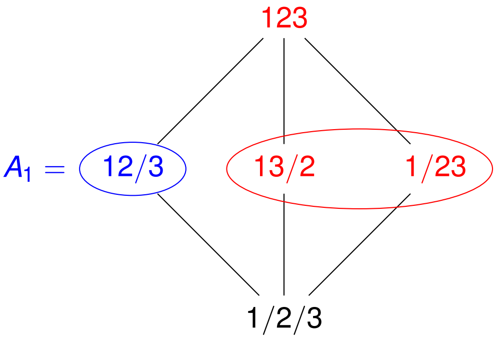

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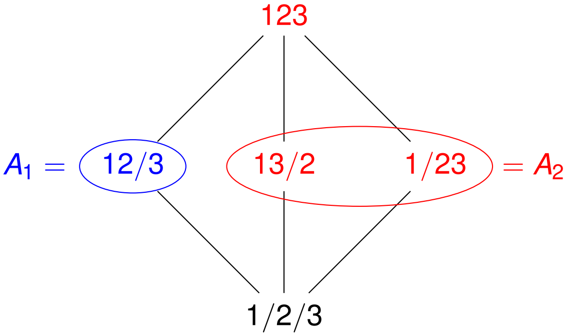

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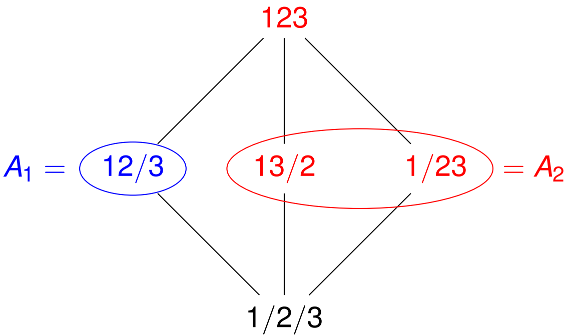

In Π_n, our partition is induced by $\hat{0} < [2] < [3] < \cdots < \hat{1}$ where $[i]$ is the partition having this set as its only non-trivial block

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Under these conditions, the following are equivalent.

1. *For each $x \neq \hat{0}$ in L, there is i such that $|A_x \cap A_i| = 1$.*
2. *Chain C satisfies the meet condition.*
3. *The characteristic polynomial of L factors as*

$$\chi(L, t) = t^{\rho(L)-n} \prod_{i=1}^{n} (t - |A_i|).$$

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Corollary (Stanley, 1972)

Let L be a semimodular, supersolvable lattice and $(A_1, \ldots, A_n)$ be induced by a saturated chain of left-modular elements. Then

$$\chi(L; t) = \prod_{i=1}^{n} (t - |A_i|).$$

Outline

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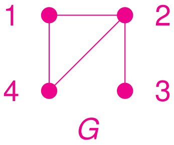
Ex. 

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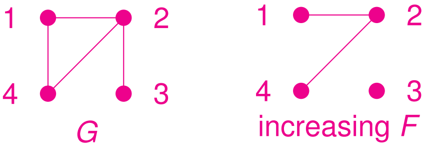

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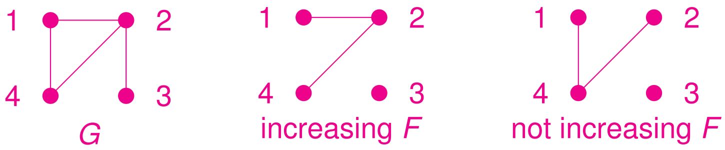

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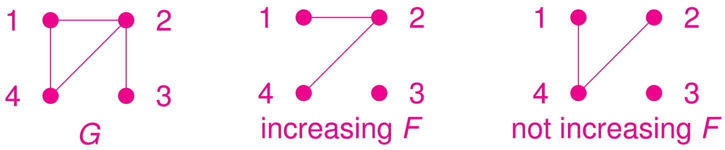

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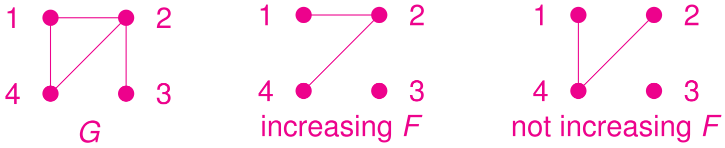

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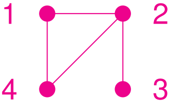
$G =$ 

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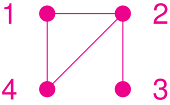
$G =$ 

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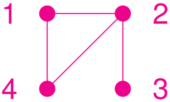
$G =$ 

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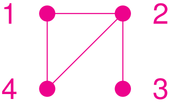
$G =$ 

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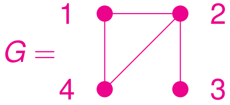

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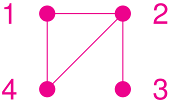
$G =$ 

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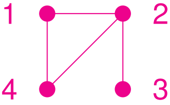
$G =$ 

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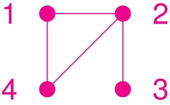

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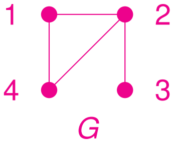

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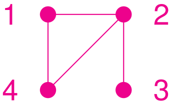

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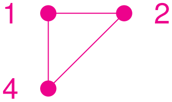

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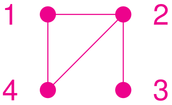

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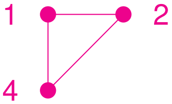

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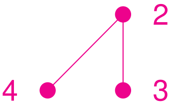

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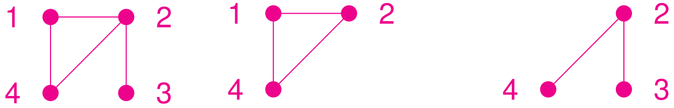

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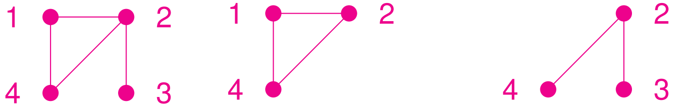

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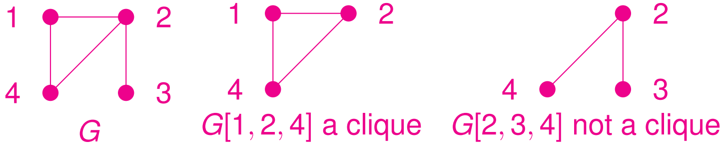

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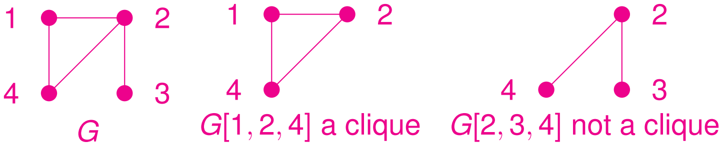

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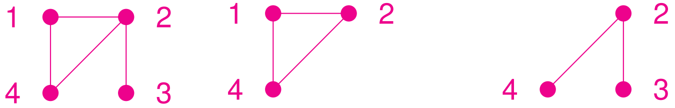
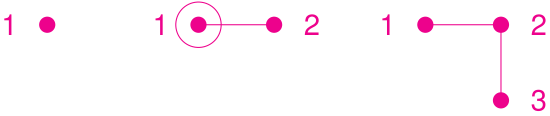


G $\quad G[1,2,4]$ a clique $\quad G[2,3,4]$ not a clique



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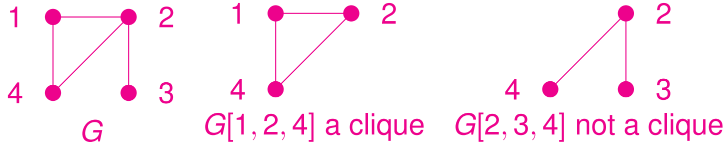

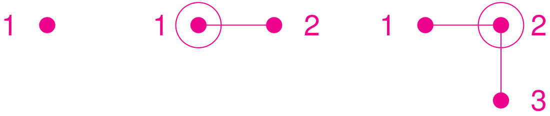

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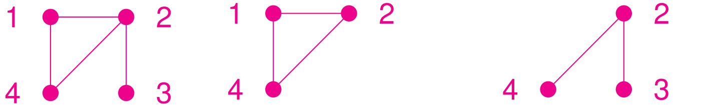

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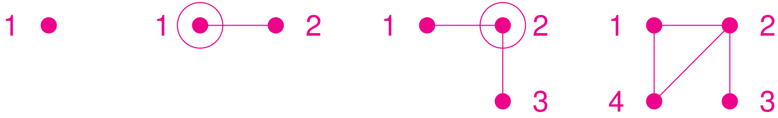

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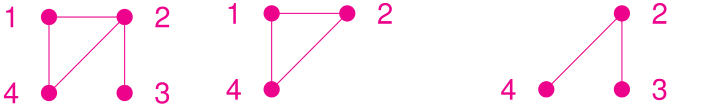

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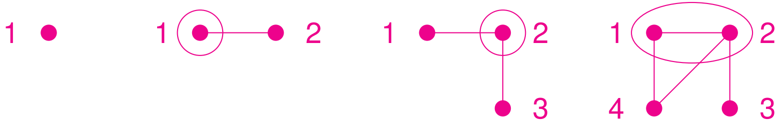

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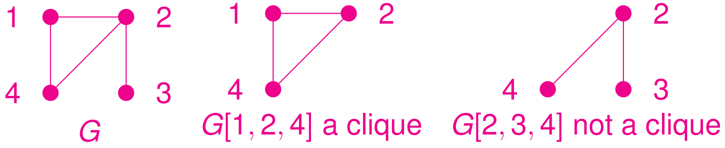

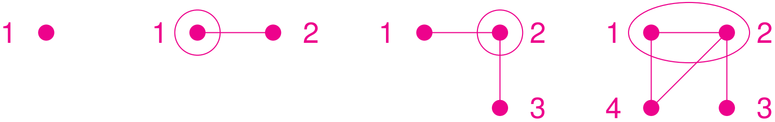

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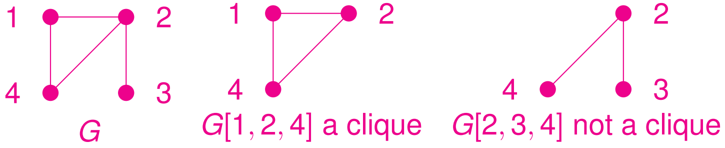

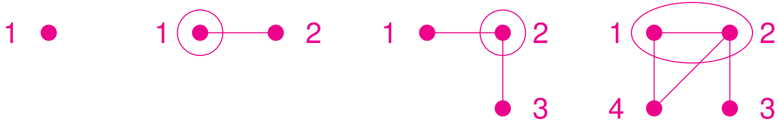

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If G is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of G with vertex set W. A *clique* in G is a subgraph $G[W]$ which is complete.

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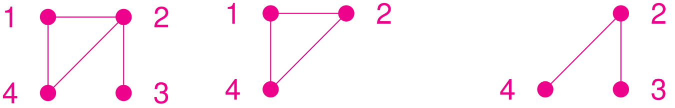

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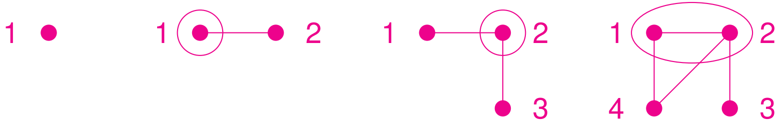

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Theorem (Hallam-S)

Let G be a graph with $V = [n]$. Then $p(G;t) = IF(G;t)$ if and only if $1, \ldots, n$ is a peo of G.