Partially Ordered Sets and their Möbius Functions IV: Factoring the Characteristic Polynomial

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Motivating Examples

Quotient Posets

The Standard Equivalence Relation

The Main Theorem

Partitions Induced by Chains

Application: Increasing Forests
Outline

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Application: Increasing Forests
This work is joint with Joshua Hallam.

All posets will be ranked.

Many ranked posets have characteristic polynomials whose roots are nonnegative integers. Why? Answers have been given by Saito and Terao, Stanley, Zaslavsky, Blass and S, as well as others. Recall that the characteristic polynomial of a ranked poset $P$ is $\chi(P) = \chi(P; t) = \sum_{x \in P} \mu(x) t^{\rho(P)} - \rho(x)$.

In some cases the factorization is easy to explain. Recall:

**Proposition**

Let $P$, $Q$ be ranked posets.

1. $P \sim Q \implies \chi(P; t) = \chi(Q; t)$.
2. $P \times Q$ is ranked and $\chi(P \times Q; t) = \chi(P; t) \chi(Q; t)$.

Ex. We have $\chi(C_1) = \mu(0) t + \mu(1) = t - 1$. Now $B_n \sim C_n$. So by the previous theorem $\chi(B_n) = \chi(C_n) = \chi(C_1)^n = (t - 1)^n$. 
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Ex. Consider the partition lattice $\Pi_3$. 

$$
\Pi_3 = \begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & \\
1 & 3 & \\
1 & 2 & 3
\end{array}
$$

$$
\chi(\Pi_3, t) = t^2 - 3t + 2 = (t-1)(t-2).
$$

Theorem $\chi(\Pi_n, t) = (t-1)(t-2) \cdots (t-n+1)$. 

But $\Pi_n$ is not a product of smaller posets.
Ex. Consider the partition lattice $\Pi_3$. 

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\[
\begin{array}{ccc}
123 & 2 \\
12/3 & \text{ } & 13/2 \\
1/2/3 & \text{ } & 1/23 \\
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\begin{align*}
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&\quad \frac{1/2}{3} 1
\end{align*}
\]

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$$CL_n = \hat{0} \quad a_1 \quad a_2 \quad \cdots \quad a_n$$

Thus $\chi(\text{CL}_n) = t - n$. So the characteristic polynomial of $\text{CL}_n$ can give us any positive integer root as $n$ varies.
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$$CL_n = \begin{array}{cccccc} -1 & a_1 & a_2 & -1 & \cdots & a_n & -1 \\ \hat{0} & 1 \end{array}$$

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& & & -1
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\hat{0} \times (a \times b \times c) = (\hat{0}, \hat{0}, \hat{0}, \hat{0})
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$$
\begin{array}{ccc}
-1 & 12/3 & -1 \\
1 & 13/2 & -1 \\
+1 & 1/23 & \\
\end{array}
$$

$$
\begin{array}{ccc}
(a, b) & +1 & (a, c) & +1 \\
(\hat{0}, b) & -1 & (a, \hat{0}) & -1 \\
(\hat{0}, \hat{0}) & +1 & \\
\end{array}
$$

$\Pi_3$

$CL_1 \times CL_2$
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\[
\begin{array}{c}
+2 & 123 \\
\downarrow & \downarrow & \downarrow \\
-1 & 12/3 & -1 & 13/2 & -1 & 1/23 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
+1 & 1/2/3 & -1 & 1/23 & -1 & 1/23 \\
\end{array}
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\begin{array}{c}
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\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(\hat{0}, \hat{0}) & +1 \\
\end{array}
\]

$CL_1 \times CL_2$

Note that the M"obius values of $(a, b)$ and $(a, c)$ added to give the M"obius value of $(a, b) \sim (a, c)$.

So $\chi(\Pi_3)$ did not change after the identification since characteristic polynomials only record the sums of the M"obius values at each rank.
Clearly $\Pi_3$ and $CL_1 \times CL_2$ are not isomorphic. What if we identify the top two elements of $CL_1 \times CL_2$?

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+2 \\
+1 \\
\end{array}
\end{array}
\begin{array}{c}123 \\
1/2/3 \\
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\end{array}
\begin{array}{c}\begin{array}{c}
+1 \\
-1 \\
\end{array}
\end{array}
\begin{array}{c}1/2/3 \\
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\end{array}
\begin{array}{c}\begin{array}{c}
(\hat{0}, b) \\
(\hat{0}, \hat{0}) \\
\end{array}
\end{array}
\begin{array}{c}+1 \\
+1 \\
\end{array}
\begin{array}{c}(a, \hat{0}) \\
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\end{array}
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\end{array}\end{array}
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\begin{array}{c}CL_1 \times CL_2 \\
\text{after identification}
\end{array}
Clearly $\Pi_3$ and $CL_1 \times CL_2$ are not isomorphic. What if we identify the top two elements of $CL_1 \times CL_2$?

Note that the Möbius values of $(a, b)$ and $(a, c)$ added to give the Möbius value of $(a, b) \sim (a, c)$. So $\chi(CL_1 \times CL_2)$ did not change after the identification since characteristic polynomials only record the sums of the Möbius values at each rank.
General Method.

Suppose \( P \) is a ranked poset and we wish to prove \( \chi(P) = (t - r_1) \ldots (t - r_n) \) where \( r_1, \ldots, r_n \) are positive integers.

1. Construct the poset \( Q = \text{CL}^{r_1} \times \cdots \times \text{CL}^{r_n} \).

2. Identify elements of \( Q \) to form a poset \( Q/\sim \) in such a way that
   - \( \chi(Q/\sim) = \chi(Q) = (t - r_1) \ldots (t - r_n) \),
   - \( Q/\sim \sim = P \).

3. It follows that \( \chi(P) = \chi(Q/\sim) = (t - r_1) \ldots (t - r_n) \).
General Method.
Suppose $P$ is a ranked poset and we wish to prove

$$\chi(P) = (t - r_1) \ldots (t - r_n)$$

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**General Method.**

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   $$
   Q = CL_{r_1} \times \cdots \times CL_{r_n}.
   $$

2. **Identify elements of $Q$ to form a poset $Q/\sim$ in such a way that**

   (a) $\chi(Q/\sim) = \chi(Q)$
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1. Construct the poset

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2. Identify elements of $Q$ to form a poset $Q/\sim$ in such a way that

(a) $\chi(Q/\sim) = \chi(Q) = (t - r_1) \ldots (t - r_n),$
(b) $(Q/\sim) \cong P.$
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Suppose $P$ is a ranked poset and we wish to prove

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General Method.
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**General Method.**

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Application: Increasing Forests
Let $P$ be a poset and let $\sim$ be an equivalence relation on $P$. We define the quotient, $P/\sim$, to be the set of equivalence classes with the binary relation $\leq$ defined by $X \leq Y$ in $P/\sim \iff x \leq y$ in $P$ for some $x \in X$ and some $y \in Y$. Quotients of posets need not be posets. Ex. Consider $C_3 = \{0, 1, 2, 3\}$. Put an equivalence relation on $C_3$ with classes $X = \{0, 2\}$, $Y = \{1\}$. Then $X < Y$ since $0 < 1$ and $Y < X$ since $1 < 2$. 
Let $P$ be a poset and let $\sim$ be an equivalence relation on $P$. We define the quotient, $P/\sim$, to be the set of equivalence classes with the binary relation $\leq$ defined by

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**Ex.** Consider

\[
C_3 = \begin{array}{c}
& & 2 \\
& 1 \\
0 & & \\
\end{array}
\]
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Then $X < Y$ since $0 < 1$. 


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**Ex.** Consider

$$C_3 = \begin{array}{c} \text{2} \\ \downarrow \text{1} \\ \text{0} \end{array}$$

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Then $X < Y$ since $0 < 1$ and $Y < X$ since $1 < 2$. 
Let $P$ be a poset and let $\sim$ be an equivalence relation on $P$.
Let $P$ be a poset and let $\sim$ be an equivalence relation on $P$. We say the quotient $P/\sim$ is a *homogeneous quotient* if

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**Lemma (Hallam-S)**

If $P/\sim$ is a homogeneous quotient then $P/\sim$ is a poset.

**Proof.** Reflexivity and transitivity in $\leq$ in $P/\sim$ are easy. To prove antisymmetry, suppose that $X \leq Y$ and $Y \leq X$. By definition, there is an $x \in X$ and $y \in Y$ with $x \leq y$. Since $Y \leq X$ there is an $x' \in X$ with $x \leq y \leq x'$. Since $X \leq Y$ there is a $y' \in Y$ with $x \leq y \leq x' \leq y'$. Continuing, we get a chain $x \leq y \leq x' \leq y' \leq ...$ If some inequality is an equality, then we have a common element of $X$ and $Y$ which implies $X = Y$. If all are strict, then we would have an infinite chain in $P$. But this contradicts the fact that $P$ is finite, so this case can not happen.
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How do we determine a suitable equivalence relation?
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\[
\begin{array}{ccc}
12/3 & 13/2 & 1/23 \\
\times & \\ \\
\hat{0} & \hat{0} & \hat{0}
\end{array}
\]
How do we determine a suitable equivalence relation? If $P$ is a lattice, then there is a canonical choice.

Let us revisit $\Pi_3$. Label the atoms of $CL_1 \times CL_2$ with atoms from $\Pi_3$ as follows:

\[
\begin{align*}
\hat{0} & \quad 12/3 & \quad 13/2 & \quad 1/23 & \quad (\hat{0}, 13/2) & \quad (12/3, \hat{0}) & \quad (\hat{0}, 1/23) \\
\times & \quad \times & \quad \times & \quad \times & \quad \times & \quad \times & \quad \times \\
\hat{0} & \quad \hat{0} & \quad \hat{0} & \quad (\hat{0}, \hat{0}) & \quad (\hat{0}, \hat{0}) & \quad (\hat{0}, \hat{0}) & \quad (\hat{0}, \hat{0})
\end{align*}
\]
Now relabel each element of the product with the join of its two coordinates.

Finally, identify elements with the same label to obtain the same quotient we did before.

Not only is the quotient isomorphic to $\Pi_3$, it even has the same labeling.
Now relabel each element of the product with the join of its two coordinates.

(12/3, 13/2) (12/3, 1/23)

(0, 13/2) (12/3, 0) (0, 1/23)

(0, 0)

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Now relabel each element of the product with the join of its two coordinates.

\[(\hat{0}, \frac{13}{2}), (\frac{12}{3}, \hat{0}), (\hat{0}, 1/23)\]

\[=\]

\[(\frac{12}{3}, \frac{13}{2}), (\frac{12}{3}, 1/23), (\frac{12}{3}, \hat{0})\]

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\[(\hat{0}, \frac{13}{2}) \quad \hat{0}, \frac{1}{23} \quad \frac{12}{3}, \hat{0} \quad \hat{0}, \hat{0}\]

\[(\hat{0}, \hat{0}) \quad \frac{12}{3}, \hat{0} \quad \hat{0}, \frac{1}{23} \quad \frac{12}{3}, \frac{13}{2}\]

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\[(12/3, 13/2) \quad (12/3, 1/23)\]

\[\text{=} \quad 123\]

\[\text{=} \quad 13/2 \quad 12/3 \quad 1/23\]

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\[
\begin{align*}
(12/3, 13/2) & \quad (12/3, 1/23) \\
(\hat{0}, 13/2) & \quad (12/3, \hat{0}) & (\hat{0}, 1/23) \\
(\hat{0}, \hat{0}) & & & \\
\end{align*}
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(0, 0) & \\
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Finally, identify elements with the same label to obtain the same quotient we did before. Not only is the quotient isomorphic to \( \Pi_3 \), it even has the same labeling.
An ordered partition of a set $\mathcal{A}$ is a sequence of subsets $(A_1, \ldots, A_n)$ with $\bigcup_i A_i = \mathcal{A}$.
An ordered partition of a set \( \mathcal{A} \) is a sequence of subsets \((A_1, \ldots, A_n)\) with \( \bigcup_i A_i = \mathcal{A} \). We write \((A_1, \ldots, A_n) \vdash \mathcal{A}\).
An ordered partition of a set $\mathcal{A}$ is a sequence of subsets $(A_1, \ldots, A_n)$ with $\bigcup_i A_i = \mathcal{A}$. We write $(A_1, \ldots, A_n) \vdash \mathcal{A}$.

**Ex.** $(A_1, A_2) \vdash \mathcal{A}(\Pi_3)$ where $A_1 = \{12/3\}$, $A_2 = \{13/2, 1/23\}$. 
An ordered partition of a set $A$ is a sequence of subsets $(A_1, \ldots, A_n)$ with $\bigcup_i A_i = A$. We write $(A_1, \ldots, A_n) \vdash A$. Let $(A_1, \ldots, A_n) \vdash A(L)$ for a lattice $L$. Let $CL_{A_i}$ be the claw with atom set $A_i$.

Ex. $(A_1, A_2) \vdash A(\Pi_3)$ where $A_1 = \{12/3\}$, $A_2 = \{13/2, 1/23\}$. 
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Let \((A_1, \ldots, A_n) \vdash A(L)\) for a lattice \( L \). Let \( CL_{A_i} \) be the claw with atom set \( A_i \).

**Ex.** \((A_1, A_2) \vdash A(\Pi_3)\) where \( A_1 = \{12/3\}, A_2 = \{13/2, 1/23\}\). Note that \( CL_{A_1} \) and \( CL_{A_2} \) were the claws used for \( \Pi_3 \).
An ordered partition of a set $\mathcal{A}$ is a sequence of subsets $(A_1, \ldots, A_n)$ with $\bigcup_i A_i = \mathcal{A}$. We write $(A_1, \ldots, A_n) \vdash \mathcal{A}$. Let $(A_1, \ldots, A_n) \vdash \mathcal{A}(L)$ for a lattice $L$. Let $CL_{A_i}$ be the claw with atom set $A_i$. The standard equivalence relation on $\prod_i CL_{A_i}$ is

$$t \sim s \text{ in } \prod_{i=1}^n CL_{A_i} \iff \bigvee t = \bigvee s \text{ in } L.$$ 

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Outline

Motivating Examples

Quotient Posets

The Standard Equivalence Relation

The Main Theorem

Partitions Induced by Chains

Application: Increasing Forests
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<td>0</td>
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Let $L$ be a lattice, $(A_1, \ldots, A_n) \vdash A(L)$ and $Q = \prod_i CL_{A_i}$. Suppose that for all $x \in L$ and all $t \in T_x^a$ we have $|\text{supp } t| = \rho(t)$. Then the standard equivalence relation is homogeneous, $Q/\sim$ is ranked, and $\rho(T_x^a) = \rho(x)$.  

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x \in \Pi_3 & t \in \mathcal{T}_x^a & \text{supp } t & \rho(x) \\
\hline
\hat{0} & (\hat{0}, \hat{0}) & \emptyset & 0 \\
12/3 & (12/3, \hat{0}) & \{1\} & 1 \\
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\]

Then the standard equivalence relation is homogeneous, \( Q/ \sim \) is ranked, and

\[
\rho(\mathcal{T}_x^a) = \rho(x).
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We wish to make sure that when identifying the elements in an equivalence class, the Möbius function of the class is the sum of the Möbius functions of its elements so that $\chi$ does not change.

Lemma (Hallam-S)

Let lattice $L$, $(A_1,\ldots,A_n) \vdash A(L)$ and $Q = \prod_{i \in I} A_i$ satisfy the conditions of the previous lemma. Suppose, for each $x \neq \hat{0}$ in $L$, there exists an index $i$ such that $|A_x \cap A_i| = 1$. (1) Then for any $T \ni x \in Q/\sim$ we have $\mu(T \ni x) = \sum_{t \in T \ni x} \mu(t)$. 

Ex. $\Pi_3$ with $A_1 = \{12/3\}$ and $A_2 = \{13/2, 1/23\}$.

If $x \in A(\Pi_3)$, then $A_x = \{x\}$ and (1) is clear. If $x = 123$ then $|A_x \cap A_1| = 1$. 
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**Theorem (Hallam-S)**

Let \( L \) be a lattice, \( (A_1, \ldots, A_n) \models \mathcal{A}(L) \) and \( Q = \prod_i CL_{A_i} \).

Suppose that the following three conditions hold.

1. For all \( x \in L \) we have \( T_a x \neq \emptyset \).
2. If \( t \in T_a x \) then \( |\text{supp} t| = \rho(x) \).
3. For each \( x \neq \hat{0} \) in \( L \), there is \( i \) such that \( |A_x \cap A_i| = 1 \).

Then we can conclude the following.

(a) \( (Q/\sim) \sim = L \).
(b) \( \chi(L; t) = n \prod_{i=1}^n (t - |A_i|) \).

Condition (1) is used to prove that the map \( (Q/\sim) \rightarrow L \) by \( T_a x \mapsto x \) is surjective.
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Let $L$ be a lattice, $(A_1, \ldots, A_n) \vdash A(L)$ and $Q = \prod_i CL_{A_i}$. Suppose that the following three conditions hold.

1. For all $x \in L$ we have $Ta_x \neq \emptyset$.
2. If $t \in Ta_x$ then $|\text{supp } t| = \rho(x)$.
3. For each $x \neq \hat{0}$ in $L$, there is $i$ such that $|A_x \cap A_i| = 1$.

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Then we can conclude the following.

(a) $Q / \sim \sim = L$.

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Condition (1) is used to prove that the map $(Q / \sim \sim) \rightarrow L$ by $T_x^a \mapsto x$ is surjective.
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Condition (1) is used to prove that the map $(Q/\sim) \to L$ by $\mathcal{T}_x^a \mapsto x$ is surjective.
Corollary

\[ \chi(\Pi_n; t) = (t - 1)(t - 2) \ldots (t - n + 1). \]
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**Proof.** If \( i < j \) let \( \{i, j\} \) be the atom of \( \Pi_n \) having this set as its unique non-singleton block.
Corollary

$$\chi(\Pi_n; t) = (t - 1)(t - 2) \ldots (t - n + 1).$$

**Proof.** If $i < j$ let \{i, j\} be the atom of $\Pi_n$ having this set as its unique non-singleton block. Let $(A_1, \ldots, A_{n-1}) \vdash A(\Pi_n)$ where

$$A_i = \{\{1, i + 1\}, \{2, i + 1\}, \ldots, \{i, i + 1\}\}.$$
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We will verify the three conditions for \( x = \hat{1} \).
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(1) \( (\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}) \in T_{\hat{1}}^a. \)
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(2) \( \)
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We will verify the three conditions for \( x = \hat{1} \).

(1) \( \{(1, 2), (2, 3), \ldots, (n - 1, n)\} \in T_a^1 \).

(2) With any \( t \in Q \), associate a graph \( G_t \) with \( V = [n] \) and
\[ ij \in E \iff \{i, j\} \in t. \]
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I claim \( G_t \) is a forest.
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I claim \( G_t \) is a forest. If \( C : \ldots i, m, j, \ldots \) is a cycle with \( m = \max C \), then \( \{i, m\}, \{j, m\} \in t. \)
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Proof. If \( i < j \) let \( \{i, j\} \) be the atom of \( \Pi_n \) having this set as its unique non-singleton block. Let \( (A_1, \ldots, A_{n-1}) \vdash A(\Pi_n) \) where

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Also, the vertices of the components of \( G_t \) are the blocks of \( \bigvee t. \).
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\[ \chi(\Pi_n; t) = (t - 1)(t - 2) \ldots (t - n + 1). \]

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(2) \( \text{With any } t \in Q, \text{ associate a graph } G_t \text{ with } V = [n] \text{ and } 
\begin{align*}
ij \in E & \iff \{i, j\} \in t.
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Also, the vertices of the components of \( G_t \) are the blocks of \( \sqrt{t} \).

\[ \therefore t \in T^a_1 \]
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Also, the vertices of the components of \( G_t \) are the blocks of \( \bigvee t \).

\[ \therefore t \in T_{\hat{1}}^a \implies G_t \text{ a tree} \]
Corollary
\(\chi(\Pi_n; t) = (t - 1)(t - 2) \ldots (t - n + 1).\)

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I claim \(G_t\) is a forest. If \(C: \ldots i, m, j, \ldots\) is a cycle with \(m = \max C\), then \(\{i, m\}, \{j, m\} \in t\). But \(\{i, m\}, \{j, m\} \in A_{m-1}\).

Also, the vertices of the components of \(G_t\) are the blocks of \(\bigvee t\).

\[\therefore t \in T^a_\hat{1} \implies G_t \text{ a tree} \implies |\text{supp } t| = n - 1\]
Corollary

\[ \chi(\Pi_n; t) = (t - 1)(t - 2) \ldots (t - n + 1). \]

**Proof.** If \( i < j \) let \( \{i, j\} \) be the atom of \( \Pi_n \) having this set as its unique non-singleton block. Let \( (A_1, \ldots, A_{n-1}) \vdash A(\Pi_n) \) where

\[ A_i = \{\{1, i + 1\}, \{2, i + 1\}, \ldots, \{i, i + 1\}\}. \]

We will verify the three conditions for \( x = \hat{1} \).

1. \( (\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}) \in T_{\hat{1}}^a. \)

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How do we find an appropriate atom partition?

We say \((A_1, \ldots, A_n) \models A(L)\) is induced by a chain \(\mathbf{C}: \hat{0} = x_0 < x_1 < x_2 < \cdots < x_n = \hat{1}\) such that

\[ A_i = \{ a \in A(L) : a \leq x_i \text{ and } a \neq x_i - 1 \} \].

In \(\Pi_3\), the partition with \(A_1 = \{12/3\}\) and \(A_2 = \{13/2, 1/23\}\) is induced by the chain \(\mathbf{C}: 1/2 < 12/3 < 123\).

In \(\Pi_n\), our partition is induced by \(\hat{0} < [2] < [3] < \cdots < \hat{1}\) where \([i]\) is the partition having this set as its only non-trivial block.
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```
123
/    \
12/3 13/2
/  \
1/23
```
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![Diagram of partition and chain](image-url)
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\begin{center}
\begin{tikzpicture}

% Triangle vertices
\node (123) at (0,0) {123};
\node (231) at (1,-2) {231};
\node (123) at (2,-4) {123};

% Triplet edges
\draw (123) -- (231) -- (123) -- (231); % Triplet edges
\draw (123) -- (231) -- (123); % Chain edges
\end{tikzpicture}
\end{center}

$A_1 = \boxed{12/3}$

$13/2$

$1/23$

$1/2/3$
How do we find an appropriate atom partition? We say \((A_1, \ldots, A_n) \vdash A(L)\) is \textit{induced by a chain} if there is a chain \(C : \hat{0} = x_0 < x_1 < x_2 < \cdots < x_n = \hat{1}\) such that

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Theorem (Hallam-S) 

Let $L$ be a lattice and $(A_1, \ldots, A_n)$ induced by a chain $C$. Suppose that for all $x \in L$ and $t \in T$ we have $|\text{supp} \ t| = \rho(x)$. Under these conditions, the following are equivalent. 

1. For each $x \neq 0$ in $L$, there is $i$ such that $|A_x \cap A_i| = 1$. 
2. Chain $C$ satisfies the meet condition. 
3. The characteristic polynomial of $L$ factors as $\chi(L, t) = t^n \rho(L) - n \prod_{i=1}^{n} (t - |A_i|)$. 
Let $L$ be a lattice and $C : 0 = x_0 < x_1 < x_2 < \cdots < x_n = \hat{1}$. For $x \in L$ let $i$ be the index with $x \leq x_i$ and $x \not\leq x_{i-1}$. Say that $C$ satisfies the *meet condition* if, for every $x \in L$ of rank at least 2,

$$x \land x_{i-1} \neq \hat{0}.$$
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**Theorem (Hallam-S)**

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$$\chi(L, t) = t^{\rho(L)} - n \prod_{i=1}^n (t - |A_i|).$$
Any lattice $L$ satisfies: for all $x, y, z \in L$ with $y < z$

$$y \lor (x \land z) \leq (y \lor x) \land z \quad (\text{modular inequality}).$$  \quad (2)
Any lattice $L$ satisfies: for all $x, y, z \in L$ with $y < z$

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(2)

Call $x \in L$ left-modular if, together with any $y < z$, we have equality in (2).

A lattice is supersolvable if it has a saturated chain of left-modular elements.

Lemma (Hallam-S)

Let $L$ be a lattice and $C$ a $\hat{0} - \hat{1}$ chain in $L$ inducing $(A_1, \ldots, A_n)$.

1. If $C$ is saturated and consists of left-modular elements, then $C$ satisfies the meet condition.

2. If $L$ is semimodular then for any $x \in L$ and $t \in T$ we have

$$|\text{supp } t| = \rho(x).$$

Corollary (Stanley, 1972)

Let $L$ be a semimodular, supersolvable lattice and $(A_1, \ldots, A_n)$ be induced by a saturated chain of left-modular elements. Then

$$\chi(L; t) = n \prod_{i=1}^{n} (t - |A_i|).$$
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**Lemma (Hallam-S)**

Let $L$ be a lattice and $C$ a $0$–$1$ chain in $L$ inducing $(A_1, \ldots, A_n)$.

1. If $C$ is saturated and consists of left-modular elements, then $C$ satisfies the meet condition.
Any lattice \( L \) satisfies: for all \( x, y, z \in L \) with \( y < z \)

\[
y \lor (x \land z) \leq (y \lor x) \land z \quad \text{(modular inequality).} \quad (2)
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Call \( x \in L \) **left-modular** if, together with any \( y < z \), we have equality in (2). A lattice is **supersolvable** if it has a saturated chain of left-modular elements.

**Lemma (Hallam-S)**

Let \( L \) be a lattice and \( C \) a \( \hat{0} \)–\( \hat{1} \) chain in \( L \) inducing \( (A_1, \ldots, A_n) \).

1. If \( C \) is saturated and consists of left-modular elements, then \( C \) satisfies the meet condition.

2. If \( L \) is semimodular then for any \( x \in L \) and \( t \in T_x \) we have

\[
|\text{supp } t| = \rho(x).
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Any lattice $L$ satisfies: for all $x, y, z \in L$ with $y < z$

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Corollary (Stanley, 1972)

Let $L$ be a semimodular, supersolvable lattice and $(A_1, \ldots, A_n)$ be induced by a saturated chain of left-modular elements. Then

$$\chi(L; t) = \prod_{i=1}^{n}(t - |A_i|).$$
Outline

Motivating Examples

Quotient Posets

The Standard Equivalence Relation

The Main Theorem

Partitions Induced by Chains

Application: Increasing Forests
Let $G$ be a graph with $V = [n]$ and $F$ be a spanning forest.
Let $G$ be a graph with $V = [n]$ and $F$ be a spanning forest. Then $F$ is \emph{increasing} if the vertices in any path of $F$ starting at the minimum vertex of its component form an increasing sequence.
Let $G$ be a graph with $V = [n]$ and $F$ be a spanning forest. Then $F$ is *increasing* if the vertices in any path of $F$ starting at the minimum vertex of its component form an increasing sequence.

Ex. 1

![Graph G with vertices 1, 2, 3, 4 and edges between them.](image)
Let $G$ be a graph with $V = [n]$ and $F$ be a spanning forest. Then $F$ is *increasing* if the vertices in any path of $F$ starting at the minimum vertex of its component form an increasing sequence.

Ex.

\[
\begin{array}{c}
\text{1} & \text{2} \\
\text{4} & \text{3}
\end{array}
\]

$G$

\[
\begin{array}{c}
\text{1} & \text{2} \\
\text{4} & \text{3}
\end{array}
\]

increasing $F$
Let $G$ be a graph with $V = [n]$ and $F$ be a spanning forest. Then $F$ is *increasing* if the vertices in any path of $F$ starting at the minimum vertex of its component form an increasing sequence.

![Example graphs](image)

**Ex.**

- $G$
- Increasing $F$
- Not increasing $F$
Let $G$ be a graph with $V = [n]$ and $F$ be a spanning forest. Then $F$ is *increasing* if the vertices in any path of $F$ starting at the minimum vertex of its component form an increasing sequence.

Ex. 

\[ G \]

1 \quad 2

4 \quad 3

\[ \text{increasing } F \]

1 \quad 2

4 \quad 3

\[ \text{not increasing } F \]

Define

\[ f_k(G) = \# \text{ of increasing spanning forests of } G \text{ with } k \text{ edges}. \]
Let $G$ be a graph with $V = [n]$ and $F$ be a spanning forest. Then $F$ is *increasing* if the vertices in any path of $F$ starting at the minimum vertex of its component form an increasing sequence.

Ex. 1 \quad 2 \quad 3 \quad 4

\begin{align*}
G & \quad \text{increasing } F \\
& \quad \text{not increasing } F
\end{align*}

Define

$$f_k(G) = \# \text{ of increasing spanning forests of } G \text{ with } k \text{ edges}.$$  

and

$$IF(G; t) = \sum_{k=0}^{n-1} (-1)^k f_k(G) t^{n-k}.$$
Always write \( ij = \{i, j\} \in E(G) \) with \( i < j \).
Always write $ij = \{i, j\} \in E(G)$ with $i < j$. The \textit{induced ordered partition of $E(G)$} is $(E_1, \ldots, E_{n-1})$ where

$$E_j = \{\{i, j + 1\} : \{i, j + 1\} \in E(G)\}.$$
Always write $ij = \{i, j\} \in E(G)$ with $i < j$. The *induced ordered partition of* $E(G)$ *is* $(E_1, \ldots, E_{n-1})$ *where*

$$E_j = \{\{i, j + 1\} : \{i, j + 1\} \in E(G)\}.$$  

**Ex.**

$$G = \begin{array}{ccc}
1 & \rightarrow & 2 \\
4 & \rightarrow & 3
\end{array}$$
Always write $ij = \{i, j\} \in E(G)$ with $i < j$. The *induced ordered partition of* $E(G)$ *is* $(E_1, \ldots, E_{n-1})$ where

$$E_j = \{\{i, j + 1\} : \{i, j + 1\} \in E(G)\}.$$ 

**Ex.**

The graph $G$ has partition

$E_1$
Always write $ij = \{i, j\} \in E(G)$ with $i < j$. The *induced ordered partition of $E(G)$* is $(E_1, \ldots, E_{n-1})$ where

$$E_j = \{\{i, j + 1\} : \{i, j + 1\} \in E(G)\}.$$

**Ex.**

$$G = \begin{matrix} 1 & \bullet & \bullet & 2 \\
\quad & \big/ & \big/ & \\
4 & \bullet & \bullet & 3 \\
\end{matrix}$$

has partition

$$E_1 = \{12\}, \quad E_2$$
Always write $ij = \{i, j\} \in E(G)$ with $i < j$. The *induced ordered partition of $E(G)$* is $(E_1, \ldots, E_{n-1})$ where

$$E_j = \{\{i, j + 1\} : \{i, j + 1\} \in E(G)\}.$$ 

**Ex.**

$$G = \begin{array}{ccc}
1 & \rightarrow & 2 \\
4 & \rightarrow & 3
\end{array}$$

has partition

$$E_1 = \{12\}, \quad E_2 = \{23\}, \quad E_3$$
Always write \( ij = \{i, j\} \in E(G) \) with \( i < j \). The *induced ordered partition of* \( E(G) \) *is* \((E_1, \ldots, E_{n-1})\) where
\[
E_j = \{\{i, j+1\} : \{i, j+1\} \in E(G)\}.
\]

**Ex.**

![Graph diagram](image)

This graph has partition
\[
E_1 = \{12\}, \quad E_2 = \{23\}, \quad E_3 = \{14, 24\},
\]
Always write $ij = \{i, j\} \in E(G)$ with $i < j$. The \textit{induced ordered partition of $E(G)$} is $(E_1, \ldots, E_{n-1})$ where 
\[ E_j = \{ \{i, j+1\} : \{i, j+1\} \in E(G) \} \].

\textbf{Ex.}

\[ G = \begin{array}{ccc}
1 & \quad & 2 \\
\text{} & \quad & \\
4 & \quad & 3 \\
\end{array} \]

has partition 
\[ E_1 = \{12\}, \quad E_2 = \{23\}, \quad E_3 = \{14, 24\}, \]

\textbf{Theorem (Hallam-S)}

\textit{Let $G$ have $V = [n]$ inducing partition $(E_1, \ldots, E_{n-1})$. Then}

\[ IF(G; t) = t \prod_{j=1}^{n-1} (t - |E_j|). \]
Always write $ij = \{i,j\} \in E(G)$ with $i < j$. The induced ordered partition of $E(G)$ is $(E_1, \ldots, E_{n-1})$ where

$$E_j = \{\{i, j+1\} : \{i, j+1\} \in E(G)\}.$$ 

**Ex.**

\[
\begin{array}{c}
1 \quad 2 \\
\quad 4 \quad 3
\end{array}
\]

has partition

$$E_1 = \{12\}, \quad E_2 = \{23\}, \quad E_3 = \{14, 24\},$$

and

$$IF(G; t) = t(t-1)^2(t-2)$$

**Theorem (Hallam-S)**

*Let $G$ have $V = [n]$ inducing partition $(E_1, \ldots, E_{n-1})$. Then*

$$IF(G; t) = t \prod_{j=1}^{n-1} (t - |E_j|).$$
Always write $ij = \{i, j\} \in E(G)$ with $i < j$. The \textit{induced ordered partition of $E(G)$} is $(E_1, \ldots, E_{n-1})$ where

$$E_j = \{\{i, j+1\} : \{i, j+1\} \in E(G)\}.$$ 

\textbf{Ex.}

\begin{center}
\begin{tikzpicture}[scale=0.5]
  \node (1) at (0,0) [circle, fill=red] {1};
  \node (2) at (2,0) [circle, fill=red] {2};
  \node (3) at (2,2) [circle, fill=red] {3};
  \node (4) at (0,2) [circle, fill=red] {4};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (3) -- (4);
  \draw (1) -- (4);
\end{tikzpicture}
\end{center}

has partition

$$E_1 = \{12\}, \quad E_2 = \{23\}, \quad E_3 = \{14, 24\},$$

and

$$IF(G; t) = t(t - 1)^2(t - 2) = p(G; t) \quad \text{(chromatic polynomial)}.$$ 

\textbf{Theorem (Hallam-S)}

\textit{Let $G$ have $V = [n]$ inducing partition $(E_1, \ldots, E_{n-1})$. Then}

$$IF(G; t) = t \prod_{j=1}^{n-1} (t - |E_j|).$$
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. 
If \( G \) is a graph and \( W \subseteq V(G) \), let \( G[W] \) denote the induced subgraph of \( G \) with vertex set \( W \). A **clique** in \( G \) is a subgraph \( G[W] \) which is complete.
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A **clique** in $G$ is a subgraph $G[W]$ which is complete.

**Ex.**

$$
\begin{array}{c}
1 \\
4 \\
\end{array}
\begin{array}{c}
2 \\
3 \\
\end{array}

G
$$
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A **clique** in $G$ is a subgraph $G[W]$ which is complete.

**Ex.**

\begin{align*}
G & \quad 1 \quad 2 \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A **clique** in $G$ is a subgraph $G[W]$ which is complete.

**Ex.**

- $G$
- $G[1, 2, 4]$ a clique
- $G[2, 3, 4]$ not a clique
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A **clique** in $G$ is a subgraph $G[W]$ which is complete.

**Ex.**

$$
\begin{align*}
G & \quad | \quad G[1, 2, 4] \text{ a clique} \quad | \quad G[2, 3, 4] \text{ not a clique}
\end{align*}
$$

Say $G$ has a **perfect elimination ordering (peo)** if there is an ordering $v_1, \ldots, v_n$ of $V$ such that, for all $i$, the vertices adjacent to $v_i$ in $G[v_1, \ldots, v_{i-1}]$ form a clique.
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A **clique** in $G$ is a subgraph $G[W]$ which is complete.

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If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A \textit{clique} in $G$ is a subgraph $G[W]$ which is complete.

\textbf{Ex.} \hspace{1cm} \begin{array}{c}
1 \quad \begin{array}{c}
2 \quad 1 \quad 2 \quad 3 \\
4 \quad 4 \quad 4 \quad 3
\end{array} \\
\begin{array}{c}
G \\
G[1, 2, 4] \text{ a clique} \\
G[2, 3, 4] \text{ not a clique}
\end{array}
\end{array}
\hspace{1cm}

Say $G$ has a \textit{perfect elimination ordering (peo)} if there is an ordering $v_1, \ldots, v_n$ of $V$ such that, for all $i$, the vertices adjacent to $v_i$ in $G[v_1, \ldots, v_{i-1}]$ form a clique.
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Ex.  

$\begin{align*}
G & \quad G[1, 2, 4] \text{ a clique} \\
G[2, 3, 4] \text{ not a clique}
\end{align*}$

Say $G$ has a **perfect elimination ordering (peo)** if there is an ordering $v_1, \ldots, v_n$ of $V$ such that, for all $i$, the vertices adjacent to $v_i$ in $G[v_1, \ldots, v_{i-1}]$ form a clique.
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A \textit{clique} in $G$ is a subgraph $G[W]$ which is complete.

\textbf{Ex.} \hspace{1cm} G \hspace{1cm} G[1, 2, 4] \text{ a clique} \hspace{1cm} G[2, 3, 4] \text{ not a clique}

\hspace{1cm} G \hspace{1cm} 1 \hspace{1cm} 1 \hspace{1cm} 2 \hspace{1cm} 1 \hspace{1cm} 2 \hspace{1cm} 3

Say $G$ has a \textit{perfect elimination ordering (peo)} if there is an ordering $v_1, \ldots, v_n$ of $V$ such that, for all $i$, the vertices adjacent to $v_i$ in $G[v_1, \ldots, v_{i-1}]$ form a clique.
If \( G \) is a graph and \( W \subseteq V(G) \), let \( G[W] \) denote the induced subgraph of \( G \) with vertex set \( W \). A **clique** in \( G \) is a subgraph \( G[W] \) which is complete.

**Ex.**

\[
\begin{align*}
G & \quad 1 & 2 & 1 \quad 2 \\
4 & 3 & 4 & 4 & 3
\end{align*}
\]

- \( G \)  
- \( G[1, 2, 4] \) a clique  
- \( G[2, 3, 4] \) not a clique

Say \( G \) has a **perfect elimination ordering (peo)** if there is an ordering \( v_1, \ldots, v_n \) of \( V \) such that, for all \( i \), the vertices adjacent to \( v_i \) in \( G[v_1, \ldots, v_{i-1}] \) form a clique.
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A **clique** in $G$ is a subgraph $G[W]$ which is complete.

**Ex.**

\[
\begin{align*}
G &\quad 4 & 3 & 1 & 2 & 2 \\
G[1, 2, 4] &\text{a clique} & 4 & 3 & 1 & 2 \\
G[2, 3, 4] &\text{not a clique} & 4 & 3 & 1 & 2
\end{align*}
\]

Say $G$ has a **perfect elimination ordering (peo)** if there is an ordering $v_1, \ldots, v_n$ of $V$ such that, for all $i$, the vertices adjacent to $v_i$ in $G[v_1, \ldots, v_{i-1}]$ form a clique.
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A **clique** in $G$ is a subgraph $G[W]$ which is complete.

**Ex.**

\[
\begin{array}{cccc}
1 & \bullet & 2 & \bullet \\
4 & \bullet & 3 & \bullet \\
\end{array}
\quad
\begin{array}{cccc}
1 & \bullet & 2 & \bullet \\
4 & \bullet & 3 & \bullet \\
\end{array}
\]

$G$ \quad $G[1, 2, 4]$ a clique \quad $G[2, 3, 4]$ not a clique

Say $G$ has a **perfect elimination ordering (peo)** if there is an ordering $v_1, \ldots, v_n$ of $V$ such that, for all $i$, the vertices adjacent to $v_i$ in $G[v_1, \ldots, v_{i-1}]$ form a clique.
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A **clique** in $G$ is a subgraph $G[W]$ which is complete.

![Graph Example]

Say $G$ has a **perfect elimination ordering (peo)** if there is an ordering $v_1, \ldots, v_n$ of $V$ such that, for all $i$, the vertices adjacent to $v_i$ in $G[v_1, \ldots, v_{i-1}]$ form a clique. If $G$ has a peo then $\rho(G; t)$ has roots in $\mathbb{N}$. 
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A **clique** in $G$ is a subgraph $G[W]$ which is complete.

Ex.  

\[
\begin{align*}
\text{G:} & \quad 1 & 2 & 1 & 2 \\
& \quad 4 & 3 & 4 & 3 \\
\end{align*}
\]

$G[1, 2, 4]$ a clique  

$G[2, 3, 4]$ not a clique

Say $G$ has a **perfect elimination ordering (peo)** if there is an ordering $v_1, \ldots, v_n$ of $V$ such that, for all $i$, the vertices adjacent to $v_i$ in $G[v_1, \ldots, v_{i-1}]$ form a clique. If $G$ has a peo then $p(G; t)$ has roots in $\mathbb{N}$. Using the bond lattice of $G$, we prove:
If $G$ is a graph and $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of $G$ with vertex set $W$. A **clique** in $G$ is a subgraph $G[W]$ which is complete.

**Ex.**

Ex. 1

\[ \begin{array}{ccc}
    1 & 2 & 1 \\
    4 & 3 & 4 \\
\end{array} \]

$G$ $G[1, 2, 4]$ a clique $G[2, 3, 4]$ not a clique

Say $G$ has a **perfect elimination ordering (peo)** if there is an ordering $v_1, \ldots, v_n$ of $V$ such that, for all $i$, the vertices adjacent to $v_i$ in $G[v_1, \ldots, v_{i-1}]$ form a clique. If $G$ has a peo then $p(G; t)$ has roots in $\mathbb{N}$. Using the bond lattice of $G$, we prove:

**Theorem (Hallam-S)**

Let $G$ be a graph with $V = [n]$. Then $p(G; t) = IF(G; t)$ if and only if $1, \ldots, n$ is a peo of $G$. 