Partially Ordered Sets and their Möbius Functions III: Topology of Posets

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A partition of a set $S$ is a family $\pi$ of nonempty sets $B_1, \ldots, B_k$ called blocks such that $\biguplus_i B_i = S$ (disjoint union). We write $\pi = B_1/\ldots/B_k \vdash S$ omitting braces and commas in blocks.

**Ex.** $\pi = \text{acf}/\text{bg}/\text{de} \vdash \{a, b, c, d, e, f, g\}$.

The partition lattice is $\Pi_n = \{\pi : \pi \vdash [n]\}$ ordered by $B_1/\ldots/B_k \leq C_1/\ldots/C_l$ if for each $B_i$ there is a $C_j$ with $B_i \subseteq C_j$.

If $P$ has a $\hat{0}$ and a $\hat{1}$ we write $\mu(P) = \mu_P(\hat{0}, \hat{1})$ and similarly for other elements of $I(P)$.

**Ex.**

\[
\begin{array}{c}
\text{Ex.} & 123 & 2 & \mu(\pi) \\
\Pi_3 = 12/3 -1 & 13/2 -1 & 1/23 -1 \\
& 1/2/3 1 & \\
\end{array}
\]

\[
\begin{array}{c|cccccccc}
\mu(\Pi_n) & 1 & 2 & 3 & 4 & 5 & 6 \\
n & 1 & -1 & 2 & -6 & 24 & -120 \\
\end{array}
\]

**Theorem**

*We have: $\mu(\Pi_n) = (-1)^{n-1}(n-1)!$*
An **abstract simplicial complex** is a finite nonempty family \( \Delta \) of finite sets called **faces** such that

\[
F \in \Delta \quad \text{and} \quad F' \subseteq F \quad \implies \quad F' \in \Delta.
\]

A **geometric realization** of \( \Delta \) has a \((d - 1)\)-dimensional simplex (tetrahedron) for each \(d\)-element set in \( \Delta \). The **dimension** of \( F \in \Delta \) is \( \dim F = \#F - 1 \). Face \( F \) is a **vertex or edge** if \( \dim F = 0 \) or 1, respectively.

**Ex.** \( \Delta = \{\emptyset, u, v, w, x, uv, uw, vw, wx, uvw\} \)

- \( \dim u = 0 \) a vertex,
- \( \dim uv = 1 \), an edge
- \( \dim uvw = 2 \).

\( uvw \) and \( wx \) are facets.

**Not pure.**

Face \( F \) is a **facet** if it is containment-maximal in \( \Delta \). We say \( \Delta \) is **pure of dimension** \( d \), and write \( \dim \Delta = d \), if \( \dim F = d \) for all facets \( F \) of \( \Delta \).

**Note.** A simplicial complex pure of dimension 1 is just a graph.
Let \( \Delta \) be pure of dimension \( d \). We say \( \Delta \) is \textit{shellable} if there is an ordering of its facets (a \textit{shelling}) \( F_1, \ldots, F_k \) such that for each \( j \leq k \):

\[
F_j \cap \left( \bigcup_{i<j} F_i \right) \text{ is a union of } (d - 1)\text{-dimensional faces of } F_j.
\]

\textbf{Ex.} For the graph at right

\( uw, vw, wx, uv, xy, wy \) is a shelling.

So \( \Delta \) is shellable.

Any sequence beginning \( uw, vw, xy \) is not a shelling since \( xy \cap (uw \cup vw) = \emptyset \).

In the original shelling:

\( r(uw) = \emptyset, r(vw) = v, r(wx) = x, \)

\( r(uv) = uv, r(xy) = y, r(wy) = wy. \)

\textbf{Note.} A graph is shellable iff it is connected.

Given a shelling \( F_1, \ldots, F_k \), the \textit{restriction of } \( F_j \) is

\[
r(F_j) = \{ v \text{ a vertex of } F_j : F_j - v \subseteq \left( \bigcup_{i<j} F_i \right) \}.
\]
Let $S^d$ denote the $d$-sphere (sphere of dimension $d$). To form the *bouquet* or *wedge* of $k$ spheres of dimension $d$, $\vee^k S^d$, take a point of each sphere and identify the points.

**Ex.**

\[\vee^2 S^1 = \]

\[\begin{array}{c}
\begin{array}{c}
\ast \\
\end{array}
\end{array} \cong \]

\[\begin{array}{c}
\begin{array}{c}
\ast \\
\end{array}
\end{array} \cong \]

\[u \quad F_1 \quad F_3 \quad x \\
F_4 \quad w \quad F_6 \quad y \\
v \quad F_2 \quad F_5 \\
x \quad v \quad y \quad w \quad x \quad y
\]

\[r(uw) = \emptyset, \quad r(vw) = v, \quad r(wx) = x, \quad r(uv) = uv, \]
\[r(xy) = y, \quad r(wy) = wy.\]

If topological spaces $X$ and $Y$ are *homotopic*, write $X \simeq Y$.

**Theorem**

*If $\Delta$ is a shellable simplicial complex pure of dimension $d$, then*

\[\Delta \simeq \vee^k S^d\]

*where $k$ is the number of facets satisfying $r(F) = F$ in a shelling of $\Delta$.\]
Let $X$ be a topological space, say $X \subseteq \mathbb{R}^n$ for some $n$. If $X$ has dimension $d$ then we write $X = X^d$.

**Ex.** 1. $S^d$, the $d$-sphere. For example $S^1$ is a circle.  
2. $B^d$, the closed $d$-ball. For example, $B^2$ is a closed disc.

The *boundary* of $X = X^d$, $\partial X$, is the set of $p \in X$ such that any (deformed) open $d$-ball centered at $p$ contains points both in and out of $X$.

**Ex.** 1. $\partial B^d = S^{d-1}$.  
2. $\partial S^d = \emptyset$.

Call $C = C^i \subseteq X$ an *$i$-cycle* if $\partial C = \emptyset$. Call two cycles *equivalent* if they form the boundary of a subset of $X$.

**Ex.** If $X$ is a hollow cylinder, then the two copies of $S^1$ at either end are equivalent.

The *$i$th reduced Betti number* of $X$ is

$$\tilde{\beta}_i(X) = \text{minimum number of inequivalent } i\text{-cycles which are not boundaries of some subset of } X \text{ and generate all } i\text{-cycles.}$$

If $X \simeq Y$ then $\tilde{\beta}_i(X) = \tilde{\beta}_i(Y)$ for all $i$. We use reduced Betti numbers since then $\tilde{\beta}_0(X) = 0$ for a connected $X$. 
Proposition

We have

\[ \tilde{\beta}_i(S^d) = \begin{cases} 
  1 & \text{if } i = d, \\
  0 & \text{if } i \neq d.
\end{cases} \]

Proof.

We will prove this for \( S^2 \). First consider \( i = 2 \). We have already seen that \( \partial S^2 = \emptyset \), so \( S^2 \) is a cycle. And it can not be a boundary, since if \( \partial Y = S^2 \) then \( Y \) would have dimension 3 and so \( Y \not\subseteq S^2 \). Thus \( \tilde{\beta}_2(S^2) = 1 \).

Now consider \( i = 1 \). If we have a 1-cylce \( C \subset S^2 \), then \( C = \partial D \) where \( D \subseteq S^2 \) is the disc interior to \( C \). So every 1-cycle is also a boundary and \( \tilde{\beta}_1(S^2) = 0 \).

Finally, for \( i = 0 \). \( S^2 \) is connected so \( \tilde{\beta}_0(S^2) = 0 \).

Taking wedges adds reduced Betti numbers.

Corollary

We have

\[ \tilde{\beta}_i(\vee^k S^d) = \begin{cases} 
  k & \text{if } i = d, \\
  0 & \text{if } i \neq d.
\end{cases} \]
The *reduced Euler characteristic* of $X$ is
\[
\tilde{\chi}(X) = \sum_{i \geq -1} (-1)^i \tilde{\beta}_i(X) = -\tilde{\beta}_{-1}(X) + \tilde{\beta}_0(X) - \tilde{\beta}_1(X) + \cdots
\]

By the previous proposition $\tilde{\beta}_i(\vee^k S^d) = k$ if $i = d$ and zero else.

**Corollary**

*We have* $\tilde{\chi}(\vee^k S^d) = (-1)^d k$.

The *ith face number* of a simplicial complex $\Delta$ is

$f_i(\Delta) = (# \text{ of faces of dimension } i) = (# \text{ of faces of cardinality } i + 1.)$

**Theorem**

$\tilde{\chi}(\Delta) = \sum_{i \geq -1} (-1)^i f_i(X) = -f_{-1}(X) + f_0(X) - f_1(X) + \cdots$

**Ex.** $\Delta \cong \vee^2 S^1 \overset{\text{Cor}}{\implies} \tilde{\chi}(\Delta) = \tilde{\chi}(\vee^2 S^1) = -2$.

- $\dim F = -1 \implies F = \emptyset \implies f_{-1}(\Delta) = 1$,
- $\dim F = 0 \implies F = \text{vertex} \implies f_0(\Delta) = 5$,
- $\dim F = 1 \implies F = \text{edge} \implies f_1(\Delta) = 6$,
- $i \geq 2 \implies f_i(\Delta) = 0$, \quad $\therefore \tilde{\chi}(\Delta) = -1 + 5 - 6 = -2$. 

\[\Delta = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}\]
If \( x, y \in P \) (poset) then an \( x-y \) chain of length \( l \) in \( P \) is a subposet \( C : x = x_0 < x_1 < \ldots < x_l = y \). If \( P \) is bounded, let \( \overline{P} = P - \{\hat{0}, \hat{1}\} \).

The order complex of a bounded \( P \) is

\[
\Delta(P) = \text{set of all chains in } \overline{P}.
\]

A subset of a chain is a chain so \( \Delta(P) \) is a simplicial complex.

**Ex.** \( P = C_4 \),

\[
\therefore \overline{C}_4 = \begin{array}{c} 3 \\
2 \\
1 \end{array} \quad \text{and } \Delta(C_4) = \begin{array}{c} 1 \\
2 \\
3 \end{array}
\]

In general \( \Delta(C_n) \simeq B^0 \), a point.

**Ex.** \( P = B_3 \),

\[
\therefore \overline{B}_3 = \begin{array}{c} 12 \\
13 \\
23 \end{array} \quad \text{and } \Delta(B_3) = \begin{array}{c} 1 \\
12 \\
13 \end{array}
\]

In general \( \Delta(B_n) \simeq S^{n-2} \).
Lemma

In $l(P)$: $(\zeta - \delta)^l(x, y) = \# \text{ of } x-y \text{ chains of length } l$.

**Proof.** We have $(\zeta - \delta)(x, y) = 1$ if $x < y$ and zero else. So

$$(\zeta - \delta)^l(x, y) = \sum_{x=x_0, x_1, \ldots, x_l=y} (\zeta - \delta)(x_0, x_1) \cdots (\zeta - \delta)(x_{l-1}, x_l)$$

$$= \sum_{x=x_0 < x_1 < \ldots < x_l=y} 1 = \# \text{ of } x-y \text{ chains of length } l. \quad \square$$

Theorem

In a bounded poset $P$ with $\hat{0} \neq \hat{1}$: $\mu(P) = \tilde{\chi}(\Delta(P))$.

**Proof.** Using the definition of $\mu$ and the lemma,

$$\mu(P) = \zeta^{-1}(P) = (\delta + (\zeta - \delta))^{-1}(P) = \sum_{l \geq 0} (-1)^l (\zeta - \delta)^l(P)$$

$$= \sum_{l \geq 1} (-1)^l (\# \text{ of } \hat{0}-\hat{1} \text{ chains of length } l \text{ in } P)$$

$$= \sum_{l \geq 1} (-1)^{l-2} (\# \text{ of chains of length } l - 2 \text{ in } \overline{P})$$

$$= \sum_{i \geq -1} (-1)^i f_i(\Delta(P)) = \tilde{\chi}(\Delta(P)). \quad \square$$
A poset $P$ is *graded* if it is bounded and ranked.

**Ex.** Our example posets $C_n, B_n, D_n, \Pi_n$ are all graded.

Let $E(P)$ be the edge set of the Hasse diagram of $P$. A labeling $\ell : E(P) \to \mathbb{R}$ induces a labeling of saturated chains by

$$\ell(x_0 \triangleleft x_1 \triangleleft \ldots \triangleleft x_l) = (\ell(x_0 \triangleleft x_1), \ldots, \ell(x_{l-1} \triangleleft x_l)).$$

**Ex.** For $B_n$, let

$$\ell(S \triangleleft T) = T - S.$$
Say saturated chain $C$ has a property if $\ell(C)$ has that property. An *EL-labelling* of a graded poset $P$ is $\ell : E \to \mathbb{R}$ such that, for each interval $[x, y] \subseteq P$

1. there is a unique weakly increasing $x$–$y$ chain $C_{xy}$,

2. $C_{xy}$ is lexicographically least among saturated $x$–$y$ chains.

All four of our example posets have EL-labelings. We will give the labeling and verify the two conditions for the interval $[\hat{0}, \hat{1}]$.

1. In $C_n$, let $\ell(i - 1 \Join i) = i$. Then there is only one saturated chain and $\ell(0 \Join 1 \Join \ldots \Join n) = (1, 2, \ldots, n)$.

2. In $B_n$, let $\ell(S \Join T) = T - S$. Then $\ell$ is a bijection between saturated $\hat{0}$–$\hat{1}$ chains and permutations of $[n]$

$$\ell(\hat{0} \Join \{x_1\} \Join \{x_1, x_2\} \Join \ldots \Join \hat{1}) = (x_1, x_2, \ldots, x_n).$$

There is a unique weakly increasing permutation, $(1, 2, \ldots, n)$, and it is lexicographically smaller than any other permutation.
3. In $D_n$. let $\ell(c \triangleleft d) = d/c$.

If $n = \prod_{i=1}^{k} \rho_i^{m_i}$ then $\ell$ is a bijection between saturated $\hat{0}$$\hat{1}$ chains and permutations of the multiset

$$M = \{\overbrace{\{p_1, \ldots, p_1, \ldots, p_k, \ldots, p_k\}}^{m_1}, \ldots, \overbrace{\{p_1, \ldots, p_1, \ldots, p_k, \ldots, p_k\}}^{m_k}\}.$$ 

There is a unique weakly increasing permutation of $M$ and it is lexicographically least.

4. In $\Pi_n$, if $\pi = B_1/\ldots/B_k$ and merging $B_i$ with $B_j$ forms $\sigma$ then $\ell(\pi \triangleleft \sigma) = \max\{\min B_i, \min B_j\}$.

If $C$ is a saturated $\hat{0}$$\hat{1}$ chain then $\ell(C)$ is a permutation of $\{2, \ldots, n\}$:
for all $\pi, \sigma$ we have $2 \leq \ell(\pi \triangleleft \sigma) \leq n$, 
$\#\ell(C) = n - 1 = \#\{2, \ldots, n\}$, 
and $m$ appears as a label in $C$ at most once since after merging it is no longer a min. Permutation $(2, \ldots, n)$ only occurs once: $\ell(\hat{0} \triangleleft 12/3/ \ldots /n \triangleleft 123/4/ \ldots/n \triangleleft \ldots \triangleleft \hat{1})$. 

$D_{18} = \begin{array}{c}
18 \\
3 \\
6 \\
2 \\
3 \\
2 \\
3 \\
9 \\
3 \\
2 \\
3 \\
\end{array}$
Theorem (Björner, 1980)

Let $P$ be a graded poset. If $P$ has an EL-labelling then $\Delta(P)$ is shellable. In fact, if $F_1, \ldots, F_k$ is a list of the saturated $\hat{0} - \hat{1}$ chains in lexicographic order, then $\overline{F}_1, \ldots, \overline{F}_k$ is a shelling of $\Delta(P)$. Furthermore

$$\mu(P) = (-1)^{\rho(P)}(\# \text{ of strictly decreasing } F_j). \quad (1)$$

Proof of (??). Using the first half of the theorem

$$\mu(P) = \tilde{\chi}(\Delta(P)) = (-1)^{\dim \Delta(P)}(\# \text{ of } F_j \text{ with } r(\overline{F}_j) = \overline{F}_j).$$

The power of $-1$ is as desired since $\dim \Delta(P) = \rho(P) - 2$. So it suffices to show that $\ell(F_j)$ is strictly decreasing iff $r(\overline{F}_j) = \overline{F}_j$.

$\Rightarrow$ ("\Leftarrow" is similar) Suppose $F_j : x_0 \prec \ldots \prec x_n$ is strictly decreasing. We must show that given any $x_r \in \overline{F}_j$ there is $F_i$ with $i < j$ and $F_i \cap F_j = F_j - \{x_r\}$. Now $x_{r-1} \prec x_r \prec x_{r+1}$ is strictly decreasing. Let $x_{r-1} \prec y_r \prec x_{r+1}$ be the weakly increasing chain in $[x_{r-1}, x_{r+1}]$. Then $F_i = F_j - \{x_r\} \cup \{y_r\}$ is lexicographically smaller than $F_j$. So $i < j$ and $F_i \cap F_j = F_j - \{x_r\}$. $\square$
Corollary

(a) \( \mu(C_n) = 0 \) if \( n \geq 2 \).

(b) \( \mu(B_n) = (-1)^n \).

(c) \( \mu(D_n) = \begin{cases} (-1)^k & \text{if } n = p_1 \ldots p_k \text{ distinct primes}, \\ 0 & \text{else}. \end{cases} \)

(d) \( \mu(\Pi_n) = (-1)^{n-1}(n-1)! \).

Proof. (a) \( C_n \) has a single chain which is weakly increasing. So it has no strictly decreasing chain and \( \mu(C_n) = (-1)^n \cdot 0 = 0 \). (b) The \( \ell(F_i) \) are in bijection with the permutations of \( \{1, \ldots, n\} \). The unique strictly decreasing permutation is \( (n, n-1, \ldots, 1) \). (c) Combine the proofs in (a) and (b). (d) The \( \ell(F_i) \) are permutations of \( \{2, \ldots, n\} \). Suppose \( \ell(F_i) = (n, n-1, \ldots, 2) \) where \( F_i = \pi_0 \triangleleft \pi_1 \triangleleft \ldots \triangleleft \pi_{n-1} \). Then \( \pi_1 \) is obtained from \( \pi_0 \) by merging \( \{n\} \) with another block, giving \( n-1 \) choices. So \( n-1 \) is still a minimum of some block which must be merged with one of the \( n-2 \) other blocks to form \( \pi_2 \). Continuing in this manner gives \( (n-1)! \) chains. \( \square \)