Partially Ordered Sets and their Möbius Functions I: The Möbius Inversion Theorem

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Lecture 1: The Möbius Inversion Theorem.
Introduction to partially ordered sets and Möbius functions.

Lecture 2: Graph Coloring.
The chromatic polynomial of a graph and the characteristic polynomial of its bond lattice.

Lecture 3: Topology of Posets.
The order complex and shellability.

Lecture 4: Factoring the Characteristic Polynomial
Quotients of posets and applications.
Example A: Combinatorics.
Given a set, $S$, let

$$\# S = |S| = \text{cardinality of } S.$$ 

The Principle of Inclusion-Exclusion or PIE is a very useful tool in enumerative combinatorics.

Theorem (PIE)

*Let $U$ be a finite set and $U_1, \ldots, U_n \subseteq U$. We have*

$$\left| U - \bigcup_{i=1}^{n} U_i \right| = |U| - \sum_{1 \leq i \leq n} |U_i| + \sum_{1 \leq i < j \leq n} |U_i \cap U_j| - \cdots + (-1)^n \left| \bigcap_{i=1}^{n} U_i \right|.$$ 

□
Example B: Theory of Finite Differences.

\[ \mathbb{N} = \text{the nonnegative integers.} \]
\[ \mathbb{P} = \text{the positive integers.} \]
\[ \mathbb{R} = \text{the real numbers.} \]

If one takes a function \( f : \mathbb{N} \rightarrow \mathbb{R} \) then there is an analogue of the derivative, namely the difference operator
\[
\Delta f(n) = f(n) - f(n - 1)
\]
(where \( f(-1) = 0 \) by definition). There is also an analogue of the integral, namely the summation operator
\[
Sf(n) = \sum_{i=0}^{n} f(i).
\]

The Fundamental Theorem of the Difference Calculus states:

**Theorem (FTDC)**

*If \( f : \mathbb{N} \rightarrow \mathbb{R} \) then*

\[
\Delta Sf(n) = f(n).
\]
Example C: Number Theory

If \( d, n \in \mathbb{Z} \) then write \( d \mid n \) if \( d \) divides evenly into \( n \). The number-theoretic Möbius function is \( \mu : \mathbb{P} \to \mathbb{Z} \) defined as

\[
\mu(n) = \begin{cases} 
0 & \text{if } n \text{ is not square free,} \\
(-1)^k & \text{if } n = \text{product of } k \text{ distinct primes.}
\end{cases}
\]

The importance of \( \mu \) lies in the number-theoretic Möbius Inversion Theorem or MIT.

**Theorem (Number Theory MIT)**

*Let* \( f, g : \mathbb{P} \to \mathbb{R} \) *satisfy*

\[
f(n) = \sum_{d \mid n} g(d)
\]

*for all* \( n \in \mathbb{P} \). *Then*

\[
g(n) = \sum_{d \mid n} \mu(n/d)f(d).
\]

\[\square\]
Möbius inversion over partially ordered sets (posets) is important for the following reasons.

1. It unifies and generalizes the three previous examples.
2. It makes the number-theoretic definition transparent.
3. It encodes topological information about partially ordered sets.
4. It can be used to solve combinatorial problems.
A **partially ordered set** or **poset** is a set \( P \) together with a binary relation \( \leq \) such that for all \( x, y, z \in P \):

1. (reflexivity) \( x \leq x \),
2. (antisymmetry) \( x \leq y \) and \( y \leq x \) implies \( x = y \),
3. (transitivity) \( x \leq y \) and \( y \leq z \) implies \( x \leq z \).

Given any poset notation, if we wish to be specific about the poset \( P \) involved, we attach \( P \) as a subscript. For example, using \( \leq_P \) for \( \leq \). We also adopt the usual conventions for inequalities. For example, \( x < y \) means \( x \leq y \) and \( x \neq y \). We write \( x \parallel y \) if \( x, y \) are **incomparable**, that is \( x \not\leq y \) and \( y \not\leq x \). All posets will be finite unless otherwise stated.

If \( x, y \in P \) then \( x \) is *covered by* \( y \) or \( y \) covers \( x \), written \( x \triangleleft y \), if \( x < y \) and there is no \( z \) with \( x < z < y \). The **Hasse diagram** of \( P \) is the (directed) graph with vertices \( P \) and an edge from \( x \) up to \( y \) if \( x \triangleleft y \).
Example: The Chain.
The *chain of length* $n$ is

$$C_n = \{0, 1, \ldots, n\}$$

with the usual $\leq$ on the integers.
Example: The Boolean Algebra.

Let

\[[n] = \{1, 2, \ldots, n\}\].

The *Boolean algebra* is

\[B_n = \{ S : S \subseteq [n] \}\]

partially ordered by \(S \leq T\) if and only if \(S \subseteq T\).

\[B_3 = \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\]

Note that \(B_3\) looks like a cube.
Example: The Divisor Lattice.
Given \( n \in \mathbb{P} \) the corresponding *divisor lattice* is

\[
D_n = \{ d \in \mathbb{P} : d \mid n \}
\]

partially ordered by \( c \leq_{D_n} d \) if and only if \( c \mid d \).

Note that \( D_{18} \) looks like a rectangle.
In a poset $P$, a \textit{minimal} element is $x \in P$ such that there is no $y \in P$ with $y < x$. A \textit{maximal} element is $x \in P$ such that there is no $y \in P$ with $y > x$.

\textbf{Example.} The poset on the left has minimal elements $u$ and $v$, and maximal elements $x$ and $y$.

A poset \textit{has a zero} if it has a unique minimal element, $\hat{0}$. A poset \textit{has a one} if it has a unique maximal element, $\hat{1}$. A poset is \textit{bounded} if it has both a $\hat{0}$ and a $\hat{1}$.

\textbf{Example.} Our three fundamental examples are bounded:

$\hat{0}_{C_n} = 0$, $\hat{1}_{C_n} = n$, $\hat{0}_{B_n} = \emptyset$, $\hat{1}_{B_n} = [n]$, $\hat{0}_{D_n} = 1$, $\hat{1}_{D_n} = n$.

If $x \leq y$ in $P$ then the corresponding \textit{closed interval} is

\[ [x, y] = \{ z : x \leq z \leq y \}. \]

Open and half-open intervals are defined analogously. Note that $[x, y]$ is a poset in its own right and it has a zero and a one:

$\hat{0}_{[x,y]} = x$, $\hat{1}_{[x,y]} = y$. 
Example: The Chain.
In $C_9$ we have the interval

Example: The Boolean Algebra.

In $B_7$ we have the interval

$$[[3], \{2, 3, 5, 6\}] =$$

Note that this interval looks like $B_3$. 
Example: The Divisor Lattice.
In $D_{80}$ we have the interval 

$$[2, 40] =$$

Note that this interval looks like $D_{18}$.
For posets $P$ and $Q$, an order preserving (op) map is $f : P \to Q$ with

$$x \leq_P y \implies f(x) \leq_Q f(y).$$

An isomorphism is a bijection $f : P \to Q$ such that both $f$ and $f^{-1}$ are op. In this case $P$ and $Q$ are isomorphic, written $P \cong Q$.

**Proposition**

If $i \leq j$ in $C_n$ then $[i, j] \cong C_{j-i}$.

If $S \subseteq T$ in $B_n$ then $[S, T] \cong B_{|T-S|}$.

If $c|d$ in $D_n$ then $[c, d] \cong D_{d/c}$.

**Proof for $C_n$.** Define $f : [i, j] \to C_{j-i}$ by $f(k) = k - i$. Then $f$ is op since

$$k \leq l \implies k - i \leq l - i \implies f(k) \leq f(l).$$

Also $f$ is bijective with inverse $f^{-1}(k) = k + i$. Similarly, one can prove that $f^{-1}$ is op. \qed
If $P$ and $Q$ are posets, then their \textit{product} is

$$P \times Q = \{(a, x) : a \in P, \ x \in Q\}$$

partially ordered by

$$(a, x) \leq_{P \times Q} (b, y) \iff a \leq_P b \text{ and } x \leq_Q y.$$ 

\textbf{Ex.}

\[
\begin{align*}
\begin{array}{c}
 b \\
 a
\end{array}
\times
\begin{array}{c}
 z \\
 y \\
 x
\end{array}
= \\
\begin{array}{c}
 (b, z) \\
 (b, y) \\
 (b, x) \\
 (a, z) \\
 (a, y) \\
 (a, x)
\end{array}
\cong D_{18}.
\end{align*}
\]

If $P$ is a poset then let $P^n = P \times \cdots \times P$. 
Proposition

For the Boolean algebra

\[ B_n \cong (C_1)^n. \]

If the prime factorization of \( n \) is \( n = p_1^{m_1} \cdots p_k^{m_k} \), then

\[ D_n \cong C_{m_1} \times \cdots \times C_{m_k}. \]

Proof for \( B_n \). Since \( C_1 = \{0, 1\} \), define \( f : B_n \to (C_1)^n \) by

\[ f(S) = (b_1, b_2, \ldots, b_n) \quad \text{where} \quad b_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \not\in S. \end{cases} \]

for \( 1 \leq i \leq n \). To show \( f \) is op, suppose that we have \( f(S) = (b_1, \ldots, b_n) \) and \( f(T) = (c_1, \ldots, c_n) \). Now \( S \leq T \) in \( B_n \) means \( S \subseteq T \). Equivalently, \( i \in S \) implies \( i \in T \) for every \( 1 \leq i \leq n \). So for each \( 1 \leq i \leq n \) we have \( b_i \leq c_i \) in \( C_1 \). But then \( (b_1, \ldots, b_n) \leq (c_1, \ldots, c_n) \) in \((C_1)^n\), that is, \( f(S) \leq f(T) \). Constructing \( f^{-1} \) and proving it op is similar.
The incidence algebra of a finite poset $P$ is the set

$$I(P) = \{ \alpha : P \times P \to \mathbb{R} \mid \alpha(x, y) = 0 \text{ if } x \nleq y\},$$

together with the operations:

1. (addition) $(\alpha + \beta)(x, y) = \alpha(x, y) + \beta(x, y)$,
2. (scalar multiplication) $(k\alpha)(x, y) = k \cdot \alpha(x, y)$ for $k \in \mathbb{R}$,
3. (convolution) $(\alpha \ast \beta)(x, y) = \sum_{z \in P} \alpha(x, z) \beta(z, y)$.

**Ex.** $I(P)$ has Kronecker’s delta: $\delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$

**Proposition**

For all $\alpha \in I(P)$: $\alpha \ast \delta = \delta \ast \alpha = \alpha$.

**Proof of** $\alpha \ast \delta = \alpha$. For any $x, y \in P$:

$$(\alpha \ast \delta)(x, y) = \sum_{z} \alpha(x, z) \delta(z, y) = \alpha(x, y) \delta(y, y) = \alpha(x, y).$$

**Note.** We have

$$(\alpha \ast \beta)(x, y) = \sum_{z \in [x, y]} \alpha(x, z) \beta(z, y)$$

since $\alpha(x, z) \neq 0$ implies $x \leq z$ and $\beta(z, y) \neq 0$ implies $z \leq y$. 
An *algebra* over a field $F$ is a set $A$ together with operations of sum ($+$), product ($\cdot$), and scalar multiplication ($\cdot$) such that

1. $(A, +, \cdot)$ is a ring,
2. $(A, +, \cdot)$ is a vector space over $F$,
3. $k \cdot (a \cdot b) = (k \cdot a) \cdot b = a \cdot (k \cdot b)$ for all $k \in F$, $a, b \in A$.

**Ex.** The $n \times n$ matrix algebra over $\mathbb{R}$ is

$$\text{Mat}_n(\mathbb{R}) = \text{ all } n \times n \text{ matrices with entries in } \mathbb{R}.$$

**Ex.** The Boolean algebra is an algebra over $\mathbb{F}_2$ where, for all $S, T \in B_n$:

1. $S + T = (S \cup T) - (S \cap T)$,
2. $S \cdot T = S \cap T$,
3. $0 \cdot S = \emptyset$ and $1 \cdot S = S$.

**Ex.** The incidence algebra $I(P)$ is an algebra with convolution as the product.

**Note.** Often $\cdot$ and $\bullet$ are suppressed since context makes it clear which multiplication is meant.
Let $L : x_1, \ldots, x_n$ be a list of the elements of $P$. An $L \times L$ matrix has rows and columns indexed by $L$. The matrix algebra of $P$ is

$$M(P) = \{ M \in \text{Mat}_n(\mathbb{R}) \mid M \text{ is } L \times L \text{ and } M_{x,y} = 0 \text{ if } x \not\leq y. \}$$

Note that $M(P)$ is a subalgebra of $\text{Mat}_n(\mathbb{R})$.

**Ex.** For $B_2$, let $L : \emptyset, \{1\}, \{2\}, \{1, 2\}$. Then a typical element of $M(B_2)$ is

$$M = \begin{bmatrix}
\emptyset & \{1\} & \{2\} & \{1, 2\} \\
\{1\} & 0 & \bigstar & 0 & \bigstar \\
\{2\} & 0 & 0 & \bigstar & \bigstar \\
\{1, 2\} & 0 & 0 & 0 & \bigstar
\end{bmatrix}$$

where the $\bigstar$’s can be replace by any real numbers.

The list $L : x_1, \ldots, x_n$ is a linear extension of $P$ if $x_i \leq x_j$ in $P$ implies $i \leq j$, that is, $x_i$ comes before $x_j$ in $L$. Henceforth we will take $L$ to be a linear extension. This makes each $M \in M(P)$ upper triangular:

$$i > j \implies x_i \not\leq x_j \implies M_{x_i,x_j} = 0.$$
An **isomorphism** of algebras $A$ and $B$ is a bijection $f : A \to B$ such that for all $a, b \in A$ and $k \in F$,

$$f(a + b) = f(a) + f(b), \quad f(a \bullet b) = f(a) \bullet f(b), \quad f(k \cdot a) = k \cdot f(a).$$

Given any $\alpha \in I(P)$ we let $M^\alpha$ be the matrix with entries

$$M^\alpha_{x,y} = \alpha(x, y).$$

**Ex.** We have $M^\delta = I$ where $I$ is the identity matrix.

**Theorem**

The map $\alpha \mapsto M^\alpha$ is an algebra isomorphism $I(P) \to M(P)$.

**Proof that product is preserved.** We wish to show $M^{\alpha \bullet \beta} = M^\alpha M^\beta$. But given $x, y \in P$:

$$M^{\alpha \bullet \beta}_{x,y} = (\alpha \bullet \beta)(x, y) = \sum_z \alpha(x, z) \beta(z, y) = (M^\alpha M^\beta)_{x,y}. \quad \square$$

**Proposition**

If $\alpha \in I(P)$ then $\alpha^{-1}$ exists if and only if $\alpha(x, x) \neq 0$ for all $x \in P$.

**Proof.** By the previous theorem

$$\exists \alpha^{-1} \iff \exists (M^\alpha)^{-1} \iff \det M^\alpha \neq 0 \iff \prod_{x \in P} \alpha(x, x) \neq 0. \quad \square$$
The zeta function of $P$ is $\zeta \in l(P)$ defined by

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{if } x \not\leq y. \end{cases}$$

The Möbius function of $P$ is $\mu = \zeta^{-1}$. Note that $\mu$ is well defined by the previous proposition. From the definition of $\mu$:

$$\delta(x, y) = (\mu \ast \zeta)(x, y) = \sum_{z \in [x, y]} \mu(x, z)\zeta(z, y) = \sum_{z \in [x, y]} \mu(x, z).$$

Equivalently,

if $x = y$ then $\mu(x, x) = 1$,
if $x < y$ then $\sum_{z \in [x, y]} \mu(x, z) = 0$.

Note. If $P$ has a zero then we write $\mu(y) = \mu(\hat{0}, y)$, and so

$$\mu(\hat{0}) = 1,$$
and if $y > \hat{0}$ then $\sum_{z \leq y} \mu(z) = 0$. 
\( \mu(\hat{0}) = 1, \text{ and if } y > \hat{0} \text{ then } \sum_{z \leq y} \mu(z) = 0. \)

Example The Chain.

\[
\begin{array}{c}
\text{C}_3 = \\
\begin{array}{c}
\circ \quad 3 \\
\circ \quad 2 \\
\circ \quad 1 \\
\circ \quad 0
\end{array}
\end{array}
\]

\[
\begin{align*}
\mu(0) &= \mu(\hat{0}) = 1, \\
\mu(1) + \mu(0) &= 0 \implies \mu(1) = -1, \\
\mu(2) + \mu(1) + \mu(0) &= 0 \implies \mu(2) = 0, \\
\mu(3) + \mu(2) + \mu(1) + \mu(0) &= 0 \implies \mu(3) = 0.
\end{align*}
\]

Proposition

In \( C_n \) we have \( \mu(i) = \begin{cases} 
1 & \text{if } i = 0, \\
-1 & \text{if } i = 1, \\
0 & \text{else.}
\end{cases} \)
Example: The Boolean Algebra.

\[ B_3 = \{1, 2, 3\} \]

\[ 1 \{1, 3\} \]

\[ 1 \{2, 3\} \]

\[ 1 \{1, 2\} \]

\[ \{1\} \]

\[ \{2\} \]

\[ \{3\} \]

\[ \emptyset \]

\[ 1 \]

\[ \mu(\emptyset) = \mu(\hat{0}) = 1, \]
\[ \mu(\{1\}) + \mu(\emptyset) = 0 \implies \mu(\{1\}) = -1, \]
\[ \mu(\{1, 2\}) + \mu(\{1\}) + \mu(\{2\}) + \mu(\emptyset) = 0 \implies \mu(\{1, 2\}) = 1, \]
\[ \mu(\{1, 2, 3\}) + \cdots + \mu(\emptyset) = 0 \implies \mu(\{1, 2, 3\}) = -1. \]

Conjecture

In \( B_n \) we have \( \mu(S) = (-1)^{|S|} \).
Example: The Divisor Lattice.

$D_{18} =$

$\mu(1) = \mu(\hat{0}) = 1,$
$\mu(2) = \mu(3) = -1,$
$\mu(6) + \mu(2) + \mu(3) + \mu(1) = 0 \implies \mu(6) = 1,$
$\mu(9) + \mu(3) + \mu(1) = 0 \implies \mu(9) = 0,$
$\mu(18) + \cdots + \mu(1) = 0 \implies \mu(18) = 0.$

Conjecture

If $d \in D_n$ has prime factorization $d = p_1^{m_1} \cdots p_k^{m_k}$ then

$\mu(d) = \begin{cases} (-1)^k & \text{if } m_1 = \ldots = m_k = 1, \\ 0 & \text{if } m_i \geq 2 \text{ for some } i. \end{cases}$
Theorem

1. If \( f : P \rightarrow Q \) is an isomorphism and \( x, y \in P \) then
   \[ \mu_P(x, y) = \mu_Q(f(x), f(y)). \]

2. If \( a, b \in P \) and \( x, y \in Q \) then
   \[ \mu_{P \times Q}((a, x), (b, y)) = \mu_P(a, b)\mu_Q(x, y). \] (1)

Proof for \( P \times Q \). For any poset \( R \), the equation
\[ \sum_{t \in [r, s]} \mu(r, t) = \delta(r, s) \] uniquely defines \( \mu \). So it suffices to show that the right-hand side of (1) satisfies the defining equation.

\[
\sum_{(c, z) \in [(a, x), (b, y)]} \mu_P(a, c)\mu_Q(x, z) = \sum_{c \in [a, b]} \sum_{z \in [x, y]} \mu_P(a, c)\mu_Q(x, z) \\
= \delta_P(a, b)\delta_Q(x, y) \\
= \delta_{P \times Q}((a, x), (b, y)). \]
\[ \square \]
Theorem

1. If \( S \in B_n \) then \( \mu(S) = (-1)^{|S|} \)
2. If \( d = p_1^{m_1} \cdots p_k^{m_k} \in D_n \) then

\[
\mu(d) = \begin{cases} 
(-1)^k & \text{if } m_1 = \ldots = m_k = 1, \\
0 & \text{if } m_i \geq 2 \text{ for some } i.
\end{cases}
\]

Proof for \( B_n \). We have an isomorphism \( f : B_n \to (C_1)^n \). Also
\[
\mu_{C_1}(0) = 1 \quad \text{and} \quad \mu_{C_1}(1) = -1.
\]
Now if \( f(S) = (b_1, \ldots, b_n) \) then by the previous theorem
\[
\mu_{B_n}(S) = \mu_{(C_1)^n}(b_1, \ldots, b_n)
= \prod_i \mu_{C_1}(b_i)
= (-1)^{\# \text{ of } b_i = 1}
= (-1)^{|S|}. \quad \square
\]
Theorem (M"obius Inversion Thm - MIT, Weisner (1935))

Consider a finite poset $P$ and two functions $f : P \to \mathbb{R}$ and $g : P \to \mathbb{R}$. Then the following are equivalent statements.

1. \[ f(y) = \sum_{x \leq y} g(x) \text{ for all } y \in P. \]

2. \[ g(y) = \sum_{x \leq y} \mu(x, y)f(x) \text{ for all } y \in P. \]

Proof. Let $L : x_1, \ldots, x_n$ be the linear extension used for $I(P)$. Consider vectors $v^f = [f(x_1) \ldots f(x_n)]$, $v^g = [g(x_1), \ldots, g(x_n)]$.

\[ f(y) = \sum_{x \leq y} g(x) \quad \forall y \in P \iff f(y) = \sum_{x \in P} g(x)\zeta(x, y) \quad \forall y \in P \]

\[ \iff v^f = v^g M^\zeta \iff v^g = v^f (M^\zeta)^{-1} = v^f M^\mu \]

\[ \iff g(y) = \sum_{x \in P} f(x)\mu(x, y) \quad \forall y \in P \]

\[ \iff g(y) = \sum_{x \leq y} f(x)\mu(x, y) \quad \forall y \in P. \]
Theorem (MIT)

\[ f(y) = \sum_{x \leq y} g(x) \quad \forall y \in P \iff g(y) = \sum_{x \leq y} \mu(x, y)f(x) \quad \forall y \in P. \]

Ex. Theory of Finite Differences.

For \( g : \mathbb{N} \to \mathbb{R} \):

\[ \Delta g(n) = g(n) - g(n - 1), \quad Sg(n) = \sum_{i=0}^{n} g(i). \]

Theorem (FTDC)

If \( g : \mathbb{N} \to \mathbb{R} \) then:

\[ \Delta Sg(n) = g(n). \]

Proof. Consider the chain \( C_n \) and the restriction \( g : C_n \to \mathbb{R} \). For each \( k \in C_n \), define

\[ f(k) = \sum_{i \leq k} g(i) = Sg(k). \]

Then by the MIT applied to \( C_n \)

\[ g(n) = \sum_{i \leq n} \mu(i, n)f(i) = \mu(n, n)f(n) + \mu(n - 1, n)f(n - 1) \]

\[ = f(n) - f(n - 1) = \Delta f(n) = \Delta Sg(n). \]
Theorem (Dual MIT)

\[ f(x) = \sum_{y \geq x} g(y) \quad \forall x \in P \iff g(x) = \sum_{y \geq x} \mu(x, y) f(y) \quad \forall x \in P. \]

\[ \square \]

Ex. Principle of Inclusion-Exclusion.

Theorem (PIE)

Let \( U \) be a finite set and \( U_1, \ldots, U_n \subseteq U \).

\[ \left| U - \bigcup_{i=1}^{n} U_i \right| = |U| - \sum_{1 \leq i \leq n} |U_i| + \cdots + (-1)^n \left| \bigcap_{i=1}^{n} U_i \right|. \]

Proof. For the Boolean algebra \( B_n \), define \( f, g : B_n \to \mathbb{R} \) by

\[ f(S) = \text{# of elements in all } U_i, \ i \in S, \text{ and possibly other } U_j, \]
\[ g(S) = \text{# of elements in all } U_i, \ i \in S, \text{ and no other } U_j. \]

Now \( f(S) = \left| \bigcap_{i \in S} U_i \right| \) and \( f(S) = \sum_{T \supseteq S} g(T) \). Thus

\[ \left| U - \bigcup_{i=1}^{n} U_i \right| = g(\emptyset) = \sum_{T \supseteq \emptyset} \mu(\emptyset, T) f(T) = \sum_{T \in B_n} (-1)^{|T|} \left| \bigcap_{i \in T} U_i \right|. \]

\[ \square \]
Theorem (MIT)

\[ f(y) = \sum_{x \leq y} g(x) \forall y \in P \iff g(y) = \sum_{x \leq y} \mu(x, y)f(x) \forall y \in P. \]

Ex. Number Theory

Theorem (Number Theory MIT)

Let \( f, g : \mathbb{P} \to \mathbb{R} \) satisfy \( f(n) = \sum_{d|n} g(d) \) for all \( n \in \mathbb{P} \). Then

\[ g(n) = \sum_{d|n} \mu(n/d)f(d). \]

Proof. The restrictions \( f, g : D_n \to \mathbb{R} \) satisfy, for all \( m \in D_n \):

\[ f(m) = \sum_{d|m} g(d) = \sum_{d \leq D_n m} g(d). \]

Apply the poset MIT to \( D_n \) and use \([d, n] \cong [1, n/d]\):

\[ g(n) = \sum_{d \leq D_n n} \mu(d, n)f(d) = \sum_{d|n} \mu(d, n)f(d) = \sum_{d|n} \mu(n/d)f(d). \]