# Pattern Avoidance in Set Partitions

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**Abstract**

The study of patterns in permutations is a very active area of current research. Klazar defined and studied an analogous notion of pattern for set partitions. We continue this work, finding exact formulas for the number of set partitions which avoid certain specific patterns. In particular, we enumerate and characterize those partitions avoiding any partition of a 3-element set. This allows us to conclude that the corresponding sequences are P-recursive. Finally, we define a second notion of pattern in a set partition, based on its restricted growth function. Related results are obtained for this new definition.

# 1 Introduction

The study of patterns in permutations has been very active of late; see the article of Wilf [41] for a survey. Klazar [20, 21, 22] defined and investigated what it means for a set partition to avoid a pattern, generalizing Kreweras's much-studied notion of a noncrossing partition [24]. Recently, Klazar and Marcus [23] proved a generalization of the Marcus-Tardos Theorem [27] (which itself provided a demonstration of the Füredi-Hajnal and Stanley-Wilf Conjectures), that in particular gives the asymptotic growth rate of the number of set partitions avoiding a given pattern. Here, our focus will be on exact enumeration. To make things precise, we will need some definitions.

Let $\mathbb{P}$ and $\mathbb{N}$ denote the positive and nonnegative integers, respectively. For $m, n \in \mathbb{N}$ we have the interval $[m, n] = \{m, m+1, \ldots, n\}$ with special case $[n] = [1, n]$. If $S$ is any set, then a *partition $\pi$ of $S$* is a set of nonempty subsets $B_1, B_2, \ldots, B_k$ of $S$ such that $\uplus_i B_i = S$ (disjoint union). We will write $\pi \vdash S$ and $\pi = B_1/B_2/\ldots/B_k$. The subsets are called *blocks* and the number of blocks will be denoted $b(\pi)$. Most often we will also not use set braces and commas in the blocks unless they are needed for clarity. For example, if $\pi = 14/2/356$ then $\pi \vdash [6]$ and $b(\pi) = 3$. We will use the notation

$$\Pi_n = \{\pi \ : \ \pi \vdash [n]\} \quad \text{and} \quad \Pi = \biguplus_{n \geq 0} \Pi_n. \tag{1}$$

In order to connect Klazar's definition of pattern with the usual one for permutations, it is convenient to introduce a standardization map. If $S$ is any subset of the integers with cardinality $\#S = n$ then the corresponding *standardization map* is the unique order-preserving bijection $\mathrm{st}_S : S \to [n]$. When $S$ is clear from context, we will drop the subscript. We let $\mathrm{st}_S$ act element-wise on objects built using $S$ as label set. For example, if $S = \{3, 4, 6\}$ then $\mathrm{st}(3) = 1$, $\mathrm{st}(4) = 2$, $\mathrm{st}(6) = 3$. Consequently, for the sequence $p = 4346$ we have $\mathrm{st}(p) = 2123$ and for the partition $\pi = 36/4$ we have $\mathrm{st}(\pi) = 13/2$. The definition of pattern containment for permutations can now be stated as follows: If $p = a_1 a_2 \ldots a_r$ and $q = b_1 b_2 \ldots b_s$ are permutations, then *$q$ contains $p$ as a pattern* if there is a subsequence $q' = b_{i_1} b_{i_2} \ldots b_{i_r}$ of $q$ with $\mathrm{st}(q') = p$. Otherwise *$q$ avoids $p$*. Given a pattern permutation $p$, we let

$$\mathfrak{S}_n(p) = \{q \in \mathfrak{S}_n \ : \ q \text{ avoids } p\},$$

where $\mathfrak{S}_n$ is the symmetric group on $[n]$. We will also let $\mathfrak{S} = \uplus_{n \geq 0} \mathfrak{S}_n$.

For pattern containment in set partitions, we will need the notion of a subpartition. A *subpartition* of $\sigma$ is a partition $\sigma'$ such that each block of $\sigma'$ is contained in a different block of $\sigma$. For example, $\sigma = 14/236/5$

has $\sigma' = 26/4$ as a subpartition, but not $26/3$ since both 26 and 3 are in the same block of $\sigma$. If $\pi$ and $\sigma$ are set partitions, then $\sigma$ *contains* $\pi$ *as a pattern* if there is a subpartition $\sigma'$ of $\sigma$ with $\mathrm{st}(\sigma') = \pi$. Also, $\sigma'$ is called a *copy* of $\pi$ in $\sigma$. If $\sigma$ has no copies of $\pi$ then it *avoids* $\pi$. Continuing our example, $\sigma = 14/2/356$ contains four copies of the pattern $13/2$, namely $14/2$, $14/3$, $35/4$, and $36/4$. On the other hand, $\sigma$ avoids $134/2$ since any copy of this pattern would have to have 356 as a block and then there is no integer that can take the place of the 2 in the pattern. Parallel to the notation above, given a pattern $\pi$ we let

$$\Pi_n(\pi) = \{\sigma \in \Pi_n \ : \ \sigma \text{ avoids } \pi\} \quad \text{and} \quad \Pi(\pi) = \biguplus_{n \geq 0} \Pi_n(\pi).$$

Note that the noncrossing partitions may be defined as those in $\Pi(13/24)$.

In the following section we will provide exact formulas and generating functions for $\#\Pi_n(\pi)$ for various patterns $\pi$, including all $\pi \vdash [3]$. Gessel [16] and Noonan-Zeilberger [29] initiated the study of P-recursiveness and its relationship to patterns in permutations. In section 3 we consider analogous results for set partitions. The section after that uses restricted growth functions to give a second definition of pattern in a set partition, and various results using this new notion are presented. We end with a section outlining future work and open problems.

# 2 Enumeration

As is often the case when dealing with set partitions, exponential generating functions will be useful. So we begin by setting up some notation for them.

If $I$ is a set of nonnegative integers, then let

$$F_I(x) = \sum_{i \in I} \frac{x^i}{i!}. \tag{2}$$

We will also use the following notation for a special case of (2) which will appear repeatedly

$$\exp_m(x) = \sum_{n=0}^{m} \frac{x^n}{n!}.$$

The next result follows from standard manipulation of exponential generating functions (see Wilf's book [40, Chapter 3]), so we omit the proof.

**Proposition 2.1.** *Let*

$$a_{n,l}^I = \#\{\sigma = C_1/C_2/\ldots/C_l \in \Pi_n \ : \ \#C_j \in I \text{ for } 1 \leq j \leq l\}.$$

*It follows that*

$$\sum_{n=0}^{\infty} a_{n,l}^{I} \frac{x^n}{n!} = \frac{F_I(x)^l}{l!}. \qquad \blacksquare$$

Finally, given a pattern $\pi$, we let

$$F_\pi(x) = \sum_{n=0}^{\infty} \#\Pi_n(\pi) \frac{x^n}{n!}.$$

This will cause no confusion with (2), since the context will always make it clear whether the subscript refers to an index set or a partition.

For our first result, we will consider the extreme cases where $\pi$ has only singleton blocks or is itself a single block. When $\pi = 1/2/\ldots/m$, $\sigma$ contains a copy of $\pi$ if and only if $\sigma$ has at least $m$ blocks from which to take the singletons. Similarly, if $\pi = 12\ldots m$ then a copy of $\pi$ can come from any block of $\sigma$ having at least $m$ elements. Combining these observations with the previous proposition proves the following.

**Theorem 2.2.** *We have*

$$\begin{aligned}
\Pi(1/2/\ldots/m) &= \{\sigma \in \Pi \ : \ b(\sigma) < m\}, \\
F_{1/2/\ldots/m}(x) &= \exp_{m-1}(\exp(x) - 1), \\
\Pi(12\ldots m) &= \{\sigma \in \Pi \ : \ \#C < m \textit{ for all } C \in \sigma\}, \\
F_{12\ldots m}(x) &= \exp(\exp_{m-1}(x) - 1). \qquad \blacksquare
\end{aligned}$$

In what follows, we will often abbreviate

$$\hat{0}_m = 1/2/\ldots/m \quad \text{and} \quad \hat{1}_m = 12\ldots m.$$

This is because these elements are the unique minimum and maximum of the partition lattice.

We can characterize the set partitions which avoid another infinite family of patterns. Suppose that $\sigma = C_1/C_2/\ldots/C_l \vdash S$ and $T \subseteq S$. Then the *restriction of* $\sigma$ *to* $T$ is the partition $\sigma_T$ whose blocks are the nonempty sets of the form $C_i \cap T$, $1 \le i \le l$. Using $\sigma = 14/2/356$ as usual and $T = \{3, 4, 6\}$, we obtain $\sigma_T = 36/4$. Note that $\sigma'$ is a subpartition of $\sigma$ if and only if $\sigma' = \sigma_T$ for some $T$. If $\sigma \vdash [n]$ then we will use the abbreviations $\sigma_{\le k}$ and $\sigma_{>k}$ for the cases when $T = [k]$ and $T = [k+1, n]$, respectively. In the following theorem, we use the falling factorial notation

$$\langle k \rangle_i = k(k-1)\cdots(k-i+1)$$

as well as the $a_{n,l}^{I}$ as defined in Proposition 2.1.

**Theorem 2.3.** *We have*

$$\Pi(12/3/4/\dots/m) = \{\sigma \in \Pi \,:\, \exists k \ni \sigma_{\le k} = \hat{0}_k,\, b(\sigma_{>k}) < m-1\}, \quad (3)$$

$$\#\Pi_n(12/3/4/\dots/m) = 1 + \sum_{k=1}^{n-1}\sum_{j=1}^{m-2} a^{\mathbb{P}}_{n-k,j} \sum_{i=1}^{j} \binom{j-1}{i-1}\langle k\rangle_i. \quad (4)$$

**Proof** Suppose $\sigma \vdash [n]$. If $\sigma = \hat{0}_n$ then clearly $\sigma$ is a member of both sets in (3). So assume $\sigma \neq \hat{0}_n$. Define $k$ to be the largest integer such that all the elements of $[k]$ are minima of their blocks in $\sigma$. Then $\sigma_{\le k} = \hat{0}_k$, and $k+1$ is in a block of $\sigma$ which also contains an element $s \le k$.

Suppose first that $\sigma \in \Pi(12/3/4/\dots/m)$. To show that $\sigma$ must then be in the right-hand side of (3) we assume, towards a contradiction, that $b(\sigma_{>k}) \ge m-1$. But then $\sigma_{>k}$ contains a copy, $\sigma'$, of $\hat{0}_{m-1}$ and we can take that copy to contain $k+1$ since the minima of any $m-1$ blocks will do. Inserting the element $s$ into the block of $k+1$ in $\sigma'$ gives a copy of $12/3/4/\dots/m$ in $\sigma$, a contradiction.

For the reverse inclusion, take $\sigma \neq \hat{0}_n$ in the right-hand set of (3). We again proceed by contradiction, assuming that $\sigma$ contains a copy, $\sigma'$, of $12/3/4/\dots/m$ . Let $k'$ be the element of $\sigma'$ playing the role of the 2 in $12/3/4/\dots/m$. Then $k' > k$ since the elements of $[k]$ are all in separate blocks of $\sigma$. Thus the elements of $\sigma'$ corresponding to the elements $[2,m]$ in $12/3/4/\dots/m$ are all in $\sigma_{>k}$, and are also all in separate blocks. This contradicts $b(\sigma_{>k}) < m-1$ and finishes the proof of (3).

To obtain the count (4), we enumerate the elements in the right-hand set of (3). The 1 in the sum accounts for the partition $\sigma = \hat{0}_n$. Let $k$ be as defined in the first paragraph of the proof. Then $1 \le k \le n-1$ since we are now considering $\sigma \neq \hat{0}_n$. Let $j = b(\sigma_{>k})$ so, by definition of the $\sigma$ being counted, $1 \le j \le m-2$. Every block of $\sigma$ is of one of the three forms $\{s\}$, $C$, or $C \cup \{s\}$ where $s \le k$ and $C \in \sigma_{>k}$. Let $i$ be the number of blocks of the third type. So $i \ge 1$ since, by maximality of $k$, the block of $\sigma$ containing $k+1$ must be of this form. Also $i \le j$ by their definitions. Thus we have verified the limits on the summations in (3).

To count the number of $\sigma$ for given $i, j, k$ we first note that the choice of $\sigma_{\le k}$ is unique and there are $a^{\mathbb{P}}_{n-k,j}$ choices for $\sigma_{>k}$. To determine $\sigma$ from these two subpartitions, it suffices to specify the blocks of type three. We already know that the block containing $k+1$ must be of this type, so there are $\binom{j-1}{i-1}$ ways to choose the rest of the blocks of $\sigma_{>k}$ that will be used. Now these blocks (including the one containing $k+1$) can each be unioned with a unique element $s \le k$ in a total of $\langle k\rangle_i$ ways. This gives the summand in (3) and completes the proof of this equation and of the theorem. ∎

Representing a permutation $p$ by its permutation matrix, the dihedral group of the square acts on $\mathfrak{S}_n$. The number of permutations avoiding a pattern is the same for any two patterns in the same orbit. More generally, two permutations $p, q$ are called *Wilf equivalent* if $\#\mathfrak{S}_n(p) = \#\mathfrak{S}_n(q)$ for all $n \geq 0$. For example, it is well known that any two permutations in $\mathfrak{S}_3$ are Wilf equivalent.

Only one of the symmetries for permutations remains for set partitions. Given $\pi = B_1/\ldots/B_k \vdash [m]$, define its *complement* to be the partition $\pi^c = B_1^c/\ldots/B_k^c$ where

$$B_i^c = \{m + 1 - b \;:\; b \in B_i\}$$

for $1 \leq i \leq k$. For example, $(14/2/356)^c = 63/5/421$. The proof of the next result is trivial and so is omitted.

**Lemma 2.4.** *For any pattern $\pi$, we have*

$$\begin{aligned} \Pi_n(\pi^c) &= \{\sigma^c \;:\; \sigma \in \Pi_n(\pi)\}, \\ \#\Pi_n(\pi^c) &= \#\Pi_n(\pi). \qquad \blacksquare \end{aligned}$$

We will call partitions $\pi, \sigma$ *Wilf equivalent* if $\#\Pi_n(\pi) = \#\Pi_n(\sigma)$ for all $n \geq 0$. So, for example, the preceding lemma gives us the Wilf equivalence

$$\#\Pi_n(1/2/\ldots/m-2/m-1\ m) = \#\Pi_n(12/3/4/\ldots/m)$$

with the cardinality of the right-hand side being given in (4).

We will now give a complete characterization and enumeration of $\Pi_n(\pi)$ for all $\pi \vdash [3]$. To do so, it will be useful to have a few more definitions. Call $\sigma$ a *matching* if $\#C \leq 2$ for all $C \in \sigma$. Also, define the *double factorial*

$$(2i)!! = 1 \cdot 3 \cdot 5 \cdots (2i - 1),$$

which is the number of matchings on $2i$ elements where every block has size two. Finally, we say that $\sigma$ it *layered* if it has the form

$$\sigma = [i, j]/[j + 1, k]/[k + 1, l]/\ldots/[m + 1, n]$$

for certain $i, j, k, l, \ldots, m, n$.

**Theorem 2.5.** *We have*

$$\begin{aligned}
\Pi_n(1/2/3) &= \{\sigma \in \Pi_n \ : \ b(\sigma) \le 2\}, &(5)\\
\#\Pi_n(1/2/3) &= 2^{n-1}, &(6)\\
\Pi_n(123) &= \{\sigma \in \Pi_n \ : \ \sigma \textit{ is a matching}\}, &(7)\\
\#\Pi_n(123) &= \sum_{i\ge 0} \binom{n}{2i}(2i)!!, &(8)\\
\Pi_n(12/3) &= \{\sigma \in \Pi_n \ : \exists k \ni \sigma_{\le k} = \hat{0}_k,\ \sigma_{>k} = [k+1,n]\}, &(9)\\
\#\Pi_n(12/3) &= 1 + \binom{n}{2}, &(10)\\
\Pi_n(1/23) &= \{\sigma \in \Pi_n \ : \exists k \ni \sigma_{\le k} = [1,k],\ \mathrm{st}(\sigma_{>k}) = \hat{0}_{n-k}\}, &(11)\\
\#\Pi_n(1/23) &= 1 + \binom{n}{2}, &(12)\\
\Pi_n(13/2) &= \{\sigma \in \Pi_n \ : \ \sigma \textit{ is layered}\}, &(13)\\
\#\Pi_n(13/2) &= 2^{n-1}. &(14)
\end{aligned}$$

**Proof** All these equations except the last two are easy consequences of Theorems 2.2 and 2.3 and Lemma 2.4.

To prove (13), first note that it is clear from the definition of "layered" that such a partition can not have a copy of the pattern 13/2. For the reverse direction, suppose $\sigma$ avoids 13/2 and let $C$ be the block of $\sigma$ containing 1. Also, let $i = \max C$. We claim that $C = [1,i]$. This is clear if $i = 1$. If $i > 1$ then suppose, towards a contradiction, that there is some $j$ with $1 < j < i$ and $j \notin C$. But then $1i/j$ is a copy of 13/2 in $\sigma$, a contradiction. Considering the block of $\sigma$ with minimum $i+1$ and iterating this process completes the proof of (13).

To prove (14), just note that any layered partition can be obtained from the sequence $12\ldots n$ by inserting slashes in the $n-1$ spaces between the numbers. ■

We should note that Klazar also mentioned (7) in [21, Example 1], thus showing that $\pi = 123$ is a set partition with superexponential growth rate. (Something which can not happen for permutations.)

## 3 P-recursion

We now use the results of the previous section to investigate when various sequences of the form $\#\Pi_n(\pi)$, $n \geq 0$, are P-recursive. A sequence $(a_n)_{n\geq 0}$, is *P-recursive* (polynomially recursive) if there are polynomials $P_0(n), P_1(n), \ldots, P_k(n)$ (not all zero) such that

$$P_0(n)a_n + P_1(n)a_{n+1} + \cdots + P_k(n)a_{n+k} = 0$$

for all $n \geq 0$. As a simple example, the sequence with elements $a_n = n!$ is P-recursive since we always have $(n+1)a_n - a_{n+1} = 0$.

Gessel [16] first mentioned the problem of determining for which permutations $p$ the sequence $a_n = \#\mathfrak{S}_n(p)$, $n \geq 0$, is P-recursive. Noonan and Zeilberger [29] conjectured that the sequence is P-recursive for all $p$, although later evidence has caused Zeilberger to change his mind [14] and conjecture that it is not P-recursive for $p = 1324$. For set partitions, the numbers $\#\Pi_n(\pi)$ do not always form a P-recursive sequence, as we will show shortly. To do so, we need to introduce some ideas from the theory of D-finite power series.

Let $f(x)$ be a formal power series. Then $f(x)$ is *D-finite* (differentiably finite) if there are polynomials $p_0(x), p_1(x), \ldots, p_k(x)$ (not all zero) such that

$$p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_k(x)f^{(k)}(x) = 0. \tag{15}$$

A simple example is the function $f(x) = e^x$ which satisfies $f(x) - f'(x) = 0$. Stanley [36] was the first to bring the theory of D-finite series, which had long been used for differential equations, to bear on combinatorial problems. We will need the following two results of his, the first of which can also be found in the work of Jungen [19].

**Theorem 3.1** (Jungen [19], Stanley [36])**.** *A sequence $(a_n)_{n\geq 0}$ is P-recursive if and only if its ordinary generating function $f(x) = \sum_{n\geq 0} a_n x^n$ is D-finite.* ■

**Theorem 3.2** (Stanley [36])**.** *If $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are P-recursive sequences, then so is their point-wise product $(a_n b_n)_{n\geq 0}$.* ■

**Corollary 3.3.** *A sequence $(a_n)_{n\geq 0}$ is P-recursive if and only if its exponential generating function $F(x) = \sum_{n\geq 0} a_n x^n/n!$ is D-finite.*

**Proof** We will only prove the reverse implication, as the forward direction is obtained from that proof by just reversing the steps. So suppose $F(x)$ is D-finite. Then by Theorem 3.1 the sequence $(a_n/n!)_{n\geq 0}$ is P-recursive. Also, we have already seen that the sequence $(n!)_{n\geq 0}$ is P-recursive. Thus, by Theorem 3.2, $(a_n)_{n\geq 0}$ is P-recursive. ■

For an example where the $\#\Pi_n(\pi)$ do not form a P-recursive sequence, consider the pattern $\pi = \epsilon$, the empty partition. So $\Pi_n(\epsilon) = B_n$, the $n$th Bell number.

**Proposition 3.4.** *The sequence $\#\Pi_n(\epsilon)$, $n \geq 0$, is not P-recursive.*

**Proof** Suppose, towards a contradiction, that this sequence is P-recursive. Using Proposition 2.1, we get the well-known generating function for the Bell numbers

$$F_\epsilon(x) = e^{e^x - 1}.$$

By the previous corollary, $F_\epsilon(x)$ must be D-finite and so must satisfy (15) for certain polynomials $p_i(x)$. Taking the derivatives and dividing by $F_\epsilon(x)$ which is never zero, we get an equation of the form

$$q_0(x) + q_1(x)e^x + q_2(x)e^{2x} + \cdots + q_k(x)e^{kx} = 0 \tag{16}$$

where

$$q_i(x) = p_i(x) + \sum_{j>i} a_{i,j} p_j(x)$$

for certain constants $a_{i,j}$. So since the $p_i(x)$ are polynomials which are not all zero, the same must be true of the $q_i(x)$. But this implies that $e^x$ is an algebraic function, a contradiction. ■

**Question 3.5.** *For what set partitions $\pi$ is the sequence $\Pi_n(\pi)$, $n \geq 0$, P-recursive?*

We will now show that all of the patterns considered in the previous section give rise to P-recursive sequences. To do so, we will need a few more definitions and results. In his work on the growth rate of $\#\Pi_n(\pi)$, Klazar [21] was lead to consider the following patterns. A *sufficiently restricted partition* or *srp* is a matching $\pi$ such that, if $S$ is the union of the doubletons in $\pi$, then

$$\mathrm{st}(\pi_S) = 1a_1/2a_2/\ldots/ka_k$$

for some permutation $a_1a_2\ldots a_k$ of $[k+1, 2k]$.

**Theorem 3.6** (Klazar [21])**.** *If $\pi$ is an srp then the ordinary generating function for the sequence $\#\Pi_n(\pi)$, $n \geq 0$, is rational with integer coefficients. In particular, this sequence is P-recursive.*

We will also need the following result.

**Theorem 3.7** (Stanley [36])**.** *If $f(x)$ is D-finite and $g(x)$ is algebraic with $g(0) = 0$, then the composition $f(g(x))$ is D-finite.*

**Theorem 3.8.** *For $m \geq 1$, the following sequences are P-recursive as $n$ varies over $\mathbb{N}$:*

$$\#\Pi_n(1/2/\ldots/m),\ \#\Pi_n(12\ldots m), \quad \textit{and} \quad \#\Pi_n(12/3/4/\ldots/m).$$

*Furthermore, for any $\pi \vdash [3]$ the sequence $\#\Pi_n(\pi)$, $n \geq 0$, is P-recursive.*

**Proof** The only one of these sequence which is not covered by Theorem 3.6 is the one for $12\ldots m$. But in Theorem 2.2 we noted that the exponential generating function for this pattern is $\exp(\exp_{m-1}(x)-1)$. We have already seen that $f(x) = \exp(x)$ is D-finite. And $g(x) = \exp_{m-1}(x)-1$ is algebraic since it is a polynomial. So we are done by Theorem 3.7 and Corollary 3.3. ∎

# 4 Restricted growth functions

There is a second, natural definition of pattern containment for set partitions which arises from considering them as restricted growth functions. In order to make this connection, we will write all of our partitions $\pi = B_1/B_2/\ldots/B_k$ in *canonical order* which means that the blocks are indexed so that

$$\min B_1 < \min B_2 < \ldots < \min B_k. \tag{17}$$

If $S \subseteq B_j$ for some $j$ then it will also be convenient to use the notation

$$B(S) = j. \tag{18}$$

A *restricted growth function* (RGF) is a sequence $r = a_1a_2\ldots a_n$ of positive integers such that

1. $a_1$=1, and
2. for $i \geq 2$ we have $a_i \leq 1 + \max_{j<i} a_j$.

The number of elements of $r$ is called the *length* of $r$ and denoted $l(r)$. For example, $r = 123133$ is a restricted growth functions with $l(r) = 6$, while $r = 123153$ is not an RGF because there is no 4 in the prefix before the 5. Let

$$R_n = \{r\ :\ r \text{ an RGF with } l(r) = n\} \quad \text{and} \quad R = \biguplus_{n\geq 0} R_n.$$

There is a well-known bijection $\rho : \Pi_n \to R_n$. Given $\pi \in \Pi_n$ in canonical order, we let $\rho(\pi) = a_1a_2\ldots a_n$ where

$$a_i = B(i). \tag{19}$$

If one considers the example partition $\pi = 14/2/356 = B_1/B_2/B_3$ from the introduction, then $\rho(\pi) = 123133$ (the example RGF above). The definition (17) of "canonical order" ensures that $\rho(\pi)$ is an RGF. Furthermore, it is easy to construct an inverse for $\rho$ using (19). So we can work with a partition or its RGF interchangeably.

We now define pattern containment in $R$ analogously to the way it is defined for permutations. If $r \in R_k$ is the pattern RGF, then we say that $s = b_1b_2\ldots b_n \in R_n$ *contains* $r$ if there is a subsequence $b_{i_1}b_{i_2}\ldots b_{i_k}$ of $s$ which standardizes to $r$. Otherwise *$s$ avoids $r$*. By way of illustration, if $r = 121$ then there are two copies of $r$ in $b_1b_2b_3b_4b_5b_6 = 123133$, namely $b_1b_2b_4 = 121$ and $b_1b_3b_4 = 131$.

If $\pi$ and $\sigma$ are such that $\rho(\sigma)$ contains $\rho(\pi)$ then we say that $\sigma$ *R-contains* $\pi$, and that $\sigma$ *R-avoids* $\pi$ otherwise. We will also add an "R" prefix to other terms defined in the introduction in order to refer to this new definition. We can see R-containment directly in terms of partitions as follows: $\sigma$ R-contains $\pi$ if and only if $\sigma$ has a subpartition $\sigma' = C_1'/C_2'/\ldots/C_k'$ (in canonical order) with $\mathrm{st}(\sigma') = \pi$ and

$$B(C_1') < B(C_2') < \ldots < B(C_k'). \tag{20}$$

For example, of the four copies of $13/2$ in $14/2/356$ only two of them, namely $14/2$ and $14/3$, are R-copies. In fact, this is just a restatement in terms of partitions of the example at the end of the previous paragraph. Given a set partition. $\pi$ we let

$$R_n(\pi) = \{s \in \Pi_n \ : \ \sigma \text{ R-avoids } \pi\} \quad \text{and} \quad R(\pi) = \biguplus_{n\geq 0} R_n(\pi).$$

The next proposition is clear from the definitions.

**Proposition 4.1.** *For every $\pi \in \Pi$ and every $n \geq 0$,*

$$\begin{array}{rcl} R_n(\pi) & \supseteq & \Pi_n(\pi), \\ \#R_n(\pi) & \geq & \#\Pi_n(\pi). \qquad \blacksquare \end{array}$$

Note that if $\pi = \hat{0}_m$ or $\hat{1}_m$ then (20) is automatic. So the next result follows immediately from the previous proposition and, because of Theorem 2.2, the corresponding enumerations have already been done.

**Theorem 4.2.** *We have*

$$\begin{array}{rcl} R(1/2/\ldots/m) & = & \{\sigma \in \Pi \ : \ b(\sigma) < m\}, \\ R(12\ldots m) & = & \{\sigma \in \Pi \ : \ \#C < m \textit{ for all } C \in \sigma\}. \qquad \blacksquare \end{array}$$

We now turn to R-avoidance of patterns in $\Pi_3$. In this context, complementation does not necessarily preserve the number of avoiding partitions. So it is somewhat surprising that four of the five elements of $\Pi_3$ are R-Wilf equivalent.

**Theorem 4.3.** *We have*

$$\begin{aligned}
R_n(1/2/3) &= \{\sigma \in \Pi_n \ : \ b(\sigma) \leq 2\},\\
\#R_n(1/2/3) &= 2^{n-1},\\
R_n(123) &= \{\sigma \in \Pi_n \ : \ \sigma \text{ is a matching}\},\\
\#R_n(123) &= \sum_{i\geq 0} \binom{n}{2i}(2i)!!,\\
R_n(12/3) &= \big\{\sigma \in \Pi_n \ : \ \exists k \ni \sigma_{\leq k} = \hat{0}_k,\\
&\qquad \sigma_{>k} = D_1/\ldots/D_k \text{ layered, } B(D_1) > \ldots > B(D_k)\big\},\\
\#R_n(12/3) &= 2^{n-1},\\
R_n(1/23) &= \{\sigma = C_1/\ldots/C_l \in \Pi_n \ : \ \#C_i = 1 \text{ for } i \geq 2\},\\
\#R_n(1/23) &= 2^{n-1},\\
R_n(13/2) &= \{\sigma \in \Pi_n \ : \ \sigma \text{ is layered}\},\\
\#R_n(13/2) &= 2^{n-1}.
\end{aligned}$$

**Proof** The equations involving the patterns $1/2/3$ and $123$ follow from Theorems 2.5 and 4.2.

Now suppose $\sigma \in R_n(12/3)$ and let $k$ be the largest integer such that the elements of $[k]$ are the minima of their blocks in $\sigma$. So either $k = n$ (and $\sigma = \hat{0}_n$) or $k+1$ is in a block $C_i$ of $\sigma$ where $i \leq k$. Let $m$ be the maximum of $C_i$. We claim that $C_i = \{i\} \uplus [k+1, m]$. If this were not the case then there would have to be some $l$ with $k+1 < l < m$ and with $l \in C_j$ for $j \neq i$. If $j < i$ then $jl/m$ is an R-copy of $12/3$ in $\sigma$, and if $j > i$ then $i, k+1/l$ is such an R-copy. So in either case we have a contradiction. Iterating this argument shows that $\sigma$ has the form described in the theorem. It is also clear that partitions of this form do not have any R-copies of $12/3$, so this completes the characterization of such partitions.

To enumerate $R_n(12/3)$, keep $k$ as in the previous paragraph. Then the number of $\sigma$ for a given $k$ is just the number of ways to distribute the elements of $\sigma_{>k}$ among the $k$ blocks. Since $\sigma_{>k}$ is layered, this is equivalent to counting the number of compositions (ordered integer partitions) of $n-k$ into $k$ parts where 0 is allowed as a part. It is well-known that the number of such compositions is $\binom{n-1}{k-1}$. So the total number of $\sigma$ is $\sum_k \binom{n-1}{k-1} = 2^{n-1}$.

Next consider $R_n(1/23)$. Clearly $\sigma = C_1/\ldots/C_l$ can not contain an R-copy of $1/23$ if all blocks other than $C_1$ are singletons. And if some block of $\sigma$ other than $C_1$ contains two elements $i, j$ then $1/ij$ is an R-copy of $1/23$ in $\sigma$. So this gives us the required set equality. Also, the number of such $\sigma$ is just the number of choices for $C_1$, which is $2^{n-1}$ since we must have $1 \in C_1$ for the blocks to be canonically ordered.

Finally, look at $R_n(13/2)$. From Theorem 2.5 and Proposition 4.1 we have that $R_n(13/2)$ contains every layered permutation. The proof of the reverse containment is the same as that given for the corresponding containment in (13), just noting that the copy of $13/2$ constructed there is, in fact, an R-copy. Of course, this means that the enumeration is the same as well. ■

Since no numerically new sequences have been discussed in this section, we can use Theorem 3.8 to conclude the following.

**Theorem 4.4.** *For $m \geq 1$, the following sequence are P-recursive when $n$ varies over $\mathbb{N}$:*

$$\#R_n(1/2/\ldots/m) \quad \textit{and} \quad \#R_n(12\ldots m).$$

*Also, for any $\pi \vdash [3]$ the sequence $\#R_n(\pi)$, $n \geq 0$, is P-recursive.* ■

# 5 Open problems and new directions

## 5.1 The patterns $12/3/4/\ldots/m$ and $1/23\ldots m$

We were unable to simplify the summation given for $\#\Pi_n(12/3/4/\ldots/m)$. It would be interesting to do so, or to use them to find the corresponding exponential generating function.

We have given characterizations of $\Pi(\pi)$ where $\pi$ is the minimum, maximum, or one of the atoms in the partition lattice. It is also possible to do so for the coatom $1/23\ldots m$. We did not mention this earlier because the description is not used for any of the other results presented. But we will give it here in case it turns out to be useful in later work. To describe the $\sigma = C_1/\ldots/C_l$ in $\Pi(1/23\ldots m)$ we assume, as usual, that $\sigma$ is written in canonical form. We will also need the parameter $c = c(\sigma)$ which will be the $(m-1)$st largest element of $C_1$, or 0 if $\#C_1 < m-1$. Then

$$\Pi(1/23\ldots m) =$$
$$\{\sigma = C_1/\ldots/C_l \in \Pi \ : \ \#C_i < m-1 \text{ for } i \geq 2, \text{ and } \min C_2 > c(\sigma)\}.$$

The proof of this equality is much like the one for (3), where the first restriction on $\sigma$ ensures that there can be no copy of the pattern where the subset corresponding to $23\ldots m$ is in a block of index at least two, and the second restriction does the same for $C_1$.

## 5.2 Wilf equivalence

As previously mentioned, any two permutations in $\mathfrak{S}_3$ are Wilf equivalent. Babson and West [2] showed that $123a_3\dots a_n$ and $321a_3\dots a_n$ are Wilf equivalent for any permutation $a_3\dots a_n$ of $[4,n]$. This work was later generalized by Backelin, West, and Xin [3]. Are the Wilf equivalences that appeared for both containment and R-containment isolated incidents or part of a larger picture?

## 5.3 Multiple restrictions

Let $\mathcal{P} \subseteq \mathfrak{S}$ be any set of permutations and define

$$\mathfrak{S}_n(\mathcal{P}) = \{q \in \mathfrak{S}_n \ : \ q \text{ avoids } p \text{ for all } p \in \mathcal{P}\}.$$

Simion and Schmidt [33] enumerated all such sets where $\mathcal{P} \subseteq \mathfrak{S}_3$. Similarly, for $\mathcal{P} \subseteq \Pi$ one can let

$$\Pi_n(\mathcal{P}) \ = \ \{\sigma \in \Pi_n \ : \ \sigma \text{ avoids } \pi \text{ for all } \pi \in \mathcal{P}\}.$$

Goyt [17] has considered the analogous question for $\Pi_n(\mathcal{P})$ where $\mathcal{P} \subseteq \Pi_3$. For example,

$$\Pi_n(123, 13/2) = F_n \tag{21}$$

where $F_n$ is the $n$th Fibonacci number.

## 5.4 Statistics

The *inversion number* of $p = a_1a_2\dots a_n \in \mathfrak{S}_n$ is

$$\operatorname{inv} p = \#\{(a_i, a_j) \ : \ i < j \text{ and } a_i > a_j\}.$$

Also, the *major index* of $p$ is defined to be

$$\operatorname{maj} p = \sum_{a_i > a_{i+1}} i.$$

It is well-known, and easy to prove, that if $q$ is a variable then

$$\sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{inv} \pi} = \sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{maj} \pi} = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

A statistic on $\mathfrak{S}_n$ with this generating function is said to be *Mahonian* in honor of Major Percy MacMahon [26] who made the first systematic study of inv and maj. And the polynomial product above is called a *q-analogue* of the integer $n$!

Babson and Steingrímsson [1] defined generalized permutation patterns by insisting that certain elements of the pattern be adjacent in the larger permutation. They then showed that most Mahonian statistics in the literature can be written as linear combinations of the statistics defined by generalized patterns.

The Stirling numbers of the the second kind, $S(n,k)$, count the number of set partitions of $[n]$ with $k$ blocks. Carlitz [9, 10] introduced a $q$-analogue, $S_q(n,k)$, of $S(n,k)$. Milne [28], Garsia and Remmel [15], Leroux [25], and Wachs and White [38] have all given set partition analogues of the inv statistic whose generating function is $S_q(n,k)$, possibly up to a factor of $q^{\binom{k}{2}}$. Sagan [30] and later White [39] gave maj statistics for set partitions. In the previously mentioned paper of Goyt [17], generalized patterns for set partitions are defined. He then uses them to obtain various statistics in the literature as well as enumerates the number of partitions which avoid them.

Carlitz [11] was also the first to define a $q$-analogue, $F_n(q)$, of the Fibonacci numbers. These polynomials and their generalizations have been extensively studied by Cigler [13, 12, 13] as well as Shattuck and Wagner [32]. In view of (21), one can define related $q$-analogues using the generating functions for various set partition statistics over the family $\Pi_n(123, 13/2)$. This yields a new and unified approach to the study of $F_n(q)$ and its relatives which is being pursued by Goyt and Sagan [18].

## 5.5 Partial orders

The set $\mathfrak{S}$ of all permutations becomes a poset (partially ordered set) by defining $p \le q$ if and only if there is a copy of $p$ in $q$. One of the fundamental invariants of any poset is its Möbius function, $\mu$. See Stanley's text [37, Chapter 3] for information about posets in general and the Möbius function in particular. Wilf asked the following question.

**Question 5.1** (Wilf [41]). *If $p \le q$ in $\mathfrak{S}$ then what is $\mu(p,q)$?*

This question has been partially answered as follows. Call a permutation $p$ *layered* if it has the form

$$p = i, i-1, \ldots, 1, i+j, i+j-1, \ldots, i+1, i+j+k, i+j+k-1, \ldots, i+j+1, \ldots$$

where $i, j, k, \ldots$ are called the *layer lengths of p*. There is a bijection between the layered permutations in $\mathfrak{S}_n$ and compositions (ordered integer partitions) of $n$ gotten by sending $p$ as above to the composition $(i, j, k, \ldots)$. Denoting the set of all compositions by $\mathbb{P}^*$, we have a partial order on this set induced by the pattern containment order on $\mathfrak{S}$. This partial order was first studied by Bergergon, Bousquet-Mélou, and Dulucq [4] who counted its saturated lower chains. Further work in this direction was done by Snellman [34, 35]. The Möbius function of $\mathbb{P}^*$ was obtained by Sagan and Vatter [31] in two ways, combinatorially and using discrete Morse theory. It was also rederived by Björner and Sagan [8] using the theory of regular languages. This poset turns out to be intimately related to subword order, whose Möbius function was first completely determined by Björner [5, 6] and again by Björner and Reutenauer [7].

Of course, we can partially order $\Pi$ by pattern containment and ask the same question.

**Question 5.2.** *If $\pi \leq \sigma$ in $\Pi$ then what is $\mu(\pi, \sigma)$?*

There is clearly a bijection between layered permutations and layered partitions. So the work cited above applies to this poset as well. Note that if we restrict the full poset of compositions to the compositions which only contain ones and twos, then we get a corresponding partial order on $\Pi(1/2/3, 13/2)$ having rank numbers equal to the $F_n$. Goyt [personal communication] is currently investigating what can be said in various other posets related to $\Pi$ whose rank numbers are given by certain generalized Fibonacci numbers.

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