Discrete Mathematics 99 (1992) 289–306
North-Holland

Inductive proofs of q-log concavity

Bruce E. Sagan*
Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA

Received 20 February 1989
Revised 15 June 1989

Abstract

Sagan, B.E., Inductive proofs of q-log concavity, Discrete Mathematics 99 (1992) 289–306.

We give inductive proofs of q-log concavity for the Gaussian polynomials and the q-Stirling numbers of both kinds. Similar techniques are applied to show that certain sequences of elementary and complete symmetric functions are q-log concave.

1. Introduction and definitions

Throughout this paper $\mathbb{N}$ and $\mathbb{Z}$ will stand for the natural numbers $\{0, 1, 2, \ldots\}$ and integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ respectively. A sequence of natural numbers

$$(a_k)_{k\in\mathbb{Z}} = \ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$$

is *log concave* if

$$a_{k-1}a_{k+1} \leq a_k^2 \quad \text{for all } k \in \mathbb{Z}.$$

Log concave sequences appear in algebra, combinatorics and geometry. See the survey article of Stanley [17] for details.

Now let q be an indeterminate. In order to define the q-analog of log concavity, we must first give a q version of the order relation $\leq$ on $\mathbb{N}$. Given the two polynomials $f(q), g(q) \in \mathbb{N}[q]$ with $f(q) = \sum_{i\geq 0} a_i q^i$, $g(q) = \sum_{i\geq 0} b_i q^i$, we will say that

$$f(q) \leq_q g(q) \text{ if and only if } a_i \leq b_i \text{ for all } i.$$

Equivalently $f(q) \leq_q g(q)$ whenever $g(q) - f(q) \in \mathbb{N}[q]$. Note that while $\leq$ is a total order on $\mathbb{N}$, $\leq_q$ is only a partial order on $\mathbb{N}[q]$. Still, this ordering respects the algebraic operations in $\mathbb{N}[q]$.

* Supported in part by NSF grant DMS 8805574 and a fellowship from the Institute for Mathematics and its Applications.

Elsevier Science Publishers B.V.

Lemma 1.1. *Suppose the polynomials $f(q)$, $g(q)$, $F(q)$, $G(q) \in \mathbb{N}[q]$ satisfy the inequalities $f(q) \leqslant_q F(q)$ and $g(q) \leqslant_q G(q)$ then*:

(1) $f(q) + g(q) \leqslant_q F(q) + G(q)$,

(2) $f(q)g(q) \leqslant_q F(q)G(q)$.

The proof of this lemma is straightforward.

Now the definition of q-log concavity first suggested by Stanley, should be quite natural. A sequence of polynomials in $\mathbb{N}[q]$

$$(f_k(q))_{k \in \mathbb{Z}} = \ldots, f_{-2}(q), f_{-1}(q), f_0(q), f_1(q), f_2(q), \ldots \tag{1}$$

is *q-log concave* if

$$f_{k-1}(q)f_{k+1}(q) \leqslant_q f_k(q)^2 \quad \text{for all } k \in \mathbb{Z}.$$

It is clear that this statement reduces to the one about sequences of natural numbers when we let $q = 1$. Furthermore, we say that the sequence (1) is *strongly q-log concave* if

$$f_{k-1}(q)f_{l+1}(q) \leqslant_q f_k(q)f_l(q) \quad \text{for all } l \geqslant k.$$

The reader may be puzzled by this last definition, as these two notions are equivalent for sequences of natural numbers. This is not so for arbitrary q. For example, Lemke pointed out that the sequence q^2, $q + q^2$, $1 + 2q + q^2$, $4 + 2q + q^2$ is q-log concave but not strongly q-log concave.

Gessel [7] was the first to give a combinatorial proof of the Jacobi–Trudi identity (see Section 4.4) which showed strong q-log concavity of a sequence of modified q-binomial coefficients. Butler [4] then demonstrated combinatorially that the q-binomials themselves enjoyed this property as the lower index varies. Another combinatorial proof was given by Krattenhaler [10]. In a previous paper, [14], we gave Stirling number of both kinds using induction. In the next section we will show how this method extends to the q-analogs of these sequences. This settles Butler's conjecture that the q-Stirling numbers of the second kind are strongly q-log concave when the second index varies. Finally, Leroux [12] adapted the techniques in [4] to give combinatorial demonstrations for the q-Stirling numbers.

In Section 3 we will adapt the inductive method to sequences of elementary and complete symmetric functions. Section 4 will discuss proofs of these results using injections and Schur functions as well as some remarks and open problems.

2. q-Binomial coefficients and q-Stirling numbers

The standard q-analog on $n \in \mathbb{N}$ is

$$[n] = 1 + q + q^2 + \cdots + q^{n-1}.$$

This furnishes us with our first strongly q-log concave sequence.

Lemma 2.1. *The sequence* $([n])_{n\in N}$ *is strongly* q*-log concave.*

Proof. To verify $[k-1][l+1] \leqslant_q [k][l]$ for $k \leqslant l$, merely multiply out both sides and compare like powers of q. □

Next we have the q-factorial

$$[n]! = [n][n-1]\cdots[2][1].$$

Finally we can define the *q-binomial coefficients* or *Gaussian polynomials* as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{[n]!}{[k]!\,[n-k]!} & \text{for } 0 \leqslant k \leqslant n, \\ 0 & \text{for } k < 0 \text{ or } k \geqslant n. \end{cases}$$

Note that this defines the q-binomial coefficients for all natural n and integral k. It turns out that the $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ *are* polynomials in q, although this is not instantly clear from the definition.

Since we will be dealing with inductive proofs, we will need the two usual recursions for the q-binomial coefficients.

Proposition 2.2. *For* $n \geqslant 1$, *the q-binomial coefficients satisfy the recursions*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix},$$

as well as the initial condition

$$\begin{bmatrix} 0 \\ k \end{bmatrix} = \delta_{0,k}$$

where $\delta_{0,k}$ *is the Kronecker delta.*

It is easy to prove this proposition directly from the definition of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$. As a corollary, we see immediately that the q-binomial coefficients are in $\mathbb{N}[q]$ as promised above.

We will define the q-analogs of the Stirling numbers inductively. The (*signless*) *q-Stirling numbers of the first kind* are denoted $c[n, k]$ and satisfy

$$c[n, k] = c[n-1, k-1] + [n-1]c[n-1, k] \quad \text{for } n \geqslant 1, \text{ with } c[0, k] = \delta_{0,k}. \tag{2}$$

The *q-Stirling numbers of the second kind* are defined by

$$S[n, k] = S[n-1, k-1] + [k]S[n-1, k] \quad \text{for } n \geqslant 1, \text{ with } S[0, k] = \delta_{0,k}. \tag{3}$$

These polynomials were first studied by Gould [9] and Carlitz [5, 8] respectively.

Theorem 2.3. *For fixed $n \geq 0$, the sequence $(\left[{n \atop k}\right])_{k\in\mathbb{Z}}$ is strongly q-log concave.*

Proof. It will be convenient to prove the statement

$$q^i\begin{bmatrix} n \\ k-1 \end{bmatrix}\begin{bmatrix} n \\ l+1 \end{bmatrix} \leq_q \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n \\ l \end{bmatrix} \quad \text{for all } k \leq l \text{ and } 0 \leq i \leq 2(l-k+1). \tag{4}$$

The upper bound on the power of q enters because the difference in degree between $\left[{n \atop k}\right]\left[{n \atop l}\right]$ and $\left[{n \atop k-1}\right]\left[{n \atop l+1}\right]$ is exactly $2(l-k+1)$. The equation is clearly true when $n=0$, so assume $n \geq 1$.

We first consider the case where $1 \leq i \leq 2l-2k+1$. Expanding the left hand side of (4) using the first recursion in Proposition 2.2 we obtain

$$\begin{aligned} q^i&\left(\begin{bmatrix} n-1 \\ k-2 \end{bmatrix} + q^{k-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right)\left(\begin{bmatrix} n-1 \\ l \end{bmatrix} + q^{l+1}\begin{bmatrix} n-1 \\ k+1 \end{bmatrix}\right) \\ &= q^i\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}\begin{bmatrix} n-1 \\ l \end{bmatrix} + q^{l+i+1}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}\begin{bmatrix} n-1 \\ l+1 \end{bmatrix} \\ &\quad + q^{i+k-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ l \end{bmatrix} + q^{l+k+i}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ l+1 \end{bmatrix}. \end{aligned} \tag{5}$$

Applying the same procedure on the right yields

$$\begin{aligned} &\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ l-1 \end{bmatrix} + q^l\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ l \end{bmatrix} \\ &\quad + q^k\begin{bmatrix} n-1 \\ k \end{bmatrix}\begin{bmatrix} n-1 \\ l-1 \end{bmatrix} + q^{l+k}\begin{bmatrix} n-1 \\ k \end{bmatrix}\begin{bmatrix} n-1 \\ l \end{bmatrix}. \end{aligned} \tag{6}$$

Now compare corresponding terms of (5) and (6). After canceling various powers of q, we see (by Lemma 1.1) that it suffices to prove the following four inequalities:

$$q^i\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}\begin{bmatrix} n-1 \\ l \end{bmatrix} \leq_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ l-1 \end{bmatrix}, \tag{7}$$

$$q^{i+1}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}\begin{bmatrix} n-1 \\ l+1 \end{bmatrix} \leq_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ l \end{bmatrix}, \tag{8}$$

$$q^{i-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ l \end{bmatrix} \leq_q \begin{bmatrix} n-1 \\ k \end{bmatrix}\begin{bmatrix} n-1 \\ l-1 \end{bmatrix}, \tag{9}$$

$$q^i\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ l+1 \end{bmatrix} \leq_q \begin{bmatrix} n-1 \\ k \end{bmatrix}\begin{bmatrix} n-1 \\ l \end{bmatrix}. \tag{10}$$

Equation (7) follows from induction and the fact that $(l-1)-(k-1)+1 = l-k+1$ so that the bounds on i are the same as in (4). For (8) we need only verify that $0 \leq i+1 \leq 2\{l-(k-1)+1\}$, which again follows from (4). To get $i-1$ in the right range for (9), the fact that we are in the case where

$1 \leq i \leq 2l - 2k + 1$ comes into play. Note, also, that we may not have $k \leq l - 1$ for purposes of induction. But this only happens if $k = l$ which forces both sides of (9) to be equal. Finally, equation (10) is immediate.

To take care of the case where $i = 0$, we expand the left and right sides of (4) as

$$\left(q^{n-k+1}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right)\left(\begin{bmatrix} n-1 \\ l \end{bmatrix} + q^{l+1}\begin{bmatrix} n-1 \\ l+1 \end{bmatrix}\right)$$

and

$$\left(q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}\right)\left(\begin{bmatrix} n-1 \\ l-1 \end{bmatrix} + q^{l}\begin{bmatrix} n-1 \\ l \end{bmatrix}\right)$$

respectively. When $i = 2(l - k + 1)$ we use

$$q^{2(l-k+1)}\left(\begin{bmatrix} n-1 \\ k-2 \end{bmatrix} + q^{k-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right)\left(q^{n-l-1}\begin{bmatrix} n-1 \\ l \end{bmatrix} + \begin{bmatrix} n-1 \\ l+1 \end{bmatrix}\right)$$

and

$$\left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^{k}\begin{bmatrix} n-1 \\ k \end{bmatrix}\right)\left(q^{n-l}\begin{bmatrix} n-1 \\ l-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ l \end{bmatrix}\right).$$

The details of the comparison process are similar to those above and are left to the reader. □

It seems as if (4) is a stronger statement than the theorem itself. Because of certain properties of the q-binomial coefficients (symmetry and unimodality), they are actually equivalent. We will explain this more fully in (ii) of the last section.

The proof for the q-Stirling numbers of the first kind is particularly easy.

Theorem 2.4. *For fixed $n \geq 0$, the sequence $(c[n, k])_{k \in \mathbb{Z}}$ is strongly q-log concave.*

Proof. We again use induction on n. To eliminate a plethora of $n - 1$'s we prove

$$c[n+1, k-1]c[n+1, l+1] \leq_q c[n+1, k]c[n+1, l] \quad \text{for all } l \geq k.$$

Expanding both sides by the recursion for $c[n, k]$ and comparing corresponding terms yields a sufficient set of equations:

$$\begin{aligned}
&c[n, k-2]c[n, l] \leq_q c[n, k-1]c[n, l-1], \\
&[n]c[n, k-2]c[n, l+1] \leq_q [n]c[n, k-1]c[n, l], \\
&[n]c[n, k-1]c[n, l] \leq_q [n]c[n, k]c[n, l-1], \\
&[n]^2c[n, k-1]c[n, l+1] \leq_q [n]^2c[n, k]c[n, l].
\end{aligned}$$

These all follow from induction and Lemma 1.1. □

Theorem 2.5. *For fixed $n \geq 0$, the sequence $(S[n,k])_{k\in\mathbb{Z}}$ is strongly q-log concave.*

Proof. We need to strengthen the induction hypothesis to

$$q^iS[n+1,k-1]S[n+1,l+1] \leq_q S[n+1,k]S[n+1,l]$$

for all $l \geq k$ and $0 \leq i \leq l-k+2$. Proceed as usual. For the first and fourth terms we want to show

$$q^iS[n,k-2]S[n,l] \leq_q S[n,k-1]S[n,l-1],$$
$$q^i[k-1][l+1]S(n,k-1]S([n,l+1] \leq_q [k][l]S[n,k]S[n,l].$$

Since $[k-1][l+1] \leq_q [k][l]$ (by Lemma 2.1), Lemma 1.1 and induction finish these cases.

Combining the two middle terms, we need to prove

$$\begin{aligned} &q^i[l+1]S[n,k-2]S[n,l+1] + q^i[k-1]S[n,k-1]S[n,l] \\ &\quad \leq_q [l]S[n,k-1]S[n,l] + [k]S[n,k]S[n,l-1]. \end{aligned} \tag{11}$$

Suppose first that $1 \leq i \leq l-k+2$. Substituting $q^i[l+1] = q^{i+1}[l]+1$ and $[k] = q(k-1]+1$ above, we obtain

$$\begin{aligned} &q^{i+1}[l]S[n,k-2]S[n,l+1] + q^iS[n,k-2]S[n,l+1] \\ &\quad + q^i[k-1]S[n,k-1]S[n,l] \\ &\quad \leq_q [l]S[n,k-1]S[n,l] + S[n,k]S(n,l-1] + q[k-1]S[n,k]S[n,l-1]. \end{aligned}$$

Comparing corresponding terms we get three inequalities, all of which are true because of the bounds on i in this case (and Lemma 1.1). In particular, for the middle pair when $k<1$ we use

$$S[n,k]S(n,l-1]\begin{cases} \geq_q q^jS[n,k-1]S[n,l] & \text{for } 0 \leq j \leq l-k+1, \\ \geq_q q^jS[n,k-2]S[n,l+1] & \text{for } 0 \leq j \leq 2(l-k+2). \end{cases}$$

To conclude we need only consider $i=0$. Using $[l+1]=[l]+q^l$ and $[k] = [k-1]+q^{k-1}$, equation (11) becomes

$$\begin{aligned} &[l]S[n,k-2]S[n,l+1] + q^lS[n,k-2]S([n,l+1] \\ &\quad + [k-1]S[n,k-1]S[n,l] \\ &\quad \leq_q [l]S[n,k-1]S[n,l] + q^{k-1}S[n,k]S[n,l-1] + [k-1]S[n,k]S[n,l-1]. \end{aligned}$$

We omit the rest of the argument since it is similar to the first case. □

Unlike the case of the q-binomial coefficients, the induction hypothesis really *is* stronger here. So we have really proved the following.

Corollary 2.6. *Fix $n \geq 0$. Then for all $l \geq k$ and $0 \leq i \leq l-k+2$ we have*

$$q^iS[n,k-1]S[n,l+1] \leq_q S[n,k]S[n,l].$$

3. Elementary and complete symmetric functions

Let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ be a set of variables. The kth *elementary symmetric function* is

$$e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \quad \text{for } 0 \leq k \leq n,$$

with $e_k(x_1, x_2, \ldots, x_n) = 0$ for $k < 0$ or $k > n$. The kth *complete symmetric function* is

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \quad \text{for } k \geq 0,$$

with $h_k(x_1, x_2, \ldots, x_n) = 0$ for $k < 0$. From these definitions it is easy to verify the following proposition.

Proposition 3.1. *For $n \geq 1$, the elementary symmetric functions satisfy the recursion*

$$e_k(x_1, \ldots, x_n) = x_n e_{k-1}(x_1, \ldots, x_{n-1}) + e_k(x_1, \ldots, x_{n-1})$$

with the initial condition $e_k(\emptyset) = \delta_{0,k}$.

For $n \geq 1$, the complete symmetric functions satisfy the recursion

$$h_k(x_1, \ldots, x_n) = x_n h_{k-1}(x_1, \ldots, x_n) + h_k(x_1, \ldots, x_{n-1})$$

with the initial condition $h_k(\emptyset) = \delta_{0,k}$.

The q-binomial coefficients and q-Stirling numbers can both be expressed as specializations of these functions. In fact, we have

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{-\binom{k}{2}} e_k(1, q, \ldots, q^{n-1}) \tag{12}$$

$$= h_k(1, q, \ldots, q^{n-k}), \tag{13}$$

$$c(n, k] = e_{n-k}([1], [2], \ldots, [n-1]), \tag{14}$$

$$S(n, k] = h_{n-k}([1], [2], \ldots, [k]). \tag{15}$$

All these identities can be proven directly from the appropriate recursions. This suggests that we can prove generalizations of the theorems in Section 2 for elementary symmetric functions and complete symmetric functions. First, however, we need a few more definitions.

Given two polynomial $f(\boldsymbol{x}), g(\boldsymbol{x}) \in \mathbb{N}[\boldsymbol{x}]$ (where $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$) we define

$$f(\boldsymbol{x}) \leq_x g(\boldsymbol{x}) \text{ if and only if } g(\boldsymbol{x}) - f(\boldsymbol{x}) \in \mathbb{N}[\boldsymbol{x}].$$

It is obvious that the analog of Lemma 1.1 holds. The definition of strongly $\boldsymbol{x}$-log concave is obtained by replacing q by $\boldsymbol{x}$ everywhere in the definition of strongly

q-log concave. For the sake of brevity we will let

$$e_k(n) \stackrel{\text{def}}{=} e_k(x_1, x_2, \ldots, x_n)$$

and similarly for the complete symmetric functions.

Theorem 3.2. *For fixed $n \geq 0$, the following sequences are strongly $\mathbf{x}$-log concave:*
(1) $(e_k(n))_{k \in \mathbb{Z}}$,
(2) $(h_k(n))_{k \in \mathbb{Z}}$.

Proof. To prove (1), follow the usual procedure of applying the recursion and comparing like terms. Thus to show

$$e_{k-1}(n)e_{l+1}(n) \leq_x e_k(n)e_l(n)$$

it suffices to prove

$$\begin{aligned}
&x_n^2 e_{k-2}(n-1)e_l(n-1) \leq_x x_n^2 e_{k-1}(n-1)e_{l-1}(n-1),\\
&x_n e_{k-2}(n-1)e_{l+1}(n-1) \leq_x x_n e_{k-1}(n-1)e_l(n-1),\\
&x_n e_{k-1}(n-1)e_l(n-1) \leq_x x_n e_k(n-1)e_{l-1}(n-1),\\
&e_{k-1}(n-1)e_{l+1}(n-1) \leq_x e_k(n-1)e_l(n-1).
\end{aligned}$$

All of these are instances of the induction hypothesis.

For the complete symmetric functions, we will prove that for all $n \geq m \geq 0$ and all $l \geq k$ we have

$$h_{k-1}(n)h_{l+1}(m) \leq_x h_k(n)h_l(m). \tag{16}$$

Our method will be a double induction on l and n. To check the boundary cases, note that (16) is certainly true if $l < 0$ or $n = 0$.

Now consider $n > 0$. If $k < 0$ or $m = 0$ then both sides of (16) must be zero. Thus we may assume $k \geq 0$, $m > 0$ and use induction on these two variables as well. Expanding both sides as usual, we are reduced to verifying

$$\begin{aligned}
&x_n x_m h_{k-2}(n)h_l(m) \leq_x x_n x_m h_{k-1}(n)h_{l-1}(m),\\
&x_n h_{k-1}(n)h_{l+1}(m-1) \leq_x x_n h_{k-1}(n)h_l(m-1),\\
&x_m h_{k-1}(n-1)h_l(m) \leq_x x_m h_k(n-1)h_{l-1}(m),\\
&h_{k-1}(n-1)h_{l+1}(m-1) \leq_x h_k(n-1)h_l(m-1).
\end{aligned}$$

All of these equations follow by induction as long as $n > m$ and $l > k$ (so that the induction hypothesis will apply to the third inequality).

If $n = m$ and $l \geq k$ we can apply the recursion to only the terms involving l in (16). This yields a pair of inequalities

$$\begin{aligned}
&x_n h_{k-1}(n)h_l(n) \leq_x x_n h_k(n)h_{l-1}(n),\\
&h_{k-1}(n)h_{l+1}(n-1) \leq_x h_k(n)h_l(n-1).
\end{aligned}$$

All are true under the restrictions we have imposed in this case. Finally, to take care of the situation where $l = k$ and $n > m$, we expand only the terms with k in (16). The details are left to the reader. □

We immediately have the following corollary.

Corollary 3.3. *Let $f_1, f_2, \ldots, f_n \in \mathbb{N}[q]$ be any arbitrary sequence of polynomials. Then the sequences*

$$(e_k(f_1, f_2, \ldots, f_n))_{k\in\mathbb{Z}}$$

and

$$(h_k(f_1, f_2, \ldots, f_n))_{k\in\mathbb{Z}}$$

are strongly q-log concave.

For simplicity, we will often suppress the parameter q as we have above. Another corollary is the following.

Corollary 3.4. *The following sequences are strongly q-log concave:*

(1) $(q^{\binom{k}{2}}\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right])_{k\in\mathbb{Z}}$,
(2) $(\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right])_{n\in\mathbb{N}}$,
(3) $(c[n, k])_{k\in\mathbb{Z}}$,
(4) $(S[n, k])_{n\in\mathbb{N}}$.

Proof. Combining the previous corollary with equations (12), (13), (14) and (15) yields items (1), (2), (3) and (4) respectively. □

The strong q-log concavity of the first equation is equivalent to

$$q^{l-k+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}\begin{bmatrix} n \\ l+1 \end{bmatrix} \leqslant_q \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n \\ l \end{bmatrix} \quad \text{for all } l \geqslant k$$

upon cancellation of various powers of q. As was noted after the proof of Theorem 2.3, this is a weaker statement than the theorem itself. Item (3) was proved in Section 2 as Theorem 2.4 while the other two have not been presented previously here, although they are known results. We call (1) and (3) (respectively (2) and (4)) *strong q-log concavity in n* (*respectively k*) because that is the variable which is changing. Note that the $c[n, k]$ are not even q-log concave in n; the case $k = 1$ is a counterexample. However they do satisfy a similar condition to be given later (Corollary 4.8).

Next we investigate q-log concavity in n for the elementary and complete symmetric functions.

Theorem 3.5. *Fix $k \in \mathbb{N}$ and let $(f_n)_{n\geq 1}$, $f_n \in \mathbb{N}[q]$, be a strongly q-log concave sequence of polynomials. Then the sequences*

$$(e_k(f_1, f_2, \ldots, f_n))_{n\in\mathbb{N}}$$

and

$$(h_k(f_1, f_2, \ldots, f_n))_{n\in\mathbb{N}}$$

are strongly q-log concave.

Proof. We will need some notation. Let $S = \{s_1 < s_2 < \cdots < s_j\}$ be a subset of the positive integers. Then define

$$S + n = \{s_1 + n, s_2 + n, \ldots, s_j + n\} \quad \text{for any } n \in \mathbb{N}$$

and

$$f_S = f_{s_1} f_{s_2} \cdots f_{s_j}.$$

The cardinality of S is denoted $|S|$. As usual, $e_k(n)$ stands for $e_k(f_1, f_2, \ldots, f_n)$ and similarly for the complete functions.

For the elementary sequence, we will prove two statements. Suppose that $l \geq k$, $n \geq m \geq 0$ and S are given with $|S| = l - k$, then

$$f_{S+n+1} e_l(m-1) e_k(n+1) \leq_q f_{S+m} e_k(m) e_l(n), \tag{17}$$

$$f_{S+n} e_l(m) e_k(n) \leq_q f_{S+m} e_k(m) e_l(n), \tag{18}$$

where the second equation holds only for $l > k$ and $n > m$.

First we show that (17) holds. It is true if $k < 0$ because then the left-hand side is 0, so assume $k \geq 0$. We will use induction on m and n. If $m = 1$, then the left side can be nonzero (the only case we need worry about) only if l and k are both 0. So $S = \emptyset$ and both sides of the equation equal one. If $n = 1$ this forces $m = 1$, so the base cases for both m and n are complete.

Now suppose $n \geq m \geq 2$. Applying recursion to the terms with subscript k of (17) and breaking the result into two inequalities yields

$$f_{S+n+1} e_l(m-1) e_k(n) \leq_q f_{S+m} e_k(m-1) e_l(n), \tag{19}$$

$$f_{S+n+1} f_{n+1} e_l(m-1) e_{k-1}(n) \leq_q f_{S+m} f_m e_{k-1}(m-1) e_l(n). \tag{20}$$

When $l = k$ in (19), S becomes empty and thus both sides are equal. When $l > k$, this inequality follows from (18) with m replaced by $m - 1$ and S by $S + 1$. In the future we will use the PASCAL language replacement symbol for this, writing it as $m := m - 1$, $S := S + 1$. Equation (20) is also a special case of (18) where $m := m - 1$, $k := k - 1$ and $S := (S + 1) \cup \{1\}$ (note that this makes the cardinality of S correct).

In a similar manner, the proof of (18) reduces to demonstrating

$$f_{S+n} e_l(m-1) e_k(n) \leq_q f_{S+m} e_k(m) e_l(n-1),$$

$$f_{S+n} f_m e_{l-1}(m-1) e_k(n) \leq_q f_{S+m} f_n e_k(m) e_{l-1}(n-1).$$

The first of these is (17) with $n := n - 1$. The second will also follow from (17) using $n := n - 1$, $l := l - 1$ and $S := S \setminus \{s_1\}$ (S with its smallest element s_1 deleted) provided we can take care of the left-over terms on both sides. But this amounts to showing that $f_{s_1+n} f_m \leqslant_q f_{s_1+m} f_n$ which is true since the f-sequence is q-log concave.

The proof that the complete symmetric functions are strongly q-log concave in n is similar to the one for the elementaries. The induction hypothesis is slightly different and S must be permitted to be a multiset (= set with repetitions), but the reader will have no trouble supplying the details. $\square$

Using equations (13)–(15) in conjunction with this theorem gives the following.

Corollary 3.6. *For fixed $k \geqslant 0$, the following sequences are strongly q-log concave:*

(1) $(\left[\begin{smallmatrix} n \\ n-k \end{smallmatrix}\right])_{n \in \mathbb{N}}$,
(2) $(c[n, n-k])_{n \in \mathbb{N}}$,
(3) $(S[n, n-k])_{n \in \mathbb{N}}$.

Of course, the first of these three results is the same as item (2) of Corollary 3.4 because of the symmetry of the Gaussian coefficients. These results are referred to as q-log concavity in $n - k$. It is not true that the sequence $(e_{n-k}(n))_{n \in \mathbb{N}}$ is strongly q-log concave when the variables are arbitrary polynomials in q, e.g., consider $(e_n(n))_{n \in \mathbb{N}}$. However, it is conjectured that a related property holds; see Section 4.5.

The following theorem simultaneously generalizes Theorems 2.3 and 2.5.

Theorem 3.7. *Fix $b, c \in \mathbb{N}[q]$ such that $c \geqslant_q b$ or $c = 0$ and b is arbitrary. Consider the sequence defined by $f_n = bq^{n-1} + c[n-1]$ for all $n \geqslant 1$. Then both the sequences $(f_n)_{n \geqslant 1}$ and*

$$h_0(f_1, \ldots, f_n),\ h_1(f_1, \ldots, f_{n-1}),\ \ldots,\ h_n(\emptyset)$$

are strongly q-log concave.

Proof. Showing that the first sequence is strongly q-log concave is a routine calculation, merely expand each f_i in terms of powers of q with coefficients that are polynomials in b and c. Comparing corresponding terms finishes the argument.

To deal with the second sequence, we will use induction on n to show that

$$q^i h_{k-1}(n-k+1) h_{l+1}(n-l-1) \leqslant_q h_k(n-k) h_l(n-l)$$

for all $l \geqslant k$ and all i, $0 \leqslant i \leqslant l - k + 2$. Traveling our accustomed path, we are led to verify

$$q^i f_{n-k+1} f_{n-l-1} h_{k-2}(n-k+1) h_l(n-l-1)$$
$$\leqslant_q f_{n-k} f_{n-l} h_{k-1}(n-k) h_{l-1}(n-l),$$
$$q^i h_{k-1}(n-k) h_{l+1}(n-l-2) \leqslant_q h_k(n-k-1) h_l(n-l-1)$$

and

$$\begin{aligned} &q^i f_{n-k+1} h_{k-2}(n-k+1) h_{l+1}(n-l-2) + q^i f_{n-l-1} h_{k-1}(n-k) h_l(n-l-1) \\ &\quad \leq_q f_{n-k} h_{k-1}(n-k) h_l(n-l-1) + f_{n-l} h_k(n-k-1) h_{l-1}(n-l). \end{aligned} \tag{21}$$

Only the last of these causes any trouble.

Assume first that $1 \leq i \leq l-k+2$. Substituting $f_{n-k+1} = qf_{n-k} + c$; $f_{n-l} = qf_{n-l-1} + c$ into equation (21) and splitting the result into three inequalities yields

$$\begin{aligned} &q^{i+1} f_{n-k} h_{k-2}(n-k+1) h_{l+1}(n-l-2) \leq_q f_{n-k} h_{k-1}(n-k) h_l(n-l-1), \\ &q^i c f_{n-k} h_{k-2}(n-k+1) h_{l+1}(n-l-2) \leq_q c h_k(n-k-1) h_{l-1}(n-l), \\ &q^i f_{n-l-1} h_{k-1}(n-k) h_l(n-l-1) \leq_q q f_{n-l-1} h_k(n-k-1) h_{l-1}(n-l). \end{aligned}$$

All of these are true by induction.

When $i = 0$ we need to split the argument up into two parts depending on the assumptions on b and c. If $C \geq_q b$, then we can use

$$f_{j+1} = f_j + bq^{j+1} + (c-b)q^j \quad \text{for all } j \geq 1,$$

to replace f_{n-k+1} and f_{n-l}. Since $b, c-b \in \mathbb{N}[q]$ the four resultant inequalities will all hold, finishing this case.

If $c = 0$ then $f_k = bq^{k-1}$, and so $h_k(n-k) = b^k \left[{n \atop k} \right]$ by equation (13). Thus the q-log concavity of the sequence of complete symmetric functions is equivalent to Theorem 2.3. □

4. Remarks and open problems

The study of q-log concavity is relatively young. So there are many questions that still need to be answered.

4.1. Related concepts

A sequence $(a_k)_{k \in \mathbb{Z}}$, $a_k \in \mathbb{N}$, is unimodal if there is an index j such that

$$\cdots \leq a_{j-2} \leq a_{j-1} \leq a_j \geq a_{j+1} \geq a_{j+2} \geq \cdots.$$

The connection with q-log concavity is the following well-known theorem.

Theorem 4.1. *Let $(a_k)_{k \in \mathbb{Z}}$ be a sequence of positive integers. If $(a_k)_{k \in \mathbb{Z}}$ is log concave then it is unimodal.*

The q-unimodality of a sequence of polynomials can be defined by replacing $\leq$ by $\leq_q$ everywhere in the above definition. However, the analog of Theorem 4.1 is false as is seen by the counterexample

$$1 + 3q + 3q^2, \quad 2 + 2q + 3q^2, \quad 1 + 3q + q^2. \tag{22}$$

Is there some strengthening of q-log concavity that will guarantee q-unimodality? Butler [3] and later Rabau (private communication) have found proofs that the q-binomial coefficients are q-unimodal in k. In fact, Butler proved a much stronger result. Let $\alpha_\lambda(k;p)$ be the number of subgroups of order p^k in a finite abelian p-group of type λ, where λ is a partition of n. The main result of [3] is that the $\alpha_\lambda(k;p)$ are p-unimodal in k. The case where $\lambda=(1^n)$ gives the case of the Gaussian polynomials. It has been noted that q-unimodality in k is false for $c[5,k]$ [3] and $S[9,k]$ [12]. The q-unimodality question is open for the other sequences considered in this paper.

There are other possible candidates for the q-analogs of the definitions of q-log concavity and q-unimodal. Consider a sequence $(f_k(q))_{k\in\mathbb{Z}}$, where

$$f_k(q)=\sum_{i\in\mathbb{N}} a_{i,k}q^i$$

for all k. We will say the sequence is *componentwise log concave* (*respectively, componentwise unimodal*) if, for each fixed i, the sequence of coefficients $(a_{k,i})_{k\in\mathbb{Z}}$ is log concave (respectively, unimodal). Although q-log concavity of the sequence of $f_k(q)$ implies log concavity of the sequence of constant terms $(a_{0,k})_{k\in\mathbb{Z}}$, it does not even imply unimodality for the other coefficient sequences as is seen by our example (22). Is it possible to add some condition to q-log concavity so that it will give componentwise log concavity? For unimodality, it is easy to see that the following proposition is true.

Proposition 4.2. *The sequence $(f_k(q))_{k\in\mathbb{Z}}$ is q-unimodal if and only if it is componentwise unimodal and there is some value $k=k_0$ such that all the coefficient sequences have their maximum at k_0.*

4.2. Internal properties

We say a polynomial is *internally log concave* (*respectively, internally unimodal*) if the sequence of its coefficients is log concave (respectively, unimodal). The q-binomial coefficients have long been known to be internally log concave (and hence internally unimodal). White (private communication) has checked internal log concavity of the $c[n,k]$ and $S[n,k]$ for $n\leqslant 20$ and has conjectured that this holds in general.

A polynomial $f(q)=a_0+a_1q+\cdots+a_nq^n$ is *symmetric* if $a_k=a_{n-k}$ for all $0\leqslant k\leqslant n$. It is not hard to show that the product of two symmetric unimodal polynomials with positive coefficients is again a symmetric unimodal polynomial with positive coefficients (see [17, Proposition 1.2]). Note, also, that if $f(q)$ and $g(q)$ are symmetric unimodal polynomials with positive coefficients then

$$f(q)\leqslant_q g(q)\ \Rightarrow\ q^if(q)\leqslant_q g(q)$$

for all i with $0\leqslant i\leqslant \deg g(q)-\deg f(q)$. This observation from [2] explains the remarks after Theorem 2.3 and Corollary 3.4.

4.3. p, q-analogs

If p is another indeterminate, then we can define p, q-analogs of many of the concepts in this paper as follows. The p, q-analog of $n \in \mathbb{N}$ is

$$[n]_{p,q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + \cdots + q^{k-1}.$$

The p, q-binomial coefficients and Stirling numbers of both kinds are obtained by replacing $[n]$ by $[n]_{p,q}$ everywhere in their definitions. These polynomials have been studied by Wachs and White [19] among others. All the results of Sections 2 and 3 about polynomial sequences in $\mathbb{N}[q]$ have the obvious two variable analogs. In fact exactly the same proofs work with minor modifications. The only places where the statement of the p, q-analog might not be immediately apparent is in Corollary 2.6 and Theorem 3.7 which become as in the following propositions.

Proposition 4.3. *Fix $n \geqslant 0$. Then for all $l \geqslant k$ and $0 \leqslant i + j \leqslant l - k + 2$ we have*

$$q^i p^j S_{p,q}[n, k-1] S_{p\cdot q}[n, l+1] \leqslant_{p,q} S_{p,q}[n, k] S_{p\cdot q}[n, l].$$

Proposition 4.4. *Fix $b, c \in \mathbb{N}[p, q]$ such that $c \geqslant_{p,q} b$ or $c = 0$ and b is arbitrary. Consider the sequence defined by $f_k = bq^{k-1} + c[k-1]_{p,q}$ for all $k \geqslant 1$. Then both the sequences $(f_k)_{k \geqslant 1}$ and*

$$h_0(f_1, \ldots, f_n), h_1(f_1, \ldots, f_{n-1}), \ldots, h_n(f_1)$$

are strongly p, q-log concave.

4.4. Jacobi–Trudi proofs

Stanton (private communication) has observed that some of our results can be proved using techniques from the theory of symmetric functions. For example, the fact that $h_k(n) = h_k(x_1, x_2, \ldots, x_n)$ is strongly $\boldsymbol{x}$-log concave in k is equivalent to the determinantal condition

$$\begin{vmatrix} h_l(n) & h_{l+1}(n) \\ h_{k-1}(n) & h_k(n) \end{vmatrix} \in \mathbb{N}[\boldsymbol{x}].$$

This is a corollary of the well-known Jacobi–Trudi identity [13, p. 25; 15, Theorem 6.1.3].

Theorem 4.5. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ be a partition. Then the $d \times d$ determinant*

$$s_\lambda(n) \stackrel{\text{def}}{=} \det(h_{\lambda_i - i + j}(n))$$

is the generating function for all Young tableaux of shape λ with part size bounded by n. The function $s_\lambda(n)$ is called the Schur function associated with λ.

There is also a dual form of this theorem which expresses a Schur function as a determinant of elementary symmetric functions.

Since these determinants count tableaux, they must have nonnegative coefficients. It follows that Theorem 3.2 (and thus Corollaries 3.3 and 3.4) comes from the case $\lambda = (l, k)$ of Jacobi–Trudi. It is interesting to note that the strengthened induction hypothesis (16) also comes from the analog of Theorem 4.5 for flagged Schur functions [7, 11, 18].

However, we have been unable to find proofs of the other results of Section 3 using an analog of Theorem 4.5. It would be interesting to do so. Also, it would be nice to generalize these identities to ones about $d \times d$ determinants.

4.5. Combinatorial proofs

We have already mentioned that Butler [4] and Leroux [12] give combinatorial proofs for the results in Section 2. These proofs rely on interpretations of the q-binomial coefficients and q-Stirling numbers in terms statistics on Ferrers diagrams. Sagan [14] used certain digraphs discovered by Wilf [20–21] to demonstrate log concavity of these quantities in the case $q = 1$. Gessel [7] and later Gessel and Viennot [8] found combinatorial methods involving lattice paths for proving determinantal identities like Jacobi–Trudi. In certain cases, all of these techniques are really the same. Such an example will be sketched next.

We begin with another way to attack Theorem 3.2. As in the proof of Theorem 3.5, with each subset $S \subseteq \{1, 2, \ldots, n\}$, we can associate a monomial or *weight*

$$\boldsymbol{x}_S = \prod_{t \in S} x_t.$$

Thus $e_k(n)$ is just the generating function for $B_{n,k}$, the set of all k-subsets of $\{1, 2, \ldots, n\}$. Thus to show that

$$e_{k-1}(n)e_{l+1}(n) \leq_q e_k(n)e_l(n)$$

it suffices to find a weight-preserving injection

$$\beta : B_{n,k-1} \times B_{n,l+1} \hookrightarrow B_{n,k} \times B_{n,l}.$$

Given the set S, let $S_i = S \cap \{1, 2, \ldots, i\}$ and $S_i^c = S \setminus S_i$ (set difference). Now, if $(S, T) \in B_{n,k-1} \times B_{n,l+1}$, then there must be a largest index j such that $|T_j| = |S_j| + 1$. Thus define

$$\beta(S, T) = (S_j \cup T_j^c, T_j \cup S_j^c).$$

It is easy to check that $\beta(S, T) \in B_{n,k} \times B_{n,l}$, and that β is injective and weight preserving.

The map is a special case of the Gessel/Viennot lattice path injection [7, 8]. (It is also implicit in Bhatt and Leiserson [1].) Butler stated it explicitly in [2] and Sagan rediscovered the map in [14].

We can also use β to give a combinatorial proof of the following well-known theorem [17].

Theorem 4.6. *Suppose that $f(q) \in \mathbb{N}[q]$ has only negative real roots, then $f(q)$ is internally log concave.*

Proof. The coefficients of $f(q)$ are elementary symmetric functions of the negatives of the roots which, by assumption, are positive numbers. Hence this result is equivalent to the $\boldsymbol{x}$-log concavity in k of the $e_k(n)$ which was proved combinatorially above. □

Slight modifications of the β-map can be used to prove all the results of Section 3 [16]. In addition, we believe these methods could also be applied to show that the elementary symmetric functions are 'almost' strongly q-log concave in $n-k$.

Conjecture 4.7. Let $(f_n)_{n\geqslant 1}$, $f_n \in \mathbb{N}[q]$ be a sequence of polynomials. Suppose there exists a constant polynomial b with $bf_n \geqslant_q f_{n+1}$ for all $n \geqslant 1$. Then for all $n \geqslant 0$ and $k \leqslant l$ we have

$$e_{k-1}(f_1, \ldots, f_{n+k-1})e_{l+1}(f_1, \ldots, f_{n+l+1}) \leqslant_q b^{l-k+1}e_k(f_1, \ldots, f_{n+k})e_l(f_1, \ldots, f_{n+l}).$$

De Medecis has given a combinatorial proof of a special case of the above conjecture due to Leroux. In particular, when $f_n = n$ and $b = 2$.

Proposition 4.8. *If $n \geqslant 1$ then $c[n-1, k]c[n+1, k] \leqslant 2c[n, k]^2$.*

We can give a direct combinatorial proof of the q-log concavity in k for the $c[n, k]$ without passing through the elementary symmetric functions. Consider the set $C_{n,k}$ consisting of all permutations π of $\{1, 2, \ldots, n\}$ whose disjoint cycle decomposition contains exactly k cycles $P_1, P_2, \ldots, P_k$. We will agree to write all our permutations in *standard form*, $\pi = P_1P_2 \cdots P_k$ where

- every P_i will be written wth its smallest element first, and
- $\min P_1 < \min P_2 < \cdots < \min P_k$.

An *inversion* in a permutation π is a pair (r, s), $r, s \in \pi$, such that r appears to the left of s and $r > s$. For example,

$$\pi = (1, 5, 3)(2, 7)(4)$$

has 5 inversions: $(5, 3)$, $(5, 2)$, $(5, 4)$, $(3, 2)$ and $(7, 4)$. If we let $\operatorname{inv}\pi$ denote the number of inversions in π then it is not hard to prove that

$$c[n \cdot k] = \sum_{\pi \in C_{n,k}} q^{\operatorname{inv}\pi}.$$

(Merely show that the right-hand side satisfies equation (2).)

Now to prove strong q-log concavity in k for the $c[n, k]$ we need an injection

$$\gamma : C_{n,k-1} \times C_{n,l+1} \hookrightarrow C_{n,k} \times C_{n,l},$$

which preserves the inv weighting. Such a bijection is given in [14]. Roughly, it uses the β injection restricted to each cycle in a way that preserves the relative ordering of elements. Details will be found in the paper just cited.

4.6. Generalization of a $q = 1$ result

In [14] we derived all the results in Section 2 for $q = 1$ as immediate corollaries to a single theorem. The analog of that result is true for arbitrary q. It states the following.

Proposition 4.9. *Let $(f_{n,k}(q))_{n\in\mathbb{N},k\in\mathbb{Z}}$ be an array of polynomials in $\mathbb{N}[q]$ satisfying the boundary condition*

$$f_{0,k}(q) = 0 \quad \textit{for } k \neq 0$$

and recursion

$$f_{n,k}(q) = c_{n,k}(q)f_{n-1,k-1}(q) + d_{n,k}(q)f_{n-1,k}(q)$$

for $n \geqslant 1$ and $c_{n,k}(q)$, $d_{n,k}(q) \in \mathbb{N}[q]$. Now suppose that:

(1) *$c_{n,k}(q)$ and $d_{n,k}(q)$ are q-log concave in k, and*

(2) *$c_{n,k-1}(q)d_{n,k+1}(q) + c_{n,k+1}(q)d_{n,k-1}(q) \leqslant_q 2c_{n,k}(q)$ for all $n \geqslant 1$,*

then $f_{n,k}(q)$ is q-log concave in k.

Unfortunately, this proposition can only be applied to the $c[n, k]$ since they are the only polynomials whose recursion satisfies items (1) and (2). Is there a strengthening of Proposition 4.9 that would give the q-log concavity in k of the q-binomial coefficients and q-Stirling numbers of the second kind as well?

Acknowledgment

The research for the paper was done while the author was visiting the IMA. I would like to thank the Institute and its staff for their gracious support during that period. I would also like to thank my colleagues for stimulating discussions and corrections of the manuscript, particularly Lynne Butler, Pierre Leroux, Dennis Stanton and Dennis White.

References

[1] S.N. Bhatt and C.E. Leiserson, How to assemble tree machines, in: Proc. 14th Symp. on Theory of Computing, San Francisco, May 5–7 (1982) 77–84.

[2] L.M. Butler, Combinatorial properties of partially ordered sets associated with partitions and finite abelian groups, Ph.D. Thesis, M.I.T., 1986.

[3] L.M. Butler, A unimodality result in the enumeration of subgroups of a finite abelian group, Proc. Amer. Math. Soc. 101 (1987) 771–775.

[4] L.M. Butler, The q-log concavity of q-binomial coefficients, preprint.

[5] L. Carlitz, On Abelian fields, Trans. Amer. Math. Soc. 35 (1933) 122–136.
[6] L. Carlitz, q-Bernoulli numbers and polynomials, Duke Math. J. 15 (1948) 987–1000.
[7] I. Gessel, Determinants and plane partitions, unpublished manuscript.
[8] I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, Adv. in Math 58 (1985) 300–321.
[9] H.W. Gould, The q-Stirling numbers of the first and second kinds, Duke Math. J. 28 (1961) 281–289.
[10] C. Krattenhaler, On the q-log-concavity of Gaussian binomial coefficients, preprint.
[11] A. Lascoux and M.P. Schützenberger, Géométrie algébrique-Polynômes de Schubert, C.R. Acad. Sci. Paris 294 (1982) 447–450.
[12] P. Leroux, Reduced matrices and q-log concavity properties of q-Stirling numbers, preprint.
[13] I.G. Macdonald, Symmetric functions and Hall polynomials (Oxford Univ. Press, Oxford, 1979).
[14] B.E. Sagan, Inductive and injective proofs of log concavity results, Discrete Math. 68 (1988) 281–292.
[15] B.E. Sagan, The ubiquitous Young tableau, preprint.
[16] B.E. Sagan, Log concave sequences of symmetric functions and analogs of the Jacobi–Trudi determinant, in preparation.
[17] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, preprint.
[18] M.L. Wachs, Flagged Schur functions, Schubert polynomials, and symmetrizing operators, J. Combin. Theory Ser. A 40 (1985) 276–289.
[19] M.L. Wachs and D.E. White, p,q-Stirling numbers and set partition statistics, preprint.
[20] H.S. Wilf, A unified setting for sequencing, ranking and selection algorithms for combinatorial objects, Adv. in Math. 24 (1977) 281–291.
[21] H.S. Wilf, A unified setting for selection algorithms (II), Ann. Discrete Math., Vol. 2 (North-Holland, Amsterdam, 1978) 135–148.