DIHEDRAL TRANSPORTATION AND $(0,1)$-MATRIX CLASSES

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Abstract. Let R and S be two vectors of real numbers whose entries have the same sum. In the transportation problems one wishes to find a matrix A with row sum vector R and column sum vector S. If, in addition, the two vectors only contain nonnegative integers then one wants the same to be true for A. This can always be done and the transportation algorithm gives a method for explicitly calculating A. We can restrict things even further and insist that A have only entries zero and one. In this case, the Gale-Ryser Theorem gives necessary and sufficient conditions for A to exist and this result can be proved constructively. One can let the dihedral group D_4 of the square act on matrices. Then a subgroup of D_4 defines a set of matrices invariant under the subgroup. So one can consider analogues of the transportation and $(0,1)$ problems for these sets of matrices. For every subgroup, we give conditions equivalent to the existence of the desired type of matrix.

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1. Introduction

Let D_4 be the dihedral group of the square. Write ρ_θ for rotation counter-clockwise through θ radians and r_m for reflection in a line of slope m. Then

$$D_4 = \{\rho_0, \rho_{\pi/2}, \rho_\pi, \rho_{3\pi/2}, r_0, r_{+1}, r_{-1}, r_\infty\}.$$

The non-identity elements of D_4 are uniquely identified by their subscripts, and we let $D_b \leq D_4$ be the cyclic subgroup generated by the element with subscript b. There are also two subgroups of D_4 isomorphic to the Klein 4-group, namely

$$D_\times = \{\rho_0, \rho_\pi, r_{+1}, r_{-1}\}$$

and

$$D_+ = \{\rho_0, \rho_\pi, r_0, r_\infty\}.$$

The subscripts of $D_\times$ and D_+ are mnemonic, geometrically representing the two reflection lines in each subgroup. A complete list of non-identity subgroups of D_4 is

$$D_{\pi/2} = D_{3\pi/2}, D_\pi, D_0, D_{+1}, D_{-1}, D_\infty, D_\times, D_+, D_4.$$

For each of these subgroups D_b (now including $D_\times$, D_+, and D_4) acting on $m \times n$ matrices (where it is implicitly assumed that $m = n$ if one of $\rho_{\pi/2}$, r_{+1} or r_{-1} is in D_b), we consider the transportation (both real and integral) and $(0,1)$-problems for those matrices invariant under D_b. We call the resulting classes of matrices *dihedral matrix classes*. The cases D_π and $D_\times$ were considered in a paper of Brualdi and Ma [BM]. The invariant matrices for D_π are the so-called *centrosymmetric* matrices. Since D_π is a subgroup of $D_\times$, the invariant matrices for $D_\times$ are also centrosymmetric. As pointed out in [BM], there are centrosymmetric matrices that are not invariant under $D_\times$. For example, the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is centrosymmetric but is not invariant under either of the two reflections r_{+1} and r_{-1}.

Given a real matrix A we let $R = R(A)$ and $S = S(A)$ be the row sum and column sum vectors of A with components $r_i = r_i(A)$ and $s_j = s_j(A)$, respectively. We let $\mathcal{T}(R,S)$ denote the corresponding *transportation class* which consists of all nonnegative real matrices with row sum vector R and column sum vector S. We also use the notation

$$\mathcal{T}^b(R,S) = \{A \in \mathcal{T}(R,S) \mid D_b A = A\}$$

and

$$\mathcal{T}^b_{\mathbb{Z}}(R,S) = \{A \in \mathcal{T}^b(R,S) \mid A \in \mathbb{Z}^{m\times n}\}.$$

For the $(0,1)$-problem, $\mathcal{A}(R,S)$ and $\mathcal{A}^b(R,S)$ denote the subsets of $\mathcal{T}(R,S)$ and $\mathcal{T}^b(R,S)$, respectively, whose entries are 0 and 1. In all cases we assume, without specific mention, the obvious necessary condition for our classes to be nonempty, namely that $\Sigma R = \Sigma S$ where, for any matrix X, ΣX is the sum of the entries of X. We assume, also without specific mention, that in discussing $\mathcal{T}^b_{\mathbb{Z}}(R,S)$ and $\mathcal{A}^b(R,S)$, the vectors R, S have nonnegative integral components. Finally, for $\mathcal{A}^b(R,S)$, we always assume that R and S have no component bigger than n and m, respectively.

Recall that we can obtain an element $T \in \mathcal{T}(R, S)$ by letting

$$t_{i,j} = \frac{r_i s_j}{N} \tag{1}$$

where $N = \Sigma R = \Sigma S$.

If we wish to construct a matrix $T \in \mathcal{T}_{\mathbb{Z}}(R, S)$, then we can use the transportation algorithm. Pick any r_i and s_j. If $r_i \leq s_j$ then let $t_{i,j} = r_i$, remove the ith row of T and the corresponding component of R, and replace S by the vector obtained by decreasing its jth component by s_j. If $s_j \leq r_i$ then we apply the same construction with the roles of the rows and columns reversed. (If $r_i = s_j$ it does not matter which of the two possibilities we use.) We then iterate the process until all row and column sums are as they should be.

For $\mathcal{A}(R, S)$ one must be more careful. Given a nonnegative integral vector R, we let $R^{\downarrow}$ denote the weakly decreasing rearrangement of R, and we let R^* denote the conjugate of $R^{\downarrow}$ viewed as an integer partition. Note that R^* is weakly decreasing by definition. Given two weakly decreasing vectors $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$, we say R *majorizes* S and write $R \succeq S$, if for all indices ℓ

$$r_1 + r_2 + \cdots + r_\ell \geq s_1 + s_2 + \cdots + s_\ell \tag{2}$$

and $\Sigma R = \Sigma S$. We also write $S \preceq R$ and say that S is *majorized* by R. If R, S are not necessarily weakly decreasing, then we define $R \succeq S$ (or $S \preceq R$) to mean $R^{\downarrow} \succeq S^{\downarrow}$. The Gale-Ryser theorem (see e.g. [Bru06]) asserts that $\mathcal{A}(R, S) \neq \emptyset$ if and only

$$S \preceq R^* \quad \text{(the \textit{Gale-Ryser condition})}. \tag{3}$$

If (3) holds, then we can construct an element $A \in \mathcal{A}(R, S)$ using the Gale-Ryser algorithm as follows.

(1) Pick any j and set the entries in column j with the largest s_j row sums equal to one and the rest of the entries equal to zero, breaking ties arbitarily,
(2) Replace R by the vector obtained by decreasing its largest s_j entries by one (using tie breaking as determined in (1)). Replace S by the vector obtained by removing s_j and return to the first step until both vectors are zeroed out.

It will be helpful to have the following notation. For a nonegative integer n, let

$$\check{n} = \lfloor n/2 \rfloor \quad \text{and} \quad \hat{n} = \lceil n/2 \rceil.$$

Also, if A is a matrix, then R_i and S_j will always denote the ith row and jth column of A, respectively.

Our goal in this paper is to determine under what conditions the various dihedral matrix classes, as determined by the subgroups of D_4, are nonempty.

2. The rotation ρ_π

As mentioned in the introduction, these centrosymmetric matrices were considered in [BM]. So here we content ourselves with stating their results. In order to state them more clearly, we assume some obvious necessary conditions. Clearly a matrix invariant under ρ_π must have palindromic row and column sum vectors. We say that a palindromic vector $R = (r_1, r_2, \ldots, r_n)$ is *initially nonincreasing* provided that $r_1 \geq r_2 \geq \cdots \geq r_{\check{n}}$. By permuting within upper rows and within lower rows, and similarly for the columns, a centrosymmetric matrix can always be assumed to have initially-nonincreasing row and column sum vectors.

Theorem 1. *We have $\mathcal{T}^\pi(R, S) \neq \emptyset$ if and only if R and S are palindromic. The same is true for $\mathcal{T}_\mathbb{Z}^\pi(R, S)$.* □

Theorem 2. (i) *Let m and n be even. Then $\mathcal{A}^\pi(R, S) \neq \emptyset$ if and only if R and S are palindromic and $S \preceq R^*$.*

(ii) *Let m be odd and n be even, the case where m is even and n odd being similar. Assume that R and S are initially nonincreasing, palindromic vectors with $r_{\hat{m}}$ even. Let vectors R' and S' be obtained, respectively, by deleting $r_{\hat{m}}$ from R and by decreasing by one the first and last $r_{\hat{m}}/2$ entries of S. Then $\mathcal{A}^\pi(R, S) \neq \emptyset$ if and only if $\mathcal{A}^\pi(R', S') \neq \emptyset$.*

(iii) *Let m and n both be odd, and assume that R and S are initially nonincreasing, palindromic vectors with $r_{\hat{m}}$ and $s_{\hat{n}}$ of the same parity. Let vectors R' and S' be obtained, respectively, by deleting $r_{\hat{m}}$ and by decreasing by 1 the first and last $\lfloor s_{\hat{n}}/2 \rfloor$ entries of R, and by deleting $s_{\hat{n}}$, and by decreasing by 1 the first and last $\lfloor r_{\hat{m}}/2 \rfloor$ entries of S. Then $\mathcal{A}^\pi(R, S) \neq \emptyset$ if and only if $\mathcal{A}^\pi(R', S') \neq \emptyset$.* □

3. The reflections r_{-1} and r_{+1}

In this section we will consider the subgroups D_{-1}, D_{+1}, and $D_\times$ generated by the reflections r_{-1} and/or r_{+1}.

Theorem 3. *We have $\mathcal{T}^{-1}(R, S) \neq \emptyset$ if and only if $R = S$ The same is true for $\mathcal{T}_{\mathbb{Z}}^{-1}(R, S)$.*

Proof. The proofs for the arbitrary and integral cases are the same. To see the forward implication, it suffices to observe that r_{-1}, which is ordinary matrix transposition, interchanges the row and column sum vectors of a matrix. For the reverse, merely note that if $R = S$ then the diagonal matrix $\text{diag}(r_1, \dots, r_n)$ provides a desired matrix. □

Given a vector $S = (s_1, s_2, \dots, s_n)$, we denote its reversal by

$$S^r = (s_n, \dots, s_2, s_1).$$

The next result follows from Theorem 3 and the fact that if $r_{+1}A = A$ if and only if A can be obtained by rotation through $\pi/2$ radians of a matrix A' with $r_{-1}A' = A'$ (i.e. transposition with respect to the antidiagonal).

Theorem 4. *We have $\mathcal{T}^{+1}(R, S) \neq \emptyset$ if and only if $S = R^r$ The same is true for $\mathcal{T}_{\mathbb{Z}}^{+1}(R, S)$.* □

Now we consider what happens for the subgroup $D_\times = \{\rho_0, \rho_\pi, r_{+1}, r_{-1}\}$.

Theorem 5. *We have $\mathcal{T}^\times(R, S) \neq \emptyset$ if and only if*

(a) *$R = S$, and*
(b) *R is palindromic.*

The same is true for $\mathcal{T}_{\mathbb{Z}}^\times(R, S)$.

Proof. We will do both the arbitrary and integral cases at the same time. The forward direction follows immediately from Theorems 3 and 4. On the other hand, if we are given (a) and (b) then it is easy to verify that

$$A = \text{diag}(r_1/2, \dots, r_n/2) + \text{antidiag}(r_1/2, \dots, r_n/2) \tag{4}$$

is an element in $\mathcal{T}^\times(R, S)$. And for $\mathcal{T}_{\mathbb{Z}}^\times(R, S)$ one merely rounds up the elements in the diagonal matrix and rounds down those in the antidiagonal matrix. □

We now deal with the case of $(0, 1)$-matrices. For r_{-1} this follows from a result of Fulkerson, Hoffman, and McAndrew [FHM65] . See [Bru06, pp. 179–182] for details.

Theorem 6. *We have $\mathcal{A}^{-1}(R,S) \neq \emptyset$ if and only if $R = S$ and $R \preceq R^*$.* □

Note that Theorem 6 is equivalent to the fact that, for $R = S$, there is a symmetric matrix in $\mathcal{A}(R,R)$ if and only if $\mathcal{A}(R,R) \neq \emptyset$.

The following result follows from the previous one in the same way that Theorem 4 follows from Theorem 3.

Theorem 7. *We have $\mathcal{A}^{+1}(R,S) \neq \emptyset$ if and only if $S = R^r$ and $R^r \preceq R^*$.* □

The nonemptiness of $\mathcal{A}^{\times}(R,R)$ was characterized in [BM] as follows.

Theorem 8. *We have $\mathcal{A}^{\times}(R,R) \neq \emptyset$ if and only if $\mathcal{A}^{\pi}(R,R) \neq \emptyset$.* □

Recall that the characterization for $\mathcal{A}^{\pi}(R,S)$, and thus for $\mathcal{A}^{\times}(R,R)$ is given in Theorem 2.

4. The reflections r_∞ and r_0

In this section we will consider the subgroups generated by the reflections r_∞ and/or r_0. First, however, we introduce some useful notation. Call an integral matrix A *even* if all its entries are even. Also let $o(A)$ be the number of odd entries of A. Given an integral vector R and an odd positive integer n, we define A^R to be the $m \times n$ (0,1)-matrix whose only nonzero entries are $a^R_{i,\hat{n}}$ for the indices i such that r_i is odd. Given an integral vector S and odd positive integer m, we define A^S in a similar way. Finally given R, S and both m and n are odd we define A^+ by

$$a^+_{i,j} = \max\{a^R_{i,j}, a^S_{i,j}\}. \tag{5}$$

In other words, $A^+ = A^R + A^S$ except in the case when the central elements of both R and S are odd in which case the central entry of the sum is too large by one.

Theorem 9. (I) *We have $\mathcal{T}^{\infty}(R,S) \neq \emptyset$ if and only if*

(a) *S is palindromic.*

(II) *We have $\mathcal{T}^{\infty}_{\mathbb{Z}}(R,S) \neq \emptyset$ if and only if* (a) *is true and*

(b) *if n is even then R is even, and if n is odd then $s_{\hat{n}} \geq o(R)$.*

Proof. (I) For the forward implication, take A such that $r_\infty A = A$. Since r_∞ exchanges columns equidistant from the vertical mid-line of A, we must have that S is a palindromic. For the other direction, it suffices to show that equation (1) defines a matrix with palindromic S-vector. Indeed, using the fact that S is palindromic,

$$t_{i,m-j+1} = \frac{r_i s_{m-j+1}}{N} = \frac{r_i s_j}{N} = t_{i,j}.$$

(II) First we note that if $r_\infty A = A$ then $a_{i,j} = a_{i,n-j+1}$ for all i, j. Thus when n is even every element in the ith row is repeated twice and R is even. On the other hand, if n is odd then r_i is odd if and only if $a_{i,\hat{n}}$ is odd. This gives the inequality in (b).

For the reverse implication, we modify the transportation matrix algorithm as follows. Let $\overline{R} = R - R(A^R)$ and $\overline{S} = S - S(A^R)$. Note that $\overline{R}$ is even by definition of A^R and $\overline{S}$ still has nonnegative entries because of (b). Construct $\overline{A} \in \mathcal{T}_{\mathbb{Z}}^\infty(\overline{R}, \overline{S})$ by letting $\overline{a}_{1,1} = \overline{a}_{1,n} = \min\{\overline{r}_1/2, \overline{s}_1\}$ and applying recursion. Now form $A \in \mathcal{T}_{\mathbb{Z}}^\infty(R, S)$ by adding one to the $\overline{a}_{i,\hat{n}}$ for all i such that r_i is odd. $\square$

The next result follows from Theorem 9 in the same way that Theorem 4 follows from Theorem 3.

Theorem 10. (I) *We have $\mathcal{T}^0(R, S) \neq \emptyset$ if and only if*
  (a) *R is palindromic.*
(II) *We have $\mathcal{T}_{\mathbb{Z}}^0(R, S) \neq \emptyset$ if and only if* (a) *is true and*
  (b) *if m is even then S is even, and if m is odd then $r_{\hat{m}} \geq o(S)$.* $\square$

We now consider the subgroup $D^+ = \{\rho_0, \rho_\pi, r_0, r_\infty\}$.

Theorem 11. *We have $\mathcal{T}^+(R, S) \neq \emptyset$ if and only if $\mathcal{T}^\infty(R, S) \neq \emptyset$ and $\mathcal{T}^0(R, S) \neq \emptyset$. The same is true in the integral case.*

Proof. The forward directions follow immediately from the fact that $\mathcal{T}^+(R, S) = \mathcal{T}^\infty(R, S) \cap \mathcal{T}^0(R, S)$. The converse for $\mathcal{T}^+(R, S)$ is proved in the usual way using (1). For $\mathcal{T}_{\mathbb{Z}}^+(R, S)$, we use a method similar to the one given in the proof of Theorem 9. We consider the vectors $\overline{R} = R - R(A^+)$ and $\overline{S} = S - S(A^+)$. We then construct a matrix $\overline{A}$ by making assignments $\overline{a}_{1,1} = \overline{a}_{1,n} = \overline{a}_{m,1} = \overline{a}_{m,n} = \min\{\overline{r}_1/2, \overline{s}_1/2\}$ and recursing. Finally, we let $A = \overline{A} + A^+$. $\square$

Theorem 12. *We have $\mathcal{A}^\infty(R, S) \neq \emptyset$ if and only if conditions* (a) *and* (b) *from Theorem* 9 *are satisfied as well as*

(c) $\overline{S} \preceq \overline{R}^*$ *where* $\overline{R}$ *is obtained from* R *by subtracting one from every odd component and* $\overline{S}$ *is* S *if* n *is even or* S *with column* $S_{\hat{n}}$ *removed if* n *is odd.*

Proof. Clearly if $A \in \mathcal{A}^\infty(R, S)$ then it must satisfy the two conditions from Theorem 9. If n is even then $\overline{R} = R$ and $\overline{S} = S$ so that $\overline{R}^* \succeq \overline{S}$ by the Gale-Ryser Theorem. In n is odd, note that the ones in column $S_{\hat{n}}$ must occur exactly in the rows with odd sums. Removing this column, we obtain a matrix $\overline{A}$ with $\overline{R}$ and $\overline{S}$ as its row and column vector. Since such a matrix exists, we must have $\overline{R}^* \succeq \overline{S}$ by the Gale-Ryser Theorem again.

For the converse we have two cases. First suppose that n is even. Then since R is even we must have every element of R^* repeated twice. Let R_1^* be R^* where we only take one out of every pair of repeated elements. Similarly, let $S_1 = (s_1, \ldots, s_{\check{n}})$. Since $\overline{R} = R$ and $\overline{S} = S$, (c) implies that $S \preceq R^*$. It follows that $S_1 \preceq R_1^*$. Now use the Gale-Ryser algorithm to create a matrix $B \in \mathcal{A}(R_1, S_1)$. It follows that we have a block matrix $A = [B \ r_\infty B] \in \mathcal{A}^\infty(R, S)$.

Now consider the case when n is odd. Since $n - 1$ is even, $\overline{R}$ is an even vector, $\overline{S}$ is palindromic, and $\overline{S} \preceq \overline{R}^*$ we can proceed as in the previous case to construct a matrix $\overline{A} \in \mathcal{A}^\infty(\overline{R}, \overline{S})$. Finally, we get the desired matrix A by inserting a middle column $S_{\hat{n}}$ in $\overline{A}$ which has ones in exactly the rows of R with odd sum. □

One might ask if (d) could be replaced by the ordinary Gale-Ryser condition $S \preceq R^*$. But this condition is not strong enough to imply $\mathcal{A}^\infty(R, S) \neq \emptyset$. For an example of this, consider $R = (6, 6, 6, 2, 1, 1)$ and $S = (4, 4, 2, 2, 2, 4, 4)$. Clearly S is palindromic and it is easy to check that $S \preceq R^*$. Now suppose, towards a contradiction, that there exists $A \in \mathcal{A}^\infty(R, S)$. Form the matrix $\overline{A}$ as in the first paragraph of the preceding proof. Then $\overline{A}$ has row and column vectors $\overline{R} = (6, 6, 6, 2)$ and $\overline{S} = (4, 4, 2, 2, 4, 4)$. But $\overline{R}^*$ does not majorize $\overline{S}$ which contradicts the Gale-Ryser Theorem.

As with previous cases, the result for symmetry under r_∞ is similar to the one for r_0.

Theorem 13. *We have* $\mathcal{A}^0 \neq \emptyset$ *if and only if conditions* (a) *and* (b) *from Theorem* 10 *are satisfied as well as*

(c) $\overline{S} \preceq \overline{R}^*$ *where* $\overline{S}$ *is obtained from* S *by subtracting one from every odd component and* $\overline{R}$ *is* R *if* m *is even or* R *with column* $R_{\hat{n}}$ *removed if* n *is odd.* □

Finally, we consider the $(0,1)$-case for D_+.

Theorem 14. *We have* $\mathcal{A}^+(R,S) \neq \emptyset$ *if and only if conditions* (a) *and* (b) *from both Theorems* 9 *and* 10 *are satisfied as well as*

(c) *if* n *is odd then* $o(R) = s_{\hat{n}}$*, if* m *is odd then* $o(S) = r_{\hat{m}}$*, and*
(d) $\check{S} \preceq \check{R}^*$ *where* $\check{R} = (\check{r}_1, \check{r}_2, \ldots, \check{r}_{\check{m}})$ *and* $\check{S} = (\check{s}_1, \check{s}_2, \ldots, \check{s}_{\check{n}})$.

Proof. Suppose first that $A \in \mathcal{A}^+(R,S)$. Then clearly conditions (a) and (b) from both Theorems 9 and 10 are satisfied. To obtain (c) of the present result, note that condition (c) of Theorem 12 must also hold. So, in particular, $\Sigma\overline{R}^* = \Sigma\overline{S}$ and this gives the desired equality when n is odd. The case when m is odd follows similarly from Theorem 13. Finally, $\check{R}$ and $\check{S}$ are the row- and column-sum vectors for the submatrix $\check{A}$ of A sitting in the first $\check{m}$ rows and the first $\check{n}$ columns. Thus $\check{R}^* \succeq \check{S}$ follows from the Gale-Ryser Theorem.

For the converse, assume first that m and n are odd. By condition (d) and the Gale-Ryser Theorem, we can construct an $\check{n} \times \check{m}$ matrix $\check{A}$ with row sum vector $\check{R}$ and column sum vector $\check{S}$. Now the current condition (c) and condition (a) from Theorems 9 and 10 imply that there is an $\check{m} \times 1$ matrix B, a $1 \times \check{n}$ matrix C, and $a_{\hat{m},\hat{n}} \in \{0,1\}$ such that the block matrix

$$A = \begin{bmatrix} \check{A} & B & r_\infty \check{A} \\ C & a_{\hat{m},\hat{n}} & r_\infty C \\ r_0 \check{A} & r_0 B & \rho_\pi \check{A} \end{bmatrix}$$

is in $\mathcal{A}^+(R,S)$. If either m or n is even then condition (b) from Theorems 9 and 10 implies that deleting the appropriate row or column in A above will give a matrix with the correct row and column sums to be in $\mathcal{A}^+(R,S)$. □

5. The case $D_{\pi/2}$

We start, as usual, with the transportation problem.

Theorem 15. (I) *We have* $\mathcal{T}^{\pi/2}(R,S) \neq \emptyset$ *if and only if*

(a) $R = S$*, and*

(b) *R is palindromic.*

(II) *We have $\mathcal{T}_{\mathbb{Z}}^{\pi/2}(R,S) \neq \emptyset$ if and only if R, S satisfy* (a) *and* (b) *as well as one of*

(c) *$r_1 + r_2 + \cdots + r_{\check{n}}$ is even, or*

(d) *n is odd and $r_{\hat{n}} \geq 2$.*

Proof. (I) For the forward direction, suppose $A \in \mathcal{T}^{\pi/2}(R,S)$. Then $\rho_{\pi/2} R_i = C_i$ which implies $R = S$. And $\rho_{\pi/2}^2 R_i = \rho_\pi R_i$ is R_{n-i} read backwards so that (b) holds.

For the converse, it suffices to show that when (a) and (b) hold then the matrix defined by (1) is invariant under $\rho_{\pi/2}$. But this follows since

$$t_{n-j+1,i} = \frac{r_{n-j+1} s_i}{N} = \frac{s_{n-j+1} r_i}{N} = \frac{r_i s_j}{N} = t_{i,j}.$$

(II) We will first consider the case when n is even. Given $A \in \mathcal{T}_Z^{\pi/2}(R,S)$, we can write A in the block form

$$A = \begin{bmatrix} B & \rho_{\pi/2}^3 B \\ \rho_{\pi/2} B & \rho_{\pi/2}^2 B \end{bmatrix} \tag{6}$$

where B is $\check{n} \times \check{n}$. Since R is palindromic by (a), it follows that

$$r_1 + r_2 + \cdots + r_{\check{n}} = \Sigma B + \Sigma(\rho_{\pi/2}^3 B) = 2\Sigma B$$

so that (c) holds.

Now suppose, for n still even, that we are given (a)–(c). For any matrix B, the matrix $A = A(B)$ defined by (6) is invariant under $\rho_{\pi/2}$. Thus it suffices to show that we can define B so that A has the given row and column sums. We will define $B = D + P$ where D is a diagonal matrix and P is a $(0,1)$-matrix with at most one 1 in every row and column. Define D by $d_{i,i} = \check{r}_i$ for $1 \leq i \leq \check{n}$. It follows that $A(D)$ has rows sums $2\check{r}_i = r_i$ if r_i is even or $r_i - 1$ if r_i is odd. We use the matrix P to correct for the odd row sums as follows. Because of (c), there are an even number of r_i which are odd, $1 \leq i \leq \check{n}$. Let those r_i be $r_{i_1}, r_{i_2}, \ldots, r_{i_{2k}}$. Let P be the $(0,1)$-matrix with 1's in positions $(i_1, i_2), \ldots, (i_{2k-1}, i_{2k})$. Now $A = A(B)$ will have one added to row i_{2j-1} by B and to row i_{2j} by $\rho_{\pi/2}^3 B$ for $1 \leq j \leq k$ and similarly for the rows below the midpoint. It follows that A has the correct row sums and we are done with the case n even.

We now deal with n odd. If $A \in \mathcal{T}_{\mathbb{Z}}^{\pi/2}(R,S)$ then, similarly to the n even case, we write

$$A = \begin{bmatrix} B & C & \rho_{\pi/2}^3 B \\ \rho_{\pi/2} C & a_{\hat{n},\hat{n}} & \rho_{\pi/2}^3 C \\ \rho_{\pi/2} B & \rho_{\pi/2}^2 C & \rho_{\pi/2}^2 B \end{bmatrix} \tag{7}$$

where B is $\check{n} \times \check{n}$ and C is $\check{n} \times 1$. If (c) holds, then we are done. If not, then consider

$$r_1 + r_2 + \cdots + r_{\check{n}} = 2\Sigma B + \Sigma C.$$

By our assumption about the left-hand side we must have ΣC odd and so, in particular, $\Sigma C \geq 1$. But then

$$r_{\hat{n}} = 2\Sigma C + a_{\hat{n},\hat{n}} \geq 2$$

and so (d) holds.

Finally, we must prove the converse when n is odd. If (c) holds, then we can construct the matrix B as when n is even, take C to be a zero matrix, and set $a_{\hat{n},\hat{n}} = r_{\hat{n}}$ to obtain a matrix with the desired row and column sums. If, instead, (d) holds then there are an odd number of r_i which are odd, $1 \leq i \leq \check{n}$. Let those r_i be $r_{i_1}, r_{i_2}, \ldots, r_{i_{2k+1}}$. Construct that matrix B as for n even using $r_{i_1}, r_{i_2}, \ldots, r_{i_{2k}}$. Let C be the matrix which is all zeros except for its i_{2k+1} entry which is one. And define $a_{\hat{n},\hat{n}} = r_{\hat{n}} - 2 \geq 0$ by the assumption in (d). It is now an easy matter to verify that we again have the desired sums in rows and columns. □

For the $(0,1)$-case we will need the following result of Brualdi and Ryser [Bru06, Theorem 6.3.2] about symmetric matrices whose entries are zeros, ones, and twos.

Theorem 16. *Let $R = (r_1, \ldots, r_n)$ be a vector of nonnegative integers. There exists a symmetric $(0,1,2)$-matrix M with row sum vector R if and only if*

$$2|I||J| \geq \sum_{i \in I} r_i - \sum_{j \notin J} r_j \tag{8}$$

for all $I, J \subseteq \{1, 2, \ldots, n\}$. □

We note that if in the previous theorem we have R weakly decreasing (and the row vector of any symmetric matrix can be brought to this form by row and column interchanges), then it suffices to check the considerably smaller set of inequalities

$$2kl \geq \sum_{i \leq k} r_i - \sum_{i > l} r_i$$

for all $1 \le k \le j \le n$.

Theorem 17. *We have $\mathcal{A}^{\pi/2}(R,S) \neq \emptyset$ if and only if conditions (a)–(d) of Theorem 15 hold and $\overline{R}$ satisfies the inequalities (8) where*

$$\overline{R} = \begin{cases} (r_1, r_2, \ldots, r_{\check{n}}) & \text{if } n \text{ is even,} \\ (r_1 - 1, r_2 - 1, \ldots, r_s - 1, r_{s+1}, r_{s+2}, \ldots, r_{\check{n}}) & \text{if } n \text{ is odd,} \end{cases}$$

and $s = \lfloor r_{\hat{n}}/2 \rfloor$.

Proof. We begin with the case when n is even. Suppose first that $A \in \mathcal{A}^{\pi/2}(R,S)$. We have already shown that conditions (a)–(c) must be satisfied. For the last condition, note that since $r_{\pi/2}A = A$ and n is even this matrix must have the form (6) for some $(0,1)$-matrix B. It follows that $M = B + B^t$ is a symmetric $(0,1,2)$-matrix. Furthermore, for $i \le n/2$ we have

$$r_i(M) = r_i(B) + r_i(B^t) = r_i(B) + c_i(B) = r_i(B) + r_i(\rho^3_{\pi/2}B) = r_i(A). \tag{9}$$

It follows from Theorem 16 that $\overline{R}$ must staisfy (8).

For the converse, using Theorem 16 again we may assume that there exists a symmetric $(0,1,2)$-matrix M with $R(M) = \overline{R}$. We claim that in fact there exists such an M with no ones on the diagonal. Indeed, using the symmetry of M we have

$$r_1 + \cdots + r_{n/2} = \Sigma M = 2\sum_{i<j} m_{i,j} + \sum_i m_{i,i}.$$

Since the left-hand side is even by condition (c), the same must be true of $\sum_i m_{i,i}$. And because the only odd entries of M are ones there must be an even number of them on M's diagonal, say the entries (i,i) for $i = i_1, i_2, \ldots, i_{2k}$. Consider the pair of ones on the diagonal in positions i_{2j-1} and i_{2j} for $1 \le j \le k$. Then there are three possibilities for the 2×2 submatrix of M in the rows and columns indexed by i_{2j-1} and i_{2j} depending on which of the three integers $0, 1, 2$ appear in the off-diagonal spots. In each case, substitute the submatix on the left in the following table with the corresponding submatrix on the right. It is easy to check that this does not change the row and column sums of M, and now M has only zeros and twos on the diagonal.

initial submatrix	substituted submatrix
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

We now write $M = B + B^t$ with the entries of the $(0,1)$-matrix B defined as in the following chart for $i \le j$. Note that from what we have just proved, if $m_{i,j} = m_{j,i} = 1$ then we must actually have $i < j$.

entries of M	entries of B
$m_{i,j} = m_{j,i} = 0$	$b_{i,j} = b_{j,i} = 0$
$m_{i,j} = m_{j,i} = 1$	$b_{i,j} = 0,\ b_{j,i} = 1$
$m_{i,j} = m_{j,i} = 2$	$b_{i,j} = b_{j,i} = 1$

Finally, we define A using the matrix B as in (6). This matrix is clearly symmetric under $r_{\pi/2}$ and has the correct row and column sum vectors by conditions (a) and (b) and the equalities in (9).

Now suppose that n is odd. By interchanging rows and columns, we can assume that R satisfies $r_1 \ge r_2 \ge \cdots \ge r_{\check{n}}$. Note that if there exists an $A \in \mathcal{A}^{\pi/2}(R,S)$ then it must have the form given in (7). First we claim that there is $A \in \mathcal{A}^{\pi/2}(R,S)$ if and only if there is such a matrix where all the ones in C precede all the zeros. To prove the forward direction (the converse being trivial), suppose that the given matrix A has a zero before a one in C. Without loss of generality we can assume the zero is in row i and the one in row $i+1$. But $r_i \ge r_{i+1}$ so that in some column of A we must have a zero followed by a one in these rows. Suppose that this column is in B as the case when it is in $\rho^3_{\pi/2}B$ is similar. So, taking account of symmetry, we have the situation depicted in (10) below:

$$A = \left[\begin{array}{ccc|c|cc}
 & & 1 & 0 & 0 & 1 \\
 & & 0 & 1 & & \\
 & & & & & \\
 & & & & & \\
\hline
0 & 1 & & & 1 & 0 \\
\hline
 & & & & & \\
1 & 0 & & & & \\
 & & & & & \\
 & & & 1 & & 0 \\
 & & & 0 & & 1 \\
 & & & & &
\end{array}\right]. \tag{10}$$

Now interchanging submatrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in four different places maintains both the symmetry and the row sum vector while exchanging $a_{i,\hat{n}} = 0$ and $a_{i+1,\hat{n}} = 1$. Continuing in this way we can put all the ones in C before all the zeros.

Note that by its definition, s is the number of ones in C. So existence of $A \in \mathcal{A}^{\pi/2}(R,S)$ is equivalent to having such an A with ones in the first s rows of C and zeros elsewhere in that submatrix. Removing the central row and column of A, we see that this is equivalent to having a matrix with an even number of rows and columns which has $\overline{R}$ as the first half of its palindromic row sum vector, where $\overline{R}$ is as given in the statement of the theorem for n odd. So the case for n odd reduces to the case when n is even and we are done. □

6. The group D_4 itself

We finally deal with the full dihedral group.

Theorem 18. (I) *We have* $\mathcal{T}^4(R,S) \neq \emptyset$ *if and only if*

(a) $R = S$, *and*

(b) *R is palindromic.*

(II) *We have $\mathcal{T}^4_{\mathbb{Z}}(R,S) \neq \emptyset$ if and only if (a) and (b) hold as well as*

(c) *if n is even then R is even, and if n is odd then $r_{\hat{n}} \geq o(R)$.*

Proof. (I) The forward direction follows from Theorem 5 and the fact that $D_\times \subseteq D_4$. For the reverse implication, it is easy to verify that if (a) and (b) are true then the matrix defined by (4) is invariant under D_4.

(II) Similar to (I), the forward implication comes from Theorems 5 and 11. For sufficiency, when n is even we use (4). When n is odd, we let $\check{A}$ be the matrix defined as in (4) but with all fractions rounded down. It follows that $A = \check{A} + A^+$ is the desired matrix, where the entries of A^+ are defined by (5). □

For our final result, we characterized the $(0,1)$-case.

Theorem 19. *We have $\mathcal{A}^4(R,S) \neq \emptyset$ if and only if conditions (a) and (b) of Theorem 18 hold as well as*

(c) *if n is even then R is even, if n is odd then $o(R) = r_{\hat{n}}$, and*

(d) *$\check{R} \preceq \check{R}^*$ where $\check{R} = (\check{r}_1, \check{r}_2, \dots, \check{r}_{\check{n}})$.*

Proof. Necessity follows from the previous result and Theorem 14. For the reverse implication, suppose first that n is even. By condition (d) and Theorem 6, there is an $\check{n} \times \check{n}$ matrix B with row and column sum vector $\check{R}$ which is symmetric under matrix transposition. It follows that the matrix A defined by 6 is invariant under D_4 and has the correct row and column sums by (c). When n is odd we construct B as in the even case, then a matrix $\check{A}$ as in 7 where C and $a_{\hat{n},\hat{n}}$ are all zero, and finally let $A = \check{A} + A^+$ with entries given by 5. Again, it is easy to see that A has the desired properties. □

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