PATTERN-AVOIDING POLYTOPES

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Abstract. The permutohedron and the Birkhoff polytope are two well-studied polytopes related to many areas of mathematics. In this paper, we generalize these polytopes by considering convex hulls of subsets of their vertices. The vertices chosen correspond to avoidance classes of permutations. We explore the combinatorial structure of certain special cases of these polytopes as well as their Ehrhart polynomials and Ehrhart series. Additionally, we find cases when the polytopes have palindromic and/or unimodal h^*-vectors. In particular, we explore connections between subpolytopes of the Birkhoff polytope, order complexes, standard Young tableaux, and (P, ω)-partitions. Multiple questions and conjectures are provided throughout.

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group on $1, 2, \ldots, n$ and $\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \cdots$. Let $\pi \in \mathfrak{S}_k$ and $\sigma \in \mathfrak{S}_n$. We say that σ *contains the pattern* π if there is some substring σ' of σ whose elements have the same relative order as those in π. Alternatively, we view σ' as *standardizing* to π by replacing the smallest element of σ' with 1, the next smallest by 2, and so on. If there is no such substring then we say that σ *avoids the pattern* π. If $\Pi \subseteq \mathfrak{S}$, then we say σ *avoids* Π if σ avoids every element of Π. We denote by

$$\mathrm{Av}_n(\Pi) := \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids } \Pi\}$$

the *avoidance class* of Π.

A polytope $P \subseteq \mathbb{R}^n$ is the convex hull of finitely many points, written $P = \mathrm{conv}\{v_1, \ldots, v_k\}$. Equivalently, a polytope may be described as an intersection of finitely many half-spaces. The *dimension* of P is the dimension of its affine span. A hyperplane $l(x)$ is called *supporting* if $l(p) \geq 0$ for every $p \in P$. If $l(x)$ is a supporting hyperplane, then the set $\{l(x) = 0\} \cap P$ is called a *face* of P and is a subpolytope of P. Faces of dimension 0 are *vertices*, faces of dimension 1 are called *edges*, and faces of dimension $\dim P - 1$ are called *facets*. Additionally, we say a polytope is *lattice* if each vertex is an element of $\mathbb{Z}^n$. Lattice polytopes have long found connections with permutations, in particular via the permutohedron and Birkhoff polytope.

The *permutohedron* is defined as

$$P_n := \mathrm{conv}\{(a_1, \ldots, a_n) \mid a_1 \cdots a_n \in \mathfrak{S}_n\}.$$

We will often make no distinction between a permutation and its corresponding point in $\mathbb{R}^n$. This polytope was first described in [25] and has connections to the geometry of flag varieties as well as representations of GL_n. We refer to [34] for general background regarding permutohedra.

The *Birkhoff polytope* is the polytope

$$B_n := \operatorname{conv}\left\{X = (x_{i,j}) \in \mathbb{R}^{n\times n} \mid \sum_{i=1}^{n} x_{i,j} = \sum_{j=1}^{n} x_{i,j} = 1 \text{ for all } i,j\right\}.$$

The Birkhoff-von Neumann Theorem states that the vertices of B_n are the permutation matrices.

In this paper, we generalize these classes of polytopes by taking convex hulls of vertices corresponding to avoidance classes of permutations. We explore the combinatorial structure for particular avoidance classes as well as their Ehrhart polynomials. Additionally, we find cases when the polytopes are compressed and conjecture formulae for their volumes and half-space descriptions.

1.1. **Ehrhart Polynomials.** For a lattice polytope $P \subseteq \mathbb{R}^n$, consider the counting function $\mathcal{L}_P(m) := |mP \cap \mathbb{Z}^n|$, where mP is the m-th dilate of P. Although not obvious, this function is a polynomial in m, called the *Ehrhart polynomial* of P. In particular, two well-known theorems due to Ehrhart [12] and Stanley [28] imply that the *Ehrhart series* of P,

$$E_P(t) := 1 + \sum_{m\geq 1} \mathcal{L}_P(m)t^m,$$

may be written in the form

$$E_P(t) = \frac{\sum_{j=0}^{d} h_j^* t^j}{(1-t)^{\dim P+1}}.$$

for some nonnegative integers $h_0^*, \ldots, h_d^*$ with $h_0^* = 1$, $h_d^* \neq 0$, and $d \leq \dim P$.

We say the polynomial $h_P^*(t) := \sum_{j=0}^{d} h_j^* t^j$ is the *h^*-polynomial* of P and the vector of coefficients, $h^*(P)$, is the *h^*-vector* of P. The h^*-vector of a lattice polytope P is a fascinating invariant, and obtaining a general understanding of h^*-vectors of lattice polytopes and their geometric/combinatorial implications is currently of great interest.

In this article, we describe a natural blending of pattern avoidance with two well-known polytopes: the permutohedron and the Birkhoff polytope. Section 2 focuses on the permutohedron case. Many interesting results reveal themselves and for a variety of avoidance classes we determine the volumes, normalized volumes, and Ehrhart polynomials for these polytopes. In Section 3, we begin analyzing a similar construction using the Birkhoff polytope with the goal of determining the behavior of its h^*-vector in special cases. This is difficult to do directly, and so we take a detour in Section 4 to study certain helpful triangulations. These triangulations connect with results in (P,ω)-partitions, which allows us to identify the behavior of the h^*-vector.

2. Permutohedra

The permutohedron has been generalized in multiple ways, including the permuto-associahedron of Kapranov [16], which was first realized as a polytope by Reiner and Ziegler [23], and the generalized permutohedra studied by Postnikov [21]. Here, we study yet another generalization of the permutohedron by looking at P_n from the perspective of pattern avoidance.

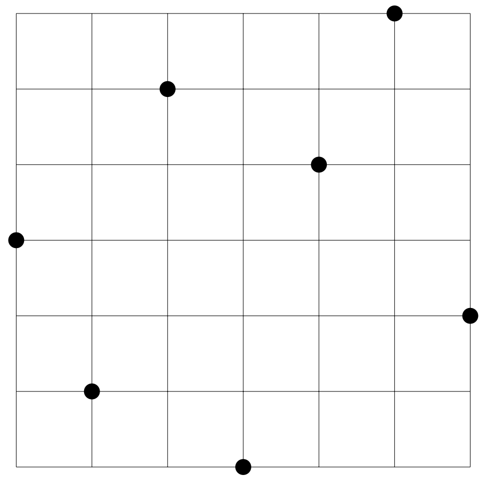


FIGURE 1. The diagram of the permutation 4261573.

Definition 2.1. Let $\Pi \subseteq \mathfrak{S}_n$ and define

$$P_n(\Pi) := \mathrm{conv}\{(a_1, \ldots, a_n) \mid a_1 \ldots a_n \in \mathrm{Av}_n(\Pi)\}$$

to be the Π-*avoiding permutohedron*. If $\Pi = \{\pi\}$ then we write $P_n(\pi)$ for $P_n(\Pi)$.

For example, if $\Pi = \emptyset$, then $P_n(\Pi) = P_n$. If $\pi \in \mathfrak{S}_3$ then it is well known that $|\mathrm{Av}_n(\pi)| = C_n$, the nth Catalan number, so that $P_n(\pi)$ has a Catalan number of vertices.

2.1. **Diagrams, Wilf Equivalence, and Grid Classes.** The *diagram* of a permutation $\pi = a_1 \cdots a_k$ is the set of points with Cartesian coordinates (i, a_i) for $i = 1, \ldots, k$. An example diagram is given in Figure 1. When no confusion will result, we make no distinction between a permutation and its diagram. Diagrams of permutations provide an easy way to see how certain permutations can be related geometrically. For example, the diagrams of π and π^{-1} are related by reflection across the line $y = x$. With both the Π-avoiding permutohedra and Π-avoiding Birkhoff polytopes (to be defined in the next section) many results will be true not only for the choice of Π in their statement, but also for certain other subsets of permutations whose diagrams are related to those in Π.

Two permutations π_1 and π_2 are called *Wilf equivalent*, written $\pi_1 \equiv \pi_2$, if $|\mathrm{Av}_n(\pi_1)| = |\mathrm{Av}_n(\pi_2)|$ for all n. For example, we have already noted that any two permutations in $\mathfrak{S}_3$ are Wilf equivalent. This is indeed an equivalence relation. Although proving $\pi_1 \equiv \pi_2$ may be quite difficult, in some instances, the Wilf equivalence of two permutations follows quickly from observing that their diagrams are related by a transformation in the dihedral group of the square.

Let $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, r_{-1}, r_0, r_1, r_\infty\}$, where R_θ is rotation counterclockwise by an angle of θ degrees and r_m is reflection across a line of slope m. A couple of these rigid motions have easy descriptions in terms of the one-line notation for permutations. If $\pi = a_1 a_2 \ldots a_k$ then its *reversal* is $\pi^r = a_k \ldots a_2 a_1 = r^\infty(\pi)$, and its *complement* is $\pi^c = (k + 1 - a_1)\ (k + 1 - a_2)\ \ldots\ (k + 1 - a_k) = r_0(\pi)$.

Note that $\sigma \in \mathrm{Av}_n(\pi)$ if and only if $f(\sigma) \in \mathrm{Av}_n(f(\pi))$ for any $f \in D_4$, hence $\pi \equiv f(\pi)$. For this reason, the equivalences induced by the dihedral action on a square are often referred to as the *trivial Wilf equivalences*.

Call polytopes P and Q *unimodularly equivalent* if one can be taken into the other by an affine transformation whose linear part has determinant ± 1. Certain trivial Wilf equivalences imply unimodular equivalence of the corresponding permutohedra.

Proposition 2.2. If $\Pi \in \mathfrak{S}$, then $P_n(f(\Pi))$ is unimodularly equivalent to $P_n(\Pi)$ for any $f \in \{R_0, R_{180}, r_0, r_\infty\}$. So their face lattices, volumes, and Ehrhart series are all equal.

Proof. For ease of notation, we prove this in the case that $\Pi = \{\pi\}$. The general demonstration is similar.

From the discussion above, $P_n(\pi^r)$ is the image of $P_n(\pi)$ under the map $f(v) = Av$, where $A = \begin{bmatrix} e_n & \cdots & e_1 \end{bmatrix}$ and the e_i are the standard unit column vectors. Since A is a permutation matrix, this is a unimodular transformation.

Also, $P_n(\pi^c)$ is the image of $P_n(\pi)$ under the map

$$g(x_1, \ldots, x_n) = (n+1-x_1, \ldots, n+1-x_n) = (n+1, \ldots, n+1) - (x_1, \ldots, x_n),$$

which is again clearly unimodular. Finally, notice that $R_{180}(\pi) = f \circ g(\pi)$ and so R_{180} gives rise to a unimodular equivalence as well. $\square$

It turns out that two permutations π and π' may be trivially Wilf equivalent without $P_n(\pi)$ and $P_n(\pi')$ being unimodularly equivalent. An explicit example is $\pi = 1423$ and $\pi' = 2431$: although these are related by a 90-degree rotation, one can compute that, while $P_5(1423)$ has 48 facets, $P_5(2431)$ only has 46.

Seeing as pattern avoidance can be unwieldy to work with for arbitrary choices of Π, let us first turn to specific avoidance classes.

Corollary 2.3. For any fixed n, all polytopes in the set $\{P_n(132), P_n(213), P_n(231), P_n(312)\}$ are unimodularly equivalent with each other. The polytopes in the set $\{P_n(123), P_n(321)\}$ are also unimodularly equivalent to each other, but these two classes are distinct.

Proof. The unimodular equivalence follows from the previous result. To show the classes are distinct, note that $P_4(123)$ has 13 facts whereas $P_4(132)$ has only 11. $\square$

In subsequent sections, it will be helpful to describe classes of permutations in the following way: Let $A = (a_{i,j})$ be a $k \times l$ matrix with entries in $\{0, \pm 1\}$. We say that a permutation σ is *A-griddable* in $\mathbb{R}^2$ if the diagram $\mathcal{C}$ of σ can be partitioned into rectangular regions $C_{i,j}$ using horizontal and vertical lines in such a way that

$$\mathcal{C} \cap C_{i,j} \text{ is } \begin{cases} \text{increasing} & \text{if } a_{i,j} = 1, \\ \text{decreasing} & \text{if } a_{i,j} = -1, \\ \text{empty} & \text{if } a_{i,j} = 0. \end{cases}$$

If $\mathcal{C} \cap C_{i,j}$ contains at most one element, it may be considered as either increasing or decreasing. For example, if

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix},$$

then $\sigma = 4261573$ is A-griddable, as demonstrated in Figure 2. For a particular matrix A, the *grid class* of A is the set of permutations that are A-griddable. We will occasionally use grid classes to more conveniently describe the structure of permutations used as the vertices polytopes.

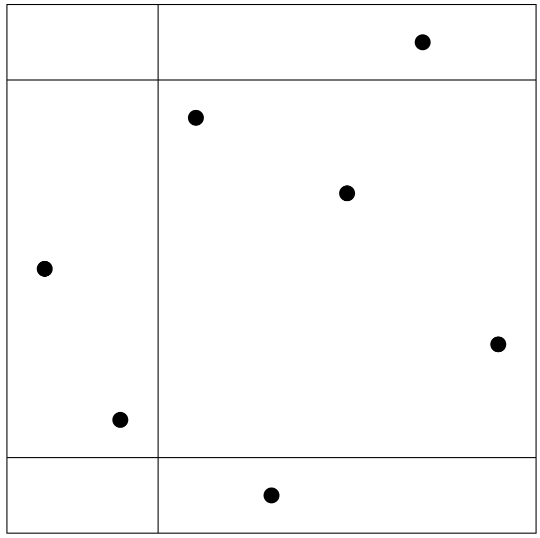


FIGURE 2. An A-gridding of 4261573.

2.2. **The case $P_n(132, 312)$.** One of the basic questions to ask about a polytope is what the face structure looks like, and we see here (and in the following subsection) that the answer can be quite pleasant.

Proposition 2.4. The polytope $P_n(132, 312)$ is a rectangular parallelepiped (*parallelotope*). Specifically, the polytope is contained in the hyperplane $\sum x_i = \binom{n}{2}$, and for each $j = 1, \dots, n-1$

$$\left|\sum_{i=1}^{j}(x_i - x_{j+1})\right| \leq \binom{j+1}{2} \tag{1}$$

are the facet-defining inequalities for $P_n(132, 312)$.

Proof. Consider the polytope P defined by the given inequalities and lying in the given hyperplane. Each inequality in (1) gives a pair of parallel faces of P because of the absolute value signs. It is also easy to check the the normal vectors are pairwise orthogonal and also orthogonal to the vector $(1, \dots, 1)$ which defines the hyperplane $\sum x_i = \binom{n}{2}$. Thus P is an $(n-1)$-dimensional parallelotope.

The polytope P will have $2^{n-1} = |\operatorname{Av}_n(132, 312)|$ vertices. So to demonstrate that $P = P_n(132, 312)$ it suffices to prove that evey $\sigma = a_1 a_2 \cdots a_n \in \operatorname{Av}_n(132, 312)$ is a vertex of P. It follows the proof of Proposition 5.2 in [11] that this class is the grid class for the matrix

$$A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Equivalently, the elements of this avoidance class are characterized by the fact that for each $j = 1, \dots, n-1$, we have a_{j+1} is either one greater than the largest previously-appearing entry or one less than the smallest previously-appearing entry. Note that if it is smaller then σ satisfies $\sum_{i=1}^{j}(x_i - x_{j+1}) = \binom{j+1}{2}$, and if it is larger then σ satisfies $\sum_{i=1}^{j}(x_i - x_{j+1}) = -\binom{j+1}{2}$. These equalities hold because the summands are exactly the integers $1, \dots, j$ in the first case and $-1, \dots, -j$ in the second. Since this is true for all j, σ is a vertex of P. $\square$

We note that when a polytope $P \subseteq \mathbb{R}^n$ is not full-dimensional, some extra care is needed when discussing volume. Usual Euclidean volume would dictate that the volume of a polytope that is not full-dimensional is zero. However, we are typically interested in the *relative volume*, that is, the volume of the polytope with respect to the lattice $(\operatorname{aff} P) \cap \mathbb{Z}^n$ where $\operatorname{aff} P$ is the affine subspace spanned by P. When P does have full dimension, the notions of volume and relative volume coincide. Throughout this paper, "volume" is understood to mean the relative volume.

The *normalized volume* of a lattice polytope $P \subseteq \mathbb{R}^n$ is $\operatorname{Vol} P := (\dim P)! \operatorname{vol}(P)$, where $\operatorname{vol}(P)$ is the usual relative volume of P. A lattice simplex $\Sigma \subseteq \mathbb{R}^n$ with vertex set $V = \{v_0, \ldots, v_k\}$ is *unimodular with respect to the lattice* L if it has smallest possible relative volume with respect to L. If L is not specified, then it is assumed that $L = (\operatorname{aff} V) \cap \mathbb{Z}^n$. Equivalently, Σ is unimodular with respect to L if the set of emanating vectors $\{v_1 - v_0, \ldots, v_k - v_0\}$ forms a $\mathbb{Z}$-basis of $L - v_0$. In particular, if P is unimodular, then it has a normalized volume of 1. We refer to Section 5.4 of [5] for a more thorough discussion of these details.

Corollary 2.5. The volume of $P_n(132, 312)$ is $(n-1)!$

Proof. By the previous proposition, the volume of $P = P_n(132, 312)$ may be computed directly by choosing a base vertex, taking the product of the lengths of the edges incident to it, and then dividing by an appropriate factor to account for the relative volume. For the scaling factor, it is well-known that for a (measurable) subset $S \subseteq \mathbb{R}^m$ and a linear function $f : \mathbb{R}^m \to \mathbb{R}^n$, with $m \leq n$,

$$\operatorname{vol}(f(S)) = \sqrt{\det A^T A} \operatorname{vol}(S),$$

where A is the matrix for f and volume is taken with respect to the usual Euclidean measure. In our case, a $\mathbb{Z}$-basis for $\operatorname{aff} P \cap \mathbb{Z}^n$ is $e_1 - e_j$ for $j = 2, \ldots, n$, so these vectors form the columns of A. It is straightforward to check that $A^T A = J_{n-1} + I_{n-1}$ where J_{n-1} is the $(n-1) \times (n-1)$ matrix with every entry 1. One may then verify that $\det(A^T A) = n$. So find the relative volume of P, we must divide the usual $(n-1)$-dimensional volume of P by $\sqrt{n}$.

Now, a convenient choice of base vertex is the permutation $\sigma = 12 \cdots n$. Using the hyperplane description of the previous result, this vertex is adjacent to the permutations $\sigma_j = 2 \cdots (j) 1 (j+1) \cdots n$ for each $j = 2, \ldots, n$. It is straightforward to compute that $|\sigma_j - \sigma| = \sqrt{j(j-1)}$, so taking the product of these lengths and then dividing by $\sqrt{n}$ results in P having relative volume $(n-1)!$ as desired. $\square$

It is worth investigating when $P_n(\Pi)$ is a subclass of a known generalization of permutohedra. In particular, we would like to know whether $P_n(132, 312)$ is a special case of Postnikov's generalized permutohedra. To answer this, we need to use a few more tools.

A *fan* $\mathcal{F}$ in $\mathbb{R}^n$ is a collection of closed cones, each containing the origin, such that the intersection of any two cones is another cone in F. Using the notation

$$|\mathcal{F}| := \bigcup_{F \in \mathcal{F}} F,$$

we say a fan $\mathcal{F}'$ *refines* $\mathcal{F}$ if $|\mathcal{F}'| = |\mathcal{F}|$ and if each cone in $\mathcal{F}'$ is contained in a cone in $\mathcal{F}$.

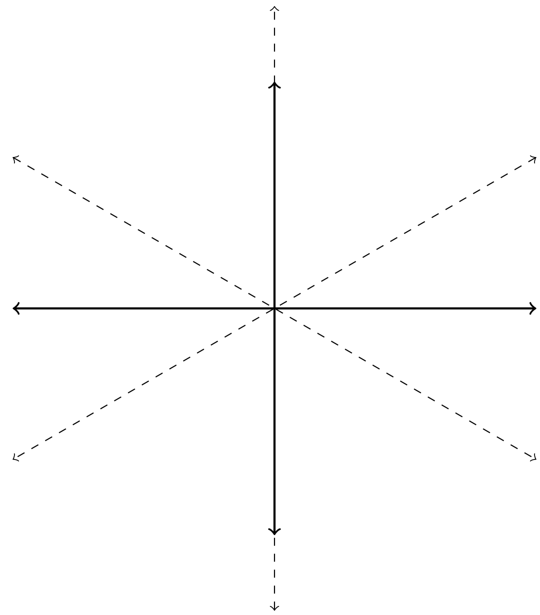


FIGURE 3. Viewed from $(1,1,1)$, the rays $N(P_3(132,312))$ are solid, while the rays of the braid arrangement fan are dashed.

Let $w \in \mathbb{R}^n$ and let $P \subseteq \mathbb{R}^n$ be any polytope. Define

$$\mathrm{face}_w(P) := \{u \in P \mid w^T u \geq w^T v \text{ for all } v \in P\}.$$

In other words, $\mathrm{face}_w(P)$ is the face of P for which the linear form defined by w is maximized. If F is a face of a polytope P, the *normal cone* of F at P is

$$N_P(F) := \{w \in \mathbb{R}^n \mid \mathrm{face}_w(P) = F\}.$$

Thus, if F is a facet of P, then $N_P(F)$ is a ray. The collection of all $N_P(F)$, ranging over all faces of P, is the *normal fan* of the polytope, and is denoted $N(P)$.

In our case, the inequalities of (1) provide the rays of the normal fan for $P_n(132,312)$. We will compare this normal fan with a certain other fan, defined in the following way. The *braid arrangement* in $\mathbb{R}^n/(1,\ldots,1)\mathbb{R}$ is the set of hyperplanes $\{x_i = x_j\}_{1\leq i<j\leq n}$. These hyperplanes partition the space into the *Weyl chambers*

$$C_\sigma : \{x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}\},$$

ranging over all $\sigma \in \mathfrak{S}_n$. The collection of these chambers and their lower-dimensional faces form the *braid arrangement fan*. The following result of Postnikov, Reiner, and Williams, allows us to see that $P_n(132,312)$ does not fall into the class of generalized permutohedra.

Proposition 2.6 (Proposition 3.2, [20]). A polytope P in $\mathbb{R}^n$ is a generalized permutohedron if and only if its normal fan, reduced by $(1,\ldots,1)\mathbb{R}$, is refined by the braid arrangement fan. $\square$

Using the hyperplane description from Proposition 2.4, we can see immediately that the rays of $P_n(132,312)$ are not all rays of the braid arrangement fan. Thus, the braid arrangement fan cannot be a refinement of $N(P_n(132,312))$. See Figure 3 for an example.

The Ehrhart polynomial of P_n is known to be $\sum_{i=0}^{n-1} F_i m^i$, where F_i is the number of forests with i edges on vertex set $\{1, 2, \ldots, n\}$ (see Exercise 4.64(a) in [30]). In the special case of $\Pi = \{132, 312\}$, we can use the same techniques to find the Ehrhart polynomial of $P_n(\Pi)$. Our first step in this direction will use the following result, due to Stanley.

Theorem 2.7 (Theorem 2.2, [29])**.** Suppose P is a lattice zonotope, that is, P can be written in the form

$$P = \{a_1 v_1 + \ldots + a_k v_k \mid 0 \le a_i \le 1\},$$

where each $v_i \in \mathbb{Z}^n$. The Ehrhart polynomial of P is

$$\mathcal{L}_P(m) = \sum_X g(X) m^{|X|} \tag{2}$$

where the sum ranges over all linearly independent subsets X of $\{v_1, \ldots, v_k\}$ and where $g(X)$ is the greatest common divisor of all full minors of the matrix whose columns are the elements of X. □

To state the next result elegantly we define, for nonnegative integers n and k, the *falling factorial*

$$n{\downarrow_k} = n(n-1)\ldots(n-k+1).$$

Proposition 2.8. The polytope $P = P_n(132, 312)$ has Ehrhart polynomial

$$\mathcal{L}_P(m) = \sum_{k=0}^{n-1} (n-1){\downarrow_k}\, m^k.$$

Proof. From the half-space and hyperplane description given in Proposition 2.4, we can see that P is, up to a translation by $(1, 2, \ldots, n)$, the zonotope

$$Z = \{a_1 v_1 + \ldots + a_{n-1} v_{n-1} \mid 0 \le a_i \le 1\} \subseteq \mathbb{R}^n$$

where $v_j = \sum_{i=1}^{j} (e_i - e_{j+1})$ for $j = 1, \ldots, n-1$. In fact, it is not difficult to see that Z is unimodularly equivalent to

$$\hat{Z} = \{a_1 w_1 + \ldots + a_{n-1} w_{n-1} \mid 0 \le a_i \le 1\} \subseteq \mathbb{R}^n$$

where

$$w_j = \sum_{i=1}^{j+1} (i-1) e_i$$

for each $j = 1, \ldots, n-1$. Note that the set of all w_j is linearly independent.

We will now complete the proof using equation (2) on the w_j basis. First, however, we need to set up some notation. For X as in (2) we will use X to stand for both the subset and the matrix whose columns are the elements of X. For any family $\mathcal{F}$ of subsets X we define

$$g(\mathcal{F}) = \sum_{X \in \mathcal{F}} g(X),$$

We also let $\mathcal{F}_{n,k}$ be the family of all k-element subsets of $w_1, \ldots, w_{n-1}$ and $g(n,k) = g(\mathcal{F}_{n,k})$. So we will be done if we can prove that $g(n,k) = (n-1){\downarrow_k}$. In fact, we will show that the following recurrence relation holds

$$g(n+1, k) = g(n,k) + n[g(n, k-1) - g(n-1, k-1)] + g(n-1, k-1). \tag{3}$$

Since it is easy to verify that $(n-1)\downarrow_k$ satisfies the same recursion, induction on n completes the proof.

Partition $\mathcal{F}_{n+1,k}$ into the following three subsets

$$\begin{aligned}
\mathcal{F}_1 &= \{X \in \mathcal{F}_{n+1,k} \mid X \text{ does not contain } w_n\},\\
\mathcal{F}_2 &= \{X \in \mathcal{F}_{n+1,k} \mid X \text{ does contains both } w_{n-1} \text{ and } w_n\},\\
\mathcal{F}_3 &= \{X \in \mathcal{F}_{n+1,k} \mid X \text{ does contains } w_n \text{ but not } w_{n-1}\}.
\end{aligned}$$

From the definitions

$$g(n+1,k) = g(\mathcal{F}_1) + g(\mathcal{F}_2) + g(\mathcal{F}_3).$$

We now show that each of these summands equals the corresponding summand in (3).

The matrices in $\mathcal{F}_1$ are the same as those for $\mathcal{F}_{n,k}$ except with a last row of zeros. Clearly this row does not contribute any nonzero minors so $g(\mathcal{F}_1) = g(n,k)$, giving the first summand.

Now consider the minors of a matrix $X \in \mathcal{F}_2$, letting M be the submatrix of the minor. If M does not contain the last row of X, then its last two columns are equal and $\det M = 0$. So the only M contributing to $g(\mathcal{F}_2)$ are those whose last row is the final row of X which is all zero except for a last entry of n. It follows that $|\det M| = n|\det M'|$ where M' is obtained by removing the last row and column of M. The possible M' which can appear are exactly those occurring in elements $X' \in \mathcal{F}_{n,k-1}$ such that $w_{n-1} \in X'$. Using the reasoning of the previous paragraph and complementation, we see that such $|\det M'|$ contribute exactly $g(n,k-1) - g(n-1,k-1)$ to the desired sum. Thus $g(\mathcal{F}_2) = n[g(n,k-1) - g(n-1,k-1)]$.

Finally take $X \in \mathcal{F}_3$ so that X ends with a sequence of at least two rows each of which has a sole nonzero entry at the end. Keeping the notation and reasoning of the previous paragraph, we see that if $\det M \neq 0$ then M must contain exactly one row from this final sequence. Let $m_1, \ldots, m_r$ be the minors which can be obtained from all nonzero minors containing the last row of X. Then for all i we have $m_i = nm_i'$ where $m_1', \ldots, m_r'$ are exactly the nonzero minors of $X' \in \mathcal{F}_{n-1,k-1}$ obtained by removing the last row and column of X. So

$$\gcd(m_1, \ldots, m_r) = n \gcd(m_1', \ldots, m_r') = ng(X').$$

Now repeat this process, but using the penultimate row of X, giving minors $m_{r+1}, \ldots, m_{2r}$ with greatest common divisor $(n-1)g(X')$. But n and $n-1$ are relatively prime, so $\gcd(m_1, \ldots, m_{2r}) = g(X')$. Continuing in this way, we see that $g(X) = g(X')$. Summing over all possible X gives $g(\mathcal{F}_3) = g(n-1,k-1)$ and completes the proof. $\square$

A standard fact from Ehrhart theory states that the leading coefficient of $\mathcal{L}_P(m)$ is the volume of P, so Corollary 2.5 is reaffirmed by the previous result. Moreover, knowing the Ehrhart polynomial allows us to deduce an interesting fact about the interior lattice points of $P_n(132, 312)$.

Corollary 2.9. The number of lattice points interior to $P_n(132, 312)$ is equal to the number of derangements in $\mathfrak{S}_{n-1}$.

Proof. Let $P = P_n(132, 312)$ and P° be the interior of P. By Ehrhart-Macdonald reciprocity,

$$\mathcal{L}_{P^\circ}(m) = (-1)^{n-1} \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!}(-m)^k.$$

Evaluating at $m = 1$, we get

$$\mathcal{L}_{P^\circ}(1) = (-1)^{n-1} \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!}(-1)^k,$$

which is the well-known inclusion-exclusion formula for derangements. □

It would be very interesting to find an explicit bijection between the interior points of $P_n(132, 312)$ and the derangements in $\mathfrak{S}_{n-1}$.

In the case of $P_n(132, 312)$, the Ehrhart polynomial was simple enough to compute directly. Since the coefficients can be explicitly determined, one may also determine the h^*-vector of $P_n(132, 312)$ by a change-of-basis although there does not seem to be a simple formula for its components.

Although finding explicit formulas for h^*-vectors is usually challenging in general, there are other methods for determining certain properties it might possess. A recent result due to Beck, Jochemko, and McCullough [3] states that lattice zonotopes always have a unimodal h^*-vector. Thus the following result follows quickly from the proof of Proposition 2.8.

Corollary 2.10. For all $n \geq 1$, $h^*(P_n(132, 312))$ is unimodal. □

Question 2.11. For which Π-avoiding permutohedra P is $h^*(P)$ unimodal?

There is not any particularly compelling reason to believe that $h^*(P)$ is always unimodal, but so far no examples with non-unimodal h^*-vectors has been found.

Let us now consider an expanded set of patterns to avoid.

Proposition 2.12. The Ehrhart polynomial for $P = P_n(123, 132, 312)$ is $(1 + m)^{n-1}$, hence $h^*_P(t)$ is the Eulerian polynomial $A_{n-1}(t)$.

Proof. As noted in [7], it is implied by [10] that the simplex P'_n whose vertices are the set

$$L_n := \{e_n\} \cup \left(\bigcup_{i=1}^{n-1} \sum_{j=i}^{n} j e_j \right)$$

has an Ehrhart polynomial of $(1 + m)^{n-1}$. Since the degree of the Ehrhart polynomial is the dimension of the polytope, P'_n is an $(n - 1)$-dimensional simplex. In particular, note that each $l \in L_n$ satisfies the equation $x_n - x_{n-1} = 1$. So, projecting P'_n to $\mathbb{R}^{n-1}$ by forgetting the last coordinate one obtains P''_n, which has the same Ehrhart polynomial as P'_n. Transforming P''_n by $f : x \mapsto Ax$, where A is the matrix with jth column $e_j - e_{j+1}$ for $j = 1, \ldots, n - 2$ and last column e_{n-1} results in the simplex whose vertices are 0 and $ie_i + \sum_{j=i+1}^{n-1} e_j$ for $i = 1, \ldots, n - 1$.

It follows from the proof of Proposition 7.2 of [11] that permutations of $\mathrm{Av}_n(123, 132, 312)$, that is, the vertices of P, are exactly those of the form $(n-1)(n-2) \cdots (j+1)(n)j(j-1) \cdots 1$ for $j = 0, \ldots, n - 1$. So $f(P''_n)$ can also be obtained from P by dropping the last coordinate and translating by $-(n - 1, n - 2, \ldots, 1)$. Since each of these operations are unimodular transformations, P has the same Ehrhart polynomial and h^*-polynomial as P'_n, which are $(1 + m)^{n-1}$ and $A_{n-1}(t)$, respectively. □

2.3. **Pitman-Stanley Polytopes.** We will next consider a Π-avoiding permutohedron whose Ehrhart polynomial is easily computable due to results of Pitman and Stanley [19]. Given a sequence of nonnegative real numbers $c = (c_1, \dots, c_n)$, there is a corresponding *Pitman-Stanley polytope* $PS_n(c)$ defined by

$$PS_n(c) := \left\{ x \in \mathbb{R}^n \mid x_i \geq 0 \text{ and } \sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} c_i \text{ for all } 1 \leq j \leq n \right\}.$$

Pitman-Stanley polytopes are connected with multiple combinatorial objects. For example, recall that a *polyhedral subdivision* of a polytope P is a collection of subpolytopes $P_1, \dots, P_k \subseteq P$ whose union is P, and $P_i \cap P_j$ is a face of both P_i and P_j for all i, j. Pitman and Stanley showed that $PS_n(c)$ has polyhedral subdivisions whose maximal elements of correspond to certain plane trees; $\mathrm{Vol}(PS_n(c))$ can be expressed in terms of parking functions; the number of lattice points of $PS_n(c)$ can be expressed in terms of plane partitions of a particular shape, whose parts are at most 2. The key result for us is the following.

Theorem 2.13 (Pitman and Stanley, [19])**.** Let a, b be positive integers, and set $c = (a, b, \dots, b) \in \mathbb{Z}^n$. The Ehrhart polynomial of $PS_n(c)$ is

$$\mathcal{L}_{PS_n(c)}(m) = \frac{am+1}{n!} \prod_{j=2}^{n} \left((a+nb)m + j\right).$$

□

Before continuing, we need a little background. The *face lattice* of a polytope is the poset of its faces ordered by inclusion. Two polytopes are *combinatorially equivalent* if their face lattices are isomorphic. As proven in Theorem 19 of [19], whenever c has positive entries, $PS_n(c)$ is combinatorially equivalent to an n-cube.

Lemma 2.14. When c has positive entries, the vertices of $PS_n(c)$ are exactly the vectors $v = (v_1, \dots, v_n)$ constructed, component-wise from left to right, by either setting $v_j = 0$ or setting $v_j = c_j + c_{j-1} + \cdots + c_i$, where v_{i-1} is the previous nonzero entry of v.

Proof. Since c has positive entries, $PS_n(c)$ is a combinatorial cube, hence the set of facets may be partitioned into n non-intersecting pairs. In particular, the pairs correspond to the hyperplanes $x_i = 0$ and $x_1 + \cdots + x_i = c_1 + \cdots + c_i$. Again, since $PS_n(c)$ is a combinatorial cube, a vertex v will lie on exactly one of the facets of each pair. Using the definition of a Pitman-Stanley polytope yields the desired conclusion. □

Theorem 2.15. The polytope $P = P_n(123, 132)$ is a combinatorial cube with an Ehrhart polynomial of

$$\mathcal{L}_P(m) = \frac{m+1}{(n-1)!} \prod_{j=2}^{n-1} (nm + j)$$

Proof. We will show that P is related to $PS_{n-1}(1, \dots, 1)$ in such a way that its face lattice and Ehrhart polynomial are preserved. Then the theorem will follow from the statement just before Lemma 2.14, and by setting $a = b = 1$ in Theorem 2.13.

We first need a description of the vertices of P. By reversing the permutations in Proposition 4.2 of [11], we note that the diagram for a vertex $v = (v_1, \dots, v_n)$ of $P_n(123, 132)$ consists of a decreasing sequence of blocks where each block is the pattern $k(k-1)\cdots 1(k+1)$ for some k. Define a function $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$ by

$$f(a_1, \ldots, a_n) = (a_1, \ldots, a_{n-1}) - (n-1, n-2, \ldots, 1).$$

We claim that f maps the vertices of P to the vertices of $PS_{n-1}(1, \ldots, 1)$. Indeed, suppose the first block of a vertex v of P is of the form $(n-1, n-2, \ldots, n-k, n)$. Then under f this maps to the sequence $(0, 0, \ldots, 0, k+1)$ with k initial zeros. But, by Lemma 2.14, this is the prefix of a vertex of $PS_{n-1}(1, \ldots, 1)$. Continuing in this way, we see that $f(v)$ will indeed be a vertex of this Pitman-Stanley polytope. Reversing the argument shows that f is, in fact, a bijection on the vertex sets.

Since P is a subpolytope of the usual permutohedron, the projection to the first $n-1$ coordinates preserves the face lattice and Ehrhart polynomial, as does lattice translation. This is what we desired to show. □

From the Ehrhart polynomial, we can immediately determine the volume and number of lattice points in the polytope.

Corollary 2.16. The normalized volume of $P_n(123, 132)$ is n^{n-2} and the number of lattice points it contains is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. To calculate the normalized volume, one takes the leading coefficient of the Ehrhart polynomial in Theorem 2.15 and multiplies by $(n-1)!$ since $\dim P_n(123, 132) = n-1$. To calculate the number of lattice points, one just plugs $m = 1$ into this polynomial. □

3. The Birkhoff Polytope

We come now to our second merging of polytopes with avoidance classes of permutations by generalizing the Birkhoff polytope B_n in the following way.

Definition 3.1. Let Π be any set of permutations. The Π-*avoiding Birkhoff polytope* is

$$B_n(\Pi) := \operatorname{conv}\{M \in \mathbb{R}^{n\times n} \mid M \text{ is the permutation matrix for some } \sigma \in \operatorname{Av}_n(\Pi)\}.$$

The Birkhoff polytope is famous for its simple description, applicability to many areas of research, and its reluctance to provide researchers with much information. For example, although its h^*-vector is known to be symmetric and unimodal [1], its volume is only known for $n \leq 10$ [4]. Moreover, although the dimension of B_n is $(n-1)^2$, the dimension of $B_n(\Pi)$ may be significantly lower, as we will see shortly.

Studying variations of the Birkhoff polytope is not uncommon. Burggraf, De Loera, and Omar [9] have studied what they call *permutation polytopes*: subpolytopes of B_n whose vertices form a subgroup of $\mathfrak{S}_n$. The Birkhoff polytope itself is a special case of a *transportation polytope* where the row/column sums are constrained by a vector rather than a single number.

Aside from having the same number of vertices, there is very little in common between $P_n(\Pi)$ and $B_n(\Pi)$. Unlike the Π-avoiding permutohedron, we will see that all trivial Wilf equivalences of permutations yield unimodular equivalences of the corresponding polytopes.

It will be convenient to use the nonstandard convention that the matrix $M = M_\sigma$ of a permutation $\sigma = a_1 \ldots a_n$ will have $m_{n-a_j+1,j} = 1$ for all j and all other entires zero.

Alternatively, M can be obtained from the diagram of σ by replacing every dot with a one and zeroing out the rest of the entries.

Proposition 3.2. If $\Pi \in \mathfrak{S}$, then $B_n(f(\Pi))$ is unimodularly equivalent to $B_n(\Pi)$ for any f in the dihedral group of the square.

Proof. Because f is a dihedral action on the square, there is an obvious corresponding action on the vertices of $B_n(\Pi)$ to obtain the vertices of $B_n(f(\Pi))$. This action is a particular permutation of the elements of each matrix, which is itself a unimodular transformation. Applying the action to the full polytope $B_n(\Pi)$ results in a unimodular transformation whose image is $B_n(f(\Pi))$. $\square$

We will be spending a significant amount of time studying $B_n(\Pi)$ for a few specific choices of Π. To begin with, we will choose the well-behaved set $\Pi = \{123, 312\}$.

Recall that the $(n-1)$-dimensional *standard simplex* is the simplex $\Delta_{n-1} \subseteq \mathbb{R}^n$ whose vertices are the standard basis vectors of $\mathbb{R}^n$. Since Δ_{n-1} lies in the hyperplane $\sum x_i = 1$, no information about the number of lattice points in Δ_{n-1} is lost by applying the projection $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ which drops the last coordinate. Thus, Δ_{n-1} is sometimes identified with $p(\Delta_{n-1}) = \operatorname{conv}\{0, e_1, \ldots, e_{n-1}\}$. In particular, they are both unimodular simplices with respect to their affine spans, so both have h^*-vectors of 1.

Proposition 3.3. The polytope $B_n(123, 312)$ is unimodularly equivalent to $p(\Delta_{\binom{n}{2}})$. Thus, for any $\Pi \subseteq \mathfrak{S}$ containing 123 and 312 we have $h^*(B_n(\Pi)) = 1$.

Proof. The vertices of $B_n(123, 312)$ are the matrices corresponding to elements of the permutation class $\mathrm{Av}_n(132, 321)$, rotated by 90 degrees clockwise. By the proof of Proposition 4.3 of [11] and induction, we see that this avoidance class is the grid class of

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Let σ be a permutation in this class. By knowing only the elements in σ corresponding to the first column of A, that is, those below the main diagonal, the remainder of σ is forced. In particular, if $\sigma = a_1 \ldots a_n \neq n(n-1)\ldots 1$ has longest initial decreasing subsequence $a_1 \ldots a_k$, then they must form an interval of integers between 1 and $n-1$. Ranging over all possible intervals for all possible $k = 1, \ldots, n-1$, the total number of permutations described in the previous sentence is $\binom{n}{2}$, for a total of $\binom{n}{2} + 1$ vertices when including $n(n-1)\ldots 1$.

Now we identify a matrix $M = (m_{i,j})$ with a point in $\mathbb{R}^{\binom{n}{2}}$ by the correspondence

$$\begin{bmatrix} * & * & * & \cdots & * & * \\ x_1 & * & * & \cdots & * & * \\ x_n & x_2 & * & \cdots & * & * \\ x_{2n-2} & x_{n+1} & x_3 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{\binom{n}{2}} & x_{\binom{n}{2}-1} & x_{\binom{n}{2}-3} & \cdots & x_{n-1} & * \end{bmatrix} \mapsto (x_1, \ldots, x_{\binom{n}{2}}).$$

Then the vertices of $B_n(123, 312)$ are identified with the columns of

$$A' = \begin{bmatrix} 0 & A_0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & A_{n-2} \end{bmatrix}$$

where A_i is the $(n-1-i) \times (n-1-i)$ matrix formed by setting each entry on or above the main diagonal to 1. Each A_i is unimodularly equivalent to an identity matrix by applying the transformation

$$\begin{bmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix},$$

so $B_n(123, 312)$ itself is unimodularly equivalent to $p(\Delta_{\binom{n}{2}})$, which has an h^*-vector of 1.

For the second claim, since $B_n(123, 312)$ is a unimodular simplex and $B_n(\Pi')$ is a subpolytope if $\{123, 312\} \subseteq \Pi'$, $B_n(\Pi')$ is a face of $B_n(123, 312)$. Thus $B_n(\Pi')$ is a unimodular lattice simplex of some dimension $k \leq n$ with respect to its affine span. So, using the equivalence we just established, any avoidance class Π' containing 123 and 312 results in $h^*(B_n(\Pi')) = 1$. $\square$

The remainder of this paper will be devoted to studying $B_n(132, 312)$ and one other class of polytopes. For this final class we will require some more definitions and notation. We say a permutation $\sigma = a_1 \cdots a_n$ is *alternating*, or *up-down*, if $a_1 < a_2 > a_3 < \cdots$. In the literature, "alternating" sometimes includes *down-up* permutations, where the previous inequalities are all reversed. It is worth noting that alternating permutations may be expressed in terms of *vincular patterns*, which are patterns requiring certain elements to occur consecutively. To indicate this, the portion of the pattern which must be consecutive is underlined. For example, 4261573 contains five instances of the vincular pattern $2\underline{31}$, which are 261, 461, 473, 573, and 673. The study of vincular patterns was introduced in [2] and has since been extended to bivincular patterns, mesh patterns, and other generalizations. We refer to [31] for more information about each of these avoidance classes, including assorted open problems.

Alternating permutations in $\mathfrak{S}_n$ are exactly the elements $\sigma = a_1 \cdots a_n \in \mathrm{Av}_n(\underline{\varepsilon 21}, \underline{123}, \underline{321})$. The "$\varepsilon$" at beginning of the vincular pattern denotes the "empty permutation" which has length 0 and is to be treated as preceding a_1. So σ containing the pattern $\underline{\varepsilon 21}$ is equivalent to $a_1 > a_2$, and avoiding it forces $a_1 < a_2$. In the interest of compact notation, we will write $\widetilde{\mathrm{Av}}_n(\Pi)$ for $\mathrm{Av}_n(\{\underline{\varepsilon 21}, \underline{123}, \underline{321}\} \cup \Pi)$ and $\widetilde{B}_n(\Pi)$ for the analogous variation of $B_n(\Pi)$.

This brings us to our final important class of polytopes, $\widetilde{B}_n(123)$. We claim that if n is even, then the number of 123-avoiding alternating permutations is the same in $\mathfrak{S}_n$ and $\mathfrak{S}_{n-1}$. To see this, note that in any permutation avoiding 123 the 1 can not be followed by two elements forming an increasing subsequence. So if n is even and $\sigma = a_1 a_2 \cdots a_n$ is alternating and 123-avoiding, then $a_{n-1} = 1$. It follows that standardizing $\sigma' = a_1 a_2 \ldots a_{n-2} a_n$ gives a bijection between the two sets of permutations in question. Thus, the projection of $\widetilde{B}_n(123)$

to $\widetilde{B}_{n-1}(123)$, defined by dropping row n and column $n-1$ of the matrices, preserves the Ehrhart polynomial.

Lemma 3.4. For all n we have $\dim B_n(132, 312) \leq \binom{n}{2}$ and $\dim \widetilde{B}_n(123) \leq \binom{\lceil n/2 \rceil}{2}$.

Proof. First consider $B_n(132, 312)$. As noted in the proof of Proposition 2.4, the vertices of $B_n(132, 312)$ are the elements of the grid class for

$$A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus, once the entries above the main diagonal of the matrix for a vertex of $B_n(132, 312)$ are known, the remaining entries are forced. Since there are $\binom{n}{2}$ coordinates above the diagonal, we get $\dim B_n(132, 312) \leq \binom{n}{2}$.

Now consider $\widetilde{B}_n(123)$ for $n = 2k$ and consider $\sigma = a_1 \ldots a_n \in \widetilde{\mathrm{Av}}_n(123)$, as well as its corresponding matrix $M = M_\sigma = (m_{i,j})$. To show that $\binom{k}{2}$ is an upper bound, first note that the subsequences $a_1 a_3 \ldots a_{n-1}$ and $a_2 a_4 \ldots, a_n$ are both decreasing. So if the values of the first are known, then the values of the second are forced. Therefore, we need only consider the contribution of odd-indexed columns of M to the dimension since then the even-indexed ones will not change the bound. Now consider the possibilities in column $2c - 1$ where $1 \leq c \leq k$. Because σ is alternating and the sequence of odd-indexed elements is decreasing, the element a_{2c-1} is smaller than the elements $a_1, a_2, \ldots, a_{2c-2}$, and larger than $a_{2c+1}, a_{2c+3}, \ldots, a_{2k-1}$. It follows that we have $k-c+1 \leq a_{2c-1} \leq 2k-2c+1$. Thus there are only $(2k-2c+1)-(k-c+1)+1 = k-c+1$ positions for the unique nonzero one in columns $2c-1$, and only these positions can contribute to the dimension. But these positions must also satisfy the linear equation that the sum of their entries is 1. This reduces the possible contribution of these positions to $k - c$. Thus

$$\dim \widetilde{B}_n(123) \leq \sum_{c=1}^{k} (k - c) = \binom{k}{2}$$

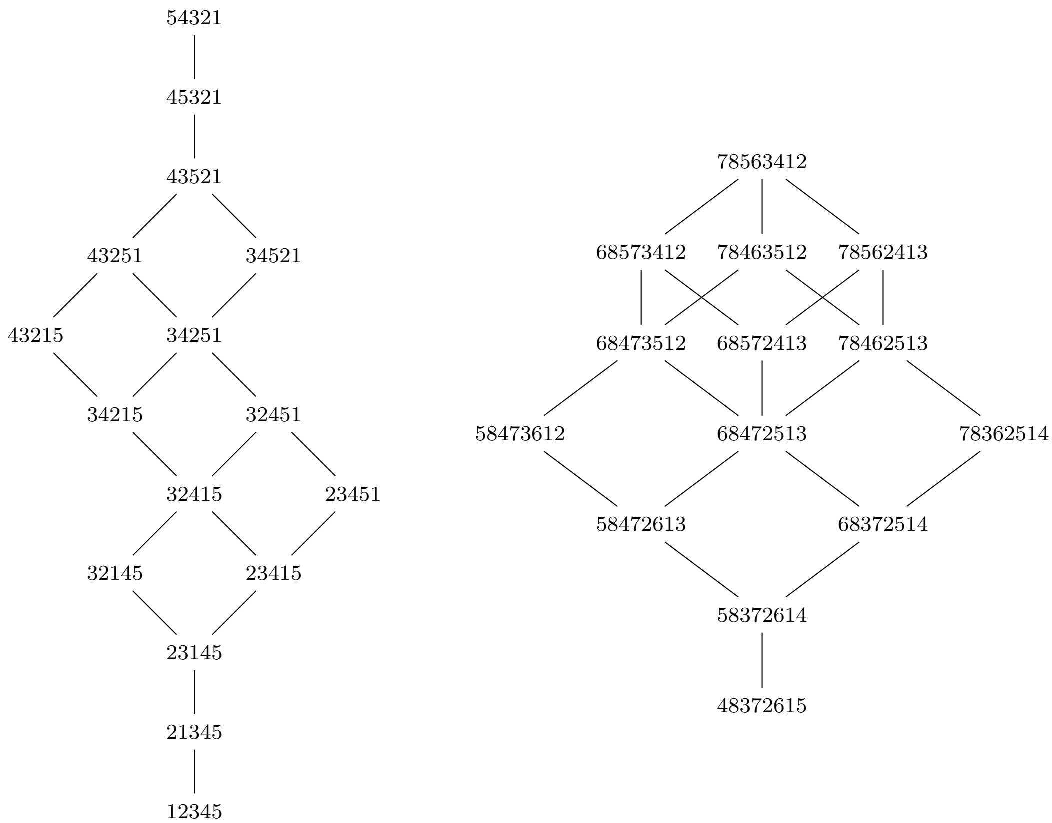
as desired.

Finally, using the bijection described prior to this result, an analogous argument holds for $n = 2k - 1$. Thus the conclusion is true for all n. □

It turns out that the upper bounds given above are also lower bounds, but the proof of this fact will be delayed until Section 3.3 since it will be more convenient to include these details when proving Proposition 3.7.

3.1. **Sublattices of the Weak Order.** In order to prove interesting results about the Ehrhart theory of $B_n(132, 312)$ and $\widetilde{B}_n(123)$, we will first show how the polytopes may be decomposed by putting a partial order on their vertex sets. These posets (partially ordered sets) are themselves highly structured and interact in a natural way with the geometry of the polytopes. We refer the reader to [30] for the necessary background regarding posets.

Recall that the *right* (respectively, *left*) *weak (Bruhat) order* on $\mathfrak{S}_n$ is defined by the cover relations $\sigma_1 \lessdot \sigma_2$ if there is a simple transposition s_i such that $\sigma_1 s_i = \sigma_2$ (respectively, $s_i \sigma_1 = \sigma_2$) and $\mathrm{inv}(\sigma_2) = \mathrm{inv}(\sigma_1) + 1$. Here, $\mathrm{inv}\,\sigma$ is the number of inversions of σ. The left and right weak orders are isomorphic by the order-preserving map $\sigma \mapsto \sigma^{-1}$, which will soon be helpful.



$Q_5(132, 312)$ $\widetilde{Q}_8(123)$

FIGURE 4. Hasse diagrams of posets $Q_5(132, 312)$ and $\widetilde{Q}_8(123)$.

Let $Q_n(\Pi)$ denote the poset obtained by restricting the right weak order to $\mathrm{Av}_n(\Pi)$. Similarly define $\widetilde{Q}_n(\Pi)$ for the left weak order on $\widetilde{\mathrm{Av}}_n(\Pi)$. Inequalities involving permutation matrices are meant to refer to these two partial orders on the corresponding permutations.

If Π is chosen arbitrarily, then there is no reason to expect these posets to have especially pleasant structure. We will see, though, that specific choices of Π may result in interesting classes of posets. Figure 4 shows the posets $Q_5(132, 312)$ and $\widetilde{Q}_8(123)$.

We will find two classes of previously-studied posets useful, so we define them now. Let $M(n)$ denote the poset on $2^{[n]}$ where, if $A = \{a_1, \ldots, a_s\}$ and $B = \{b_1, \ldots, b_t\}$ are each written in decreasing order, then $A \leq B$ if and only if $s \leq t$ and $a_i \leq b_i$ for each $1 \leq i \leq s$. Equivalently, $M(n)$ is the poset of shifted Young diagrams with largest part at most n, ordered by inclusion. These are the posets described in Exercise 3.187(a) in [30] and studied using linear algebra in [22]. In particular, $M(n)$ is a distributive lattice as proved in the previously cited exercise.

For the other class of useful posets, recall that a *Dyck path* of length $2k$ can be described as a lattice path from $(0, 0)$ to (k, k) using steps $(1, 0)$ and $(0, 1)$, which never lies below the line $y = x$. We say the steps $(1, 0)$ and $(0, 1)$ are *east* steps and *north* steps, respectively. Let D_k denote the poset of Dyck paths of length $2k$, where if $d_1, d_2 \in D_k$, then $d_1 < d_2$ if

d_1 lies entirely to the right of d_2. So, $d_1 \lessdot d_2$ if d_2 can be obtained from d_1 by replacing a single outer corner with an inner corner. The posets D_k were shown to be distributive lattices in [13], a fact we will use when proving the following poset isomorphisms. The star in this proposition indicates the dual.

Proposition 3.5. For all n, $Q_n(132,312) \cong M(n-1)$ and $\widetilde{Q}_n(123) \cong D^*_{\lceil n/2 \rceil}$. Thus, $Q_n(132,312)$ and $\widetilde{Q}_n(123)$ are distributive lattices.

Proof. First note that the statement about distributive lattices will follow immediately once we have proved the isomorphisms.

We begin by proving that $Q_n(132,312) \cong M(n-1)$. Note that if $\sigma, \tau \in \mathrm{Av}_n(132,312)$ are distinct permutations, then it follows from the grid class description of its elements in Proposition 2.4 that $\mathrm{Des}(\sigma) \neq \mathrm{Des}(\tau)$ where Des returns the descent set of a permutation. Combined with the fact that $|Q_n(132,312)| = 2^{n-1} = |M(n-1)|$, we have that $\mathrm{Des} : Q_n(132,312) \to M(n-1)$ is a bijection.

By definition, $\sigma \lessdot \tau$ in $Q_n(132,312)$ if and only if $\tau = \sigma s_i$ for some simple transposition s_i which increases the number of inversions. Using the grid class description again, we see that this transposition will either replace $i-1 \in \mathrm{Des}(\sigma)$ with $i \notin \mathrm{Des}(\sigma)$ or will replace $\mathrm{Des}(\sigma)$ with the disjoint union $\mathrm{Des}(\sigma) \uplus \{1\}$. These are both covers in $M(n-1)$, so Des is order-preserving.

To show that Des^{-1} is order-preserving, suppose $\mathrm{Des}(\sigma) = A = \{a_1 > \cdots > a_k\}$. Then, by the grid class description, σ consists of the numbers $1, \ldots, k$ in positions $a_1+1, \ldots, a_k+1$, respectively, and the numbers $k+1, \ldots, n$ placed in the remaining positions so as to form an increasing sequence. Suppose also that $B \gtrdot A$. From the description of $M(n-1)$ in terms of shifted diagrams, there are two possibilities. One is that $B = A \uplus \{1\}$. From the second sentence of this paragraph, it follows that $\tau = \mathrm{Des}^{-1} B$ satisfies $\tau = \sigma s_1$. The other possibility is that B is A with $i-1$ replaced by i, and by a similar reasoning this yields $\tau = \sigma s_i$ for $i \geq 2$.

Showing $\widetilde{Q}_n(123) \cong D^*_{\lceil n/2 \rceil}$ requires a bit more care. We will first show that $\widetilde{Q}_{2k}(123) \cong \widetilde{Q}_{2k-1}(123)$ under the map

$$\varphi(a_1 \ldots a_{2k}) = (a_1 - 1) \ldots (a_{2k-2} - 1)(a_{2k} - 1).$$

That this map is a bijection follows from the discussion prior to Proposition 3.4. Moreover, any $\sigma = a_1 \ldots a_{2k}$ always has $a_{2k-1} = 1$. So one will never apply s_1 so σ. And applying s_i, $i \geq 2$, corresponds to acting on $\varphi(\sigma)$ with s_{i-1}. From this the isomorphism follows. Therefore we may henceforth assume that $n = 2k$ for some integer k.

As noted in Proposition 3.1 of [18], each $\sigma = a_1 \ldots a_n \in \widetilde{\mathrm{Av}}_n(123)$ corresponds bijectively to a standard Young tableau (SYT) of shape $(2, \ldots, 2) = (2^k)$ which is its insertion tableau in the Robinson-Schensted algorithm. We refer the reader to [24] for background regarding standard Young tableaux and this correspondence. The entries in the resulting tableaux can be translated into Dyck paths of length $2k$ by insisting that step i be a north step if i is in the first column and an east step otherwise.

Since the patterns under consideration are alternating and 123-avoiding, it is not difficult to see that row r of the tableau contains entries $a_{2(n-r+1)-1}, a_{2(n-r+1)}$. Indeed the fact that the permutation is alternating with the odd position and even position entries forming decreasing subsequences forces the insertion of each odd (respectively, even) position entry to just push down the first (respectively, second) column of the tableau. Composing the functions of the previous paragraph, we obtain a bijection $f : \widetilde{Q}_n(123) \to D_k^*$ where $f(a_1 \dots a_{2k})$ is constructed by putting north steps in positions $a_1, a_3, \dots, a_{2k-1}$ and east steps in positions $a_2, a_4, \dots, a_{2k}$.

To show that f is order preserving, suppose $\sigma \lessdot \tau$ in $\widetilde{Q}_n(123)$. There is then some simple transposition s_i such that $s_i\sigma = \tau$ and the number of inversions increases by one so that i must occur before $i+1$ in σ. Now the positions of i and $i+1$ in σ must be odd and even, respectively, since any other choice either gives a copy of 123 or contradicts the fact that the odd and even position subsequences are decreasing. It follows that $f(\tau)$ is obtained from $f(\sigma)$ by swapping the steps in a single occurrence of a north step followed immediately by an east step, and this is a cover in D_k^*. Showing that f^{-1} is order preserving is a straightforward reversal of this argument, and so left to the reader. This completes the proof that f is an isomorphism. □

It will be helpful to observe that D_k^* can equivalently be described as the poset of (left-justified) Young diagrams fitting inside the shape $(k-1, k-2, \dots, 1)$, ordering by inclusion. This equivalence is easily seen by identifying a Dyck path with the region bounded between it, the y-axis, and the line $y = k$. So, we now have isomorphisms of both $Q_n(132, 312)$ and $\widetilde{Q}_n(123)$ with the lattices of certain Young diagrams.

For a general finite distributive lattice L of rank n, it is well-known that there exists an n-element poset P for which $L \cong J(P)$, where $J(P)$ denotes the lattice of order ideals of P. The poset P can be taken to be the join-irreducible elements of L and using the order relation from L restricted to these elements. Note that $x \in L$ is join irreducible if and only if x covers exactly one element. We denote the poset of join-irreducibles of L by $\mathrm{Irr}(L)$. To simplify matters, we will identify the join-irreducibles of $Q_n(132, 312)$ with the join-irreducibles of $M(n-1)$, and likewise identify the join-irreducibles of $\widetilde{Q}_n(123)$ and $D^*_{\lceil n/2 \rceil}$.

Let us now determine the join irreducibles of our two lattices. Using the Young diagram interpretation of both, an element is join irreducible precisely when the shape has exactly one inner corner, that is, a box in row b and column c, which we will refer to as (b, c), such that neither $(b+1, c)$ nor $(b, c+1)$ is in the shape. Identifying these diagrams with the coordinates of their unique inner corners, the induced partial order on both posets of join irreducibles is component-wise. For the remainder of this paper, the join irreducibles of $Q_n(132, 312)$ and $\widetilde{Q}_n(123)$ will be identified with the elements of these posets. See Figure 5 for an example.

3.2. **Triangulations, Shellabililty, and EL-labelings.** In this section we will use the posets $Q_n(132, 312)$ and $\widetilde{Q}_n(123)$ to carefully decompose $B_n(132, 312)$ and $\widetilde{B}_n(123)$. First, we recall some definitions and concepts in geometry and poset topology.

A *polytopal complex* $\mathcal{F}$ is a finite nonempty collection of polytopes such that

1. if $P \in \mathcal{F}$, then every face of P is in $\mathcal{F}$, and
2. if $P, Q \in \mathcal{F}$, then $P \cap Q$ is a face of both P and Q.

An commonly-considered polytopal complex is the *face complex* $\mathcal{F}(P)$ of a polytope P, whose elements are all faces of P.

A *triangulation* of a polytopal complex $\mathcal{F}$ is a geometric simplicial complex Δ with vertices those of $\mathcal{F}$ and underlying space equal to the union of the faces of $\mathcal{F}$, such that every face of

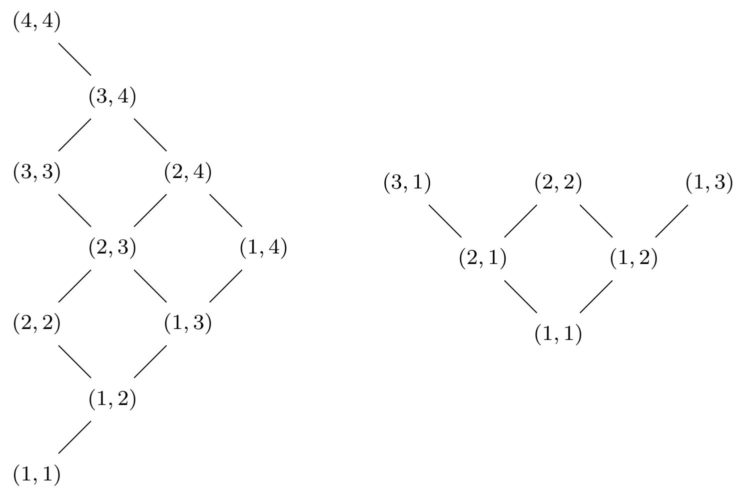


FIGURE 5. The posets $\mathrm{Irr}(Q_5(132, 312))$ and $\mathrm{Irr}(\widetilde{Q}_8(123))$.

Δ is contained in a face of $\mathcal{F}$. A triangulation of the face complex $\mathcal{F}(P)$ of a polytope P is simply called a *triangulation* of P. Therefore, if P has a unimodular triangulation $\mathcal{T}$, then its normalized volume is equal to the number of maximal simplices in $\mathcal{T}$.

Now, the *order complex* $\Delta(Q)$ of a poset Q is the simplicial complex of chains in Q. A simplicial complex is *shellable* if its maximal faces are of the same dimension and can be ordered as $F_1, \ldots, F_k$ such that for each $i = 1, \ldots, k-1$, $F_{i+1} \cap (\cup_{j=1}^{i} F_j)$ is a nonempty union of facets of F_{i+1}. A poset is called *shellable* if its order complex is shellable.

To show that $Q_n(132, 312)$ and $\widetilde{Q}_n(123)$ are shellable we will make use of the existence of a particular labeling of the edges in their Hasse diagrams.

If Q is a poset, let $E(Q)$ denote the set

$$E(Q) := \{(q_1, q_2) \in Q \times Q \mid q_1 \lessdot q_2\},$$

thought of as the edges of the Hasse diagram of Q. An *edge labeling* of Q by $\mathbb{Z}$ is a function $\lambda : E(Q) \to \mathbb{Z}$. A saturated chain $q_0 \lessdot q_1 \lessdot \cdots \lessdot q_k$ in Q is called *increasing* if $\lambda(q_0, q_1) < \lambda(q_1, q_2) < \cdots < \lambda(q_{k-1}, q_k)$. An *EL-labeling* of a poset Q is an edge labeling such that every interval $[x, y]$ in Q has a unique increasing maximal chain which lexicographically precedes all other maximal chains of $[x, y]$. Posets admitting an EL-labeling are shellable and are usually referred to as *EL-shellable*.

We will use EL-shellable posets to decompose $B_n(132, 312)$ and $\widetilde{B}_n(123)$ in specific ways in Section 4. Fortunately, specific EL-shellings of $Q_n(132, 312)$ and $\widetilde{Q}_n(123)$ are available and follow naturally from [26]. A *natural labeling* of a poset P with $|P| = n$ is an order-preserving bijection $\omega : P \to [n]$. Let L be a finite distributive lattice so that $L \cong J(P)$ where P is the poset of join irreducibles, and let ω be a natural labeling of P. Then we have a cover of order ideals $I \lessdot J$ in L if and only if $J - I = \{x\}$ for some $x \in P$. Give the cover the label $\lambda(I, J) = \omega(x)$.

Theorem 3.6 (Stanley, see [26])**.** The edge labeling of a finite distributive lattice L constructed above is an EL-labeling for L. $\square$

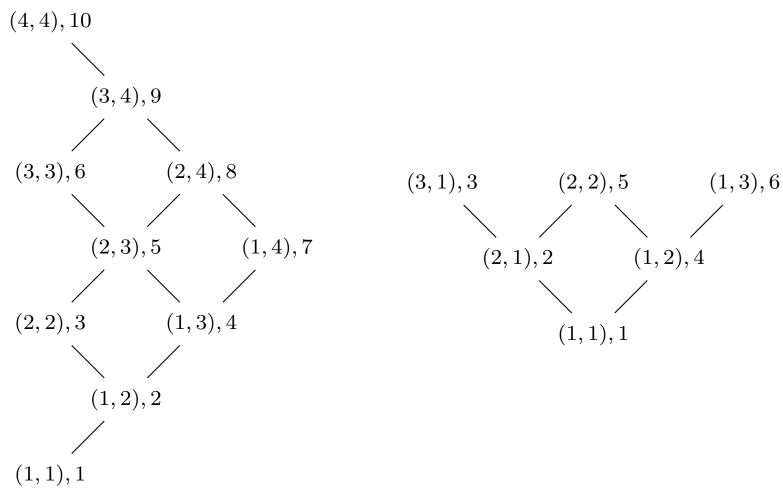


FIGURE 6. The elements of $\mathrm{Irr}(Q_5(132, 312))$ and $\mathrm{Irr}(\widetilde{Q}_8(123))$ along with their images under natural labelings.

To apply this process we will use the natural labeling of the irreducibles in both of our posets which is obtained by reading the cells (b, c) in each column of the corresponding triangular diagram, starting with the left-most column and moving to the right. Thus in $\mathrm{Irr}(Q_n(132, 321))$ this extension is given by

$$\omega(b, c) = \binom{c}{2} + b$$

and in $\mathrm{Irr}(\widetilde{Q}_n(123))$ for n even by

$$\omega(b, c) = \frac{(c-1)(n-c)}{2} + b.$$

Alternatively, one can think of both natural labeling as ordering the elements of the poset lexicographically with preference given to the second coordinate. Examples of these elements and their associated images are given in Figure 6, where the image is displayed beside each element. An application of the EL-labeling process appears for $\widetilde{Q}_8(123)$ in Figure 7. To simplify notation, we will often identify maximal chains $c : q_0 \lessdot q_1 \lessdot \cdots \lessdot q_k$ in $Q_n(132, 312)$ and $\widetilde{Q}_n(123)$ with their sequences of edge labels $\lambda(c) = (\lambda(q_0, q_1), \lambda(q_1, q_2), \ldots, \lambda(q_{k-1}, q_k))$.

We now take a first step in constructing a bridge from purely combinatorial information of these simplicial complexes to geometric information about $B_n(132, 312)$ and $\widetilde{B}_n(123)$.

Proposition 3.7. Let $f : \Delta(Q_n(132, 312)) \to \mathbb{R}^{n\times n}$ be the function

$$f(\{\sigma_1, \ldots, \sigma_u\}) = \mathrm{conv}\{M_{\sigma_1}, \ldots, M_{\sigma_u}\},$$

where M_{σ_i} is the matrix for σ_i. The collection

$$\mathcal{T}_n(132, 312) := \{f(\Gamma) \mid \Gamma \in \Delta(Q_n(132, 312))\}$$

is a set of simplices contained in $B_n(132, 312)$, where each $f(\Gamma)$ is unimodular with respect to the affine lattice $\mathrm{aff}(f(\Gamma)) \cap \mathbb{Z}^{n\times n}$. The collection $\widetilde{\mathcal{T}}_n(123)$, defined similarly, is a collection of unimodular simplices in $\widetilde{B}_n(123)$.

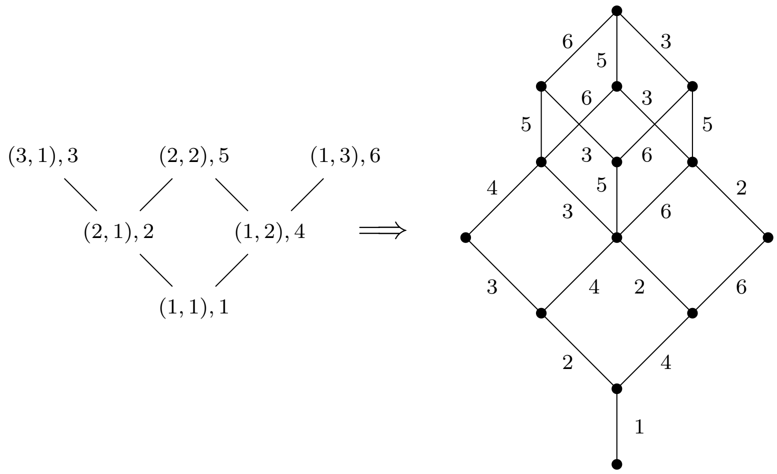


FIGURE 7. Producing an edge labeling on $\widetilde{Q}_8(123)$.

Proof. First we will focus on $\mathcal{T}_n(132, 312)$. Note that it is enough to prove the claim for the simplices in $\mathcal{T}_n(132, 312)$ of maximal dimension, since $\Gamma_1 \subseteq \Gamma_2$ in $\Delta(Q_n(132, 312))$ corresponds to an inclusion of faces $f(\Gamma_1) \subseteq f(\Gamma_2)$ in $\mathcal{T}_n(132, 312)$, and faces of unimodular simplices are again unimodular.

Arrange the maximal chains $c_1, \ldots, c_s$ in $Q_n(132, 312)$ lexicographically, and let $\Delta_q = \Delta(c_q)$ be the corresponding maximal simplex in $\Delta(Q_n(132, 312))$. We will prove our claim by induction on k.

First consider the simplex $f(\Delta_1)$. Using the grid class description of the permutations of $\mathrm{Av}_n(132, 312)$, given in Lemma 3.4, each matrix in $f(\Delta_1)$ is determined by restricting to the entries $m_{i,j}$ with $i < n - j + 1$. So, we take coordinate vectors for non-identity permutation matrices with respect to the order given by

$$\begin{bmatrix}
x_{\binom{n}{2}} & x_{\binom{n}{2}-1} & x_{\binom{n}{2}-2} & \cdots & x_{\binom{n-1}{2}+2} & x_{\binom{n-1}{2}+1} & * \\
x_{\binom{n-1}{2}} & x_{\binom{n-1}{2}-1} & x_{\binom{n-1}{2}-2} & \cdots & x_{\binom{n-2}{2}+1} & * & * \\
x_{\binom{n-2}{2}} & x_{\binom{n-2}{2}-1} & x_{\binom{n-2}{2}-2} & \cdots & * & * & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_6 & x_5 & x_4 & \cdots & * & * & * \\
x_3 & x_2 & * & \cdots & * & * & * \\
x_1 & * & * & \cdots & * & * & * \\
* & * & * & \cdots & * & * & *
\end{bmatrix}$$

and write them as columns of a matrix C, arranging the columns according to the order of elements in c_1.

Now let $\beta_{q,r}$ be the element of c_q of rank r. Because of how we defined our EL-labeling, consecutive elements in c_1 are of the form $\beta_{1,r} \lessdot \beta_{1,r} s_l$, where s_l is the lowest-indexed simple transposition that can be applied to $\beta_{1,r}$ which increases the number of inversions. It follows that after applying this operation t times, starting at the identity permutation ι, the corresponding matrix $M_{\beta_{1,t}}$ will have a 1 in position x_t and zeros in all higher indexed positions. Thus, C is an upper-unitriangular matrix. Furthermore, $\iota \in c_1$ is mapped to the zero vector by our coordinatization. Thus $f(\Delta_1)$ is a unimodular simplex and we have finished the base case of our induction. In particular, if we let L_1 be the affine span of $f(\Delta_1)$ then the vectors $f(\beta_{1,r}) - f(\iota)$ for $\beta_{1,r} \neq \iota$ in c_1 form a $\mathbb{Z}$-basis for the vector space $L_1 - f(\iota)$.

We will now consider the induction step which will be accomplished by showing that remaining maximal simplices in $\mathcal{T}_n(132, 312)$ are unimodular transformations of $f(\Delta_1)$. Recall that since $Q_n(132, 312)$ has an EL-labeling, each maximal chain c_q, $q > 1$, intersects some earlier maximal chain c_p such that they differ by a single element. So suppose c_q intersects with c_p such that $\sigma \in c_p - c_q$ and $\sigma' \in c_q - c_p$. Then σ and σ' are incomparable, and

$$\sigma \wedge \sigma' \lessdot \sigma, \sigma' \lessdot \sigma \vee \sigma'.$$

Because σ, σ' can each be obtained from simple transpositions applied to their meet, and these transpositions commute, the above relationship is captured by f via

$$f(\sigma \wedge \sigma') + f(\sigma \vee \sigma') = f(\sigma) + f(\sigma'). \tag{4}$$

We will use this relationship create a transformation $\varphi : L_p - f(\iota) \to L_q - f(\iota)$ by defining its images on the basis vectors $\beta_{p,r} - f(\iota)$ obtained from the inductive assumption. (The map φ implicitly depends on p and q even though that is not reflected in our notation.) This function will map $f(\Delta_p) - f(\iota)$ to $f(\Delta_q) - f(\iota)$, and since we will find that $\det \varphi = -1$, it will be a unimodular transformation. It follows that Δ_q is also unimodular with respect to the affine lattice $L_q \cap \mathbb{Z}^{n\times n}$.

For each r, set

$$\varphi(f(\beta_{p,r}) - f(\iota)) = f(\beta_{q,r}) - f(\iota).$$

If $\beta_{p,r} \in c_p \cap c_q$, then φ restricts to the identity on $\beta_{p,r} - f(\iota)$. Otherwise, consider the index t such that $\beta_{p,t} = \sigma$ and use equation (4) to write

$$\begin{aligned}
\varphi(f(\beta_{p,t}) - f(\iota)) &= f(\sigma') - f(\iota) \\
&= [f(\sigma \wedge \sigma') - f(\iota)] + [f(\sigma \vee \sigma') - f(\iota)] - [f(\sigma) - f(\iota)] \\
&= [f(\beta_{q,t-1}) - f(\iota)] + [f(\beta_{q,t+1}) - f(\iota)] - [f(\beta_{q,t}) - f(\iota)].
\end{aligned}$$

The way we defined φ on these basis vectors allows us to see that φ is a unimodular transformation: by taking coordinate vectors of the images $\varphi(f(\beta_{p,r}) - f(\iota))$ and arranging them according to the order of the $\beta_{p,r}$ in c_p, we see that the matrix for φ has determinant -1. Indeed, this matrix is identical to the identity matrix except in the column corresponding to σ. And in that column, because of the previously displayed equation, the only nonzero entries are a -1 on the main diagonal with a 1 just above it and another 1 just below. So, this matrix is unimodularly equivalent to the identity matrix, and Δ_q is a unimodular simplex with respect to $L_q \cap \mathbb{Z}^{n\times n}$.

We then apply induction, using the φ constructed above. Since Δ_1 is unimodular with respect to L, so are all of the images of the φ, and therefore so are all of the $f(\Delta_q)$. Thus, $\mathcal{T}_n(132, 312)$ is a collection of unimodular simplices.

The case of $\widetilde{B}_n(123)$ is similar. For $n = 2k$, let $\widetilde{\beta}_{q,r}$ be the element of $\widetilde{c}_q$ of rank r. Consecutive elements in $\widetilde{c}_1$ are of the form $\widetilde{\beta}_{1,r} \lessdot s_l\widetilde{\beta}_{1,r}$, where s_l is lowest-indexed simple transposition that can be applied to $\widetilde{\beta}_{1,r}$ which increases the number of inversions.

To most easily describe the order for which we will take coordinate vectors, we will arrange particular entries in a more easily-describable way. Recalling the equalities and inequalities from Lemma 3.4, we know that for any vertex $M = (m_{i,j})$ of $\widetilde{B}_n(123)$, the $m_{i,j}$ that contribute to the dimension of the polytope are $m_{n-i+1,j}$ where $j = 2c-1$ for $c = 1, \dots, k$ and $k-c+1 \leq i \leq 2(k-c)$. Arranging these $m_{i,j}$ in the triangle

$$\begin{array}{ccccccc}
m_{3,1} & & & & & & \\
m_{4,1} & m_{5,3} & & & & & \\
m_{5,1} & m_{6,3} & m_{7,5} & & & & \\
\vdots & \vdots & \vdots & \ddots & & & \\
m_{k-1,1} & m_{k,3} & m_{k+1,5} & \cdots & m_{n-5,n-7} & & \\
m_{k,1} & m_{k+1,3} & m_{k+2,5} & \cdots & m_{n-4,n-7} & m_{n-3,n-5} & \\
m_{k+1,1} & m_{k+2,3} & m_{k+3,5} & \cdots & m_{n-3,n-7} & m_{n-2,n-5} & m_{n-1,n-3}
\end{array}$$

then we take coordinate vectors of the matrices for the permutations in $\widetilde{c}_1$ with respect to the order

$$\begin{array}{ccccccc}
x_{\binom{k}{2}} & & & & & & \\
x_{\binom{k}{2}-2} & x_{\binom{k}{2}-1} & & & & & \\
x_{\binom{k}{2}-5} & x_{\binom{k}{2}-4} & x_{\binom{k}{2}-3} & & & & \\
\vdots & \vdots & \vdots & \ddots & & & \\
x_{2k-2} & x_{2k-1} & x_{2k} & \cdots & x_{3k-6} & & \\
x_{k} & x_{k+1} & x_{k+2} & \cdots & x_{2k-4} & x_{2k-3} & \\
x_{1} & x_{2} & x_{3} & \cdots & x_{k-3} & x_{k-2} & x_{k-1}
\end{array}$$

The maximal element of $\widetilde{c}_1$ maps to the origin, and the remaining permutation matrices map to the columns of a lower-unitriangular matrix. By creating $\widetilde{\varphi}$ as in the previous case, induction proves that $\widetilde{\mathcal{T}}_{2k}(123)$ is a collection of unimodular simplices in $\widetilde{B}_n(123)$. As usual, if n is odd, then use the isomorphism $\widetilde{Q}_n(123) \cong \widetilde{Q}_{n+1}(123)$ and proceed as in the case of even n. □

The simplices found in the previous proof have dimensions which are the lengths of the chains in the corresponding posets. So we have lower bounds $\binom{n}{2} \leq \dim B_n(132, 312)$ and $\binom{\lceil n/2 \rceil}{2} \leq \dim \widetilde{B}_n(123)$. Combined with Lemma 3.4, this gives the following result.

Theorem 3.8. For all n we have $\dim B_n(132, 312) = \binom{n}{2}$ and $\dim \widetilde{B}_n(123) = \binom{\lceil n/2 \rceil}{2}$. So the simplices in $\mathcal{T}_n(132, 312)$ are unimodular with respect to aff $B_n(132, 312) \cap \mathbb{Z}^{n \times n}$ and similarly for $\widetilde{B}_n(123)$. □

We would like to show that $\mathcal{T}_n(132, 312)$ and $\widetilde{\mathcal{T}}_n(123)$ are actually unimodular triangulations of their respective polytopes. To do so, we will use techniques from toric algebra, but first make the following note.

Not every choice of Π produces a $B_n(\Pi)$ with a unimodular triangulation, and an example will be given shortly. If a lattice polytope does have unimodular triangulation, then it follows quickly that it also has the following property.

Definition 3.9. A lattice polytope $P \subseteq \mathbb{R}^n$ is said to have the *integer decomposition property* (or *is IDP*) if, for all positive integers m and any $x \in mP \cap \mathbb{Z}^n$, there exist m points $x_1, \dots, x_m \in P \cap \mathbb{Z}^n$ such that $x = \sum x_i$.

To outline why the implication holds, suppose $v_0, \dots, v_n$ are the vertices of a unimodular simplex $S \subseteq \mathbb{R}^n$. Then $x \in mS \cap \mathbb{Z}^n$ if and only if $(x, m) \in \operatorname{cone}(S) \cap \mathbb{Z}^{n+1}$, where $\operatorname{cone}(S)$

denotes the cone in $\mathbb{R}^{n+1}$ whose ray generators are $(v_0,1),\dots,(v_n,1)$. Since S is a simplex, the lattice points in the cone are contained in a single translate of the monoid generated by $\{(v_0,1),\dots,(v_n,1)\}$, where the translates are uniquely determined by the lattice points in the half-open *fundamental parallelepiped*

$$\Phi_S := \{x \in \mathbb{R}^{n+1} \mid x = \sum_{i=0}^{n} \lambda_i (v_i,1) \text{ where } 0 \le \lambda_i < 1\}.$$

For example, given the 1-dimensional simplex $[-1,1]$, we see that $\Phi_{[-1,1]}$ contains two lattice points, which are $(0,0)$ and $(0,1)$. So, the lattice points of $\mathrm{cone}([-1,1])$ are in exactly one of the monoids $\mathbb{Z}_{\ge 0}\{(-1,1),(1,1)\}$ or $(0,1)+\mathbb{Z}_{\ge 0}\{(-1,1),(1,1)\}$.

The simplex S is unimodular if and only if Φ_S contains exactly one lattice point, which is necessarily 0. Thus the lattice points of $\mathrm{cone}(S)$ are exactly the elements of the single monoid $\mathbb{Z}_{\ge 0}\{(v_0,1),\dots,(v_n,1)\}$ which forces S to be IDP. It follows that a polytope with a unimodular triangulation must also be IDP.

Directly proving that a lattice polytope has the integer decomposition property is usually very difficult. It is more usually established as a byproduct of proving that the polytope has a unimodular triangulation, or simply a unimodular cover. As we will see in the next subsection, both $B_n(132,312)$ and $\widetilde{B}_n(123)$ are IDP. Although computer experiments suggest that it is common for $B_n(\Pi)$ to be IDP, the property is not guaranteed. For example, one can verify that

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 \end{bmatrix}$$

is a lattice point of $4B_5(2413,3124)$ but cannot be written as a sum of four lattice points from $B_5(2413,3124)$. This raises the following very broad question.

Question 3.10. For which choices of Π is $B_n(\Pi)$ IDP?

3.3. **Toric Algebra.** The methods we will use to show $\mathcal{T}_n(132,312)$ and $\widetilde{\mathcal{T}}_n(123)$ are unimodular triangulations of their respective polytopes require a bit of algebra background. First, let $\mathcal{A} = \{l_1,\dots,l_s\} \subseteq \mathbb{Z}^n$. We may define $k[\mathcal{A}] := k[x^{l_1},\dots,x^{l_s}]$, to be considered as contained in the ring of Laurent polynomials $k[x_1^{\pm},\dots,x_n^{\pm}]$, where k is a field and $x^{(v_1,\dots,v_n)} = \prod x_i^{v_i}$. It turns out that it is helpful to study $\mathcal{A}$ by first defining $T_{\mathcal{A}} = k[t_1,\dots,t_s]$ and the map $\phi : T_{\mathcal{A}} \to k[\mathcal{A}]$ by $\phi(t_i) = x^{l_i}$, since then we have

$$T_{\mathcal{A}}/\ker\phi \cong k[\mathcal{A}].$$

The ideal $I_{\mathcal{A}} := \ker\phi$ is the *toric ideal* of $\mathcal{A}$, and has been studied extensively in part due to its uses in algebraic statistics, algebraic geometry, and convex polytopes.

If P is an integral polytope then we set $\mathcal{A}_P = (P,1) \cap \mathbb{Z}^{n+1}$, and

$$k[\mathrm{cone}(P)] := k[x^a z^m \mid a \in mP \cap \mathbb{Z}^n] \subseteq k[x_1^{\pm},\dots,x_n^{\pm},z],$$

an algebra graded by the exponent of the new variable z. So when P is IDP we have $k[\mathrm{cone}(P)] = k[\mathcal{A}_P]$. However, this equality does not hold if P is not IDP, since then the monoid generated by $\mathcal{A}_P$ does not generate all elements of $\mathrm{cone}(P) \cap \mathbb{Z}^{n+1}$. To remedy this we have to introduce the *Hilbert basis* of cone(P), which is the unique minimal-cardinality set $\mathcal{H} \subseteq \text{cone}(P) \cap \mathbb{Z}^{n+1}$ such that every lattice point of cone(P) is a $\mathbb{Z}_{\geq 0}$-linear combination of elements of $\mathcal{H}$. The existence and uniqueness of the Hilbert basis can be proved using the Hilbert Basis Theorem.

This allows us to define the *toric ideal* I_P of a polytope P: Suppose the Hilbert basis of cone(P) is $\mathcal{H} = \{(v_1, w_1), \ldots, (v_r, w_r)\} \subseteq \mathbb{Z}^n \times \mathbb{Z}$. We have

$$T_{\mathcal{H}}/I_P \cong k[\text{cone}(P)],$$

where $I_P = \ker \phi$ is the toric ideal of P. So, if P is IDP, then $I_P = I_{\mathcal{A}_P}$, but in general we only have $I_P \supseteq I_{\mathcal{A}_P}$.

One significant advantage of studying the toric ideal of more general sets $\mathcal{A}$ is due to its ability to create triangulations of $\text{conv}\,\mathcal{A}$, using only the points from $\mathcal{A}$, under sufficient conditions. Specifically, if there is some $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$ such that $\nu^T l_i = 1$ for each $l_i \in \mathcal{A}$, we call $\mathcal{A}$ a *point configuration*, or simply a *configuration* if there is no risk of confusion. When $\mathcal{A}$ is a configuration, then the *positive span*

$$\text{pos}(\mathcal{A}) := \left\{ \sum_{i=1}^{s} \lambda_i l_i \mid \lambda_i \geq 0 \text{ for all } i \right\} \subseteq \mathbb{R}^n$$

is a polyhedral cone (differing from cone$(\mathcal{A}) \subseteq \mathbb{R}^{n+1}$) containing no positive-dimensional subspace, so a Hilbert basis exists. If $\mathcal{A}$ is not a configuration, then pos$(\mathcal{A})$ is still a cone but now contains a nontrivial subspace, so a Hilbert basis does not exist since a minimal generating set of $\text{pos}(\mathcal{A}) \cap \mathbb{Z}^n$ is no longer unique. Note that for any polytope P in $\mathbb{R}^n$, the set $\mathcal{A}_P$ is a configuration since it satisfies $e_{n+1}^T v = 1$ for each $v \in \mathcal{A}_P$.

Techniques from toric algebra will provide the tools for a critical step in proving that $B_n(132, 312)$ and $\widetilde{B}_n(123)$ are IDP by showing that the collections of simplices introduced in the previous section actually form unimodular triangulations of their respective polytopes. In particular, when P is one of these polytopes, we will use $I_{\mathcal{A}_P}$ to identify a triangulation of $\text{conv}\,\mathcal{A}_P$, using only the elements of $\mathcal{A}_P$. In this case, since P is a subpolytope of $[0,1]^{n \times n}$, it contains no lattice points other than its vertices. So, $\mathcal{A}_P$ consists exactly of the vertices of $(P, 1)$, and a triangulation of $\text{conv}\,\mathcal{A}_P$ is automatically a triangulation of $(P, 1)$, which in turn induces a triangulation of P by projecting each simplex back into $\mathbb{R}^{n \times n}$. The triangulation of P will be unimodular with respect to the lattice generated by $\mathbb{Z}$-linear combinations of the elements of P. Observing that this triangulation consists exactly of the simplices in $\mathcal{T}_n(132, 312)$ (respectively, $\widetilde{\mathcal{T}}_n(123)$), Theorem 3.8 will show that the triangulations are unimodular with respect to the affine lattice $B_n(132, 312) \cap \mathbb{Z}^{n \times n}$ (respectively, $\widetilde{B}_n(123) \cap \mathbb{Z}^{n \times n}$).

Returning to the general development, when $S \subseteq \mathbb{R}^n$ is a unimodular simplex, it is not difficult to show that $\mathcal{A}_S$ is the Hilbert basis of cone(S). When P is a general lattice polytope, we only know a priori that $\mathcal{A}_P$ must be contained in the Hilbert basis of cone(P). When a triangulation $\mathcal{T}$ of P is known, each lattice point $x \in \text{cone}(P)$ lies in cone(S) for some $S \in \mathcal{T}$. If S is unimodular, then x may be written as a sum of just the elements in $(S, 1) \cap \mathbb{Z}^{n+1} \subseteq \mathcal{A}_P$. Thus, if $\mathcal{T}$ is a unimodular triangulation, x can always be expressed as a sum of elements in $\mathcal{A}_P$, so $\mathcal{A}_P$ is exactly the Hilbert basis of cone(P). Therefore, in this case, any properties of $(\mathcal{T}, 1)$ as a unimodular triangulation with respect to $\text{aff}\,\mathcal{A}_P \cap \mathbb{Z}^{n+1}$ carry over to $\mathcal{T}$ as a unimodular triangulation of P.

Before continuing with toric ideals, let us first recall some additional definitions. Let Δ be an abstract simplicial complex on vertex set $\{v_1, \ldots, v_s\}$ and let $T = k[t_1, \ldots, t_s]$. The *Stanley-Reisner* ideal of Δ is

$$I_\Delta := (t_{i_1} \cdots t_{i_j} \mid \{i_1, \ldots, i_j\} \notin \Delta),$$

where the parentheses represent the ideal of T generated by these monomials. This definition leads us to the *Stanley-Reisner ring*, T/I_Δ, whose monomials are those with support corresponding to faces of Δ. The numerator of its Hilbert series is called the h-polynomial of Δ. If P is a polytope and Δ is a unimodular triangulation of P, then the h-polynomial of Δ and the h^*-polynomial of P coincide.

Note that the Stanley-Reisner ideal of a simplicial complex accounts for the combinatorial structure of the complex and does not inherently reflect any geometric properties. To overcome this limitation, we will express the Stanley-Reisner ideal as the result of operations on a different ideal, designed with geometric properties in mind.

Now, suppose $\prec$ is a monomial order on T, that is, a total well-ordering of the monomials of T which respects multiplication. Consider any ideal I of T. Each $f \in I$ then has an *initial* or *leading term* with respect to $\prec$, denoted $\mathrm{in}_\prec(f)$, which is the term of f that is greatest with respect to $\prec$. The *initial ideal* of I with respect to $\prec$ is the ideal generated by the initial terms of polynomials in I, that is,

$$\mathrm{in}_\prec(I) := (\mathrm{in}_\prec(f) \mid f \in I).$$

A *Gröbner basis* of I is a finite generating set $\mathcal{G}$ for I such that $\mathrm{in}_\prec(I) = (\mathrm{in}_\prec(g) \mid g \in \mathcal{G})$. Since I is assumed to be an ideal of a noetherian ring, a Gröbner basis always exists and may be computed from a given finite set of generators for I using the well-known Buchberger algorithm. Say $\mathcal{G}$ is *reduced* if each element has a leading coefficient of 1 and for any $g_1, g_2 \in \mathcal{G}$, $\mathrm{in}_\prec(g_1)$ does not divide any term of g_2. Given an ideal $I \subseteq T$ and a fixed monomial ordering on T, there are many Gröbner bases of I but there is exactly one reduced Gröbner basis of I.

There are many nice results connecting Gröbner bases with combinatorics, one of which involves types of triangulations that we define now. Suppose $P \subseteq \mathbb{R}^n$ is an n-dimensional lattice polytope and $P \cap \mathbb{Z}^n = \{l_1, \ldots, l_s\}$. Choose a vector $w = (w_1, \ldots, w_s) \in \mathbb{R}^s$ such that the polytope

$$P_w := \mathrm{conv}\{(l_1, w_1), \ldots, (l_s, w_s)\} \subseteq \mathbb{R}^{n+1}$$

is $(n+1)$-dimensional, i.e., P_w does not lie in an affine hyperplane of $\mathbb{R}^{n+1}$. Certain facets of P_w have outward-pointing normal vectors with a negative last coordinate; projecting these facets back to $\mathbb{R}^n$ provides the facets of a polytopal decomposition of P. If the facets are themselves simplices, then the decomposition is a triangulation. Any triangulation that can be obtained in this way by an appropriate choice of w is called *regular*, and will be denoted $\Upsilon_w(P)$.

There is a close connection between regular triangulations of $\mathrm{conv}(\mathcal{A})$, where $\mathcal{A} \subseteq \mathbb{Z}^n$ is a configuration of size s, and initial ideals of $I_\mathcal{A}$. First, we note that each monomial ordering $\prec$ on $T_\mathcal{A} = k[t_1, \ldots, t_s]$ can be represented by a sufficiently generic *weight vector* $w \in \mathbb{R}^s$ such that, for all $u, v \in \mathbb{Z}^s_{\geq 0}$, $t^u \prec t^v$ if and only if $w^T u < w^T v$. Next, we define the *initial complex* $\Delta_\prec(I)$ of an ideal $I \subseteq T_\mathcal{A}$ with respect to $\prec$ to be the simplicial complex on $[s]$ such that F is a face of $\Delta_\prec(I)$ if and only if there is no monomial in $\mathrm{in}_\prec(I)$ whose support is F. Using linear programming, one may show the following.

Theorem 3.11 (Theorem 8.3,[32])**.** Let $\mathcal{A} \subseteq \mathbb{Z}^n$ be a configuration. If w is the weight vector for a monomial order $\prec$ on $T_{\mathcal{A}}$, then $\Delta_\prec(I_{\mathcal{A}})$, an abstract simplicial complex, is geometrically the regular triangulation $\Upsilon_w(\mathrm{conv}(\mathcal{A}))$. □

Two other important connections given in [32] are summarized below.

Theorem 3.12 (Corollary 8.4 and Corollary 8.9, [32])**.** For any monomial order $\prec$ and corresponding weight vector w, the radical $\mathrm{rad}(\mathrm{in}_\prec(I_{\mathcal{A}}))$ is the Stanley-Reisner ideal of $\Upsilon_w(\mathrm{conv}(\mathcal{A}))$. Moreover, $\mathrm{in}_\prec(I_{\mathcal{A}})$ is squarefree if and only if $\Upsilon_w(\mathrm{conv}(\mathcal{A}))$ is unimodular with respect to the affine lattice generated by $\mathbb{Z}$-linear combinations of lattice points in $\mathcal{A}$. □

The triangulations $\mathcal{T}_n(132, 312)$ and $\widetilde{\mathcal{T}}_n(123)$ will turn out to have even more properties than those already discussed. A triangulation is called *flag* if its minimal nonfaces are edges. This may be detected algebraically by proving the existence of an initial ideal generated by squarefree quadratic monomials. We will demonstrate the flag property by taking the vertices of $P = B_n(132, 312)$ (respectively, $P = \widetilde{B}_n(123)$) and imposing the *graded reverse lexicographic (grevlex)* monomial ordering on $T_{\mathcal{A}_P}/I_{\mathcal{A}_P}$ induced from $Q_n(132, 312)$ (respectively, $Q = \widetilde{Q}_n(123)$) as follows. Let $T = k[t_1, \dots, t_s]$ and give the variables the total order $t_1 \succ t_2 \succ \cdots \succ t_s$. Given a monomial t^a we let $|a|$ denote the sum of the exponents. Grevlex extends the order on the variables to all monomials of $k[t_1, \dots, t_s]$ by insisting that $t^a \succ_{\text{grevlex}} t^b$ if $|a| > |b|$ or if both $|a| = |b|$ and the rightmost nonzero entry of $a - b$ is negative. To apply this to $T_{\mathcal{A}_P}$, we must first place an order on the vertices of P; for notational convenience, since our variables correspond to permutation matrices, we will frequently use the notation t_σ to denote the variable corresponding the matrix for the permutation σ. To define grevlex order on monomials in these variables, we must first specify the ordering of the variables themselves since their subscripts are not positive integers but rather permutations. We define $t_\sigma \succ_{\text{grevlex}} t_{\sigma'}$ if the permutation σ' lexicographically precedes σ as words which we denote by $\sigma' <_{\text{lex}} \sigma$. With no subscript, an inequality between permutations continues to represent the partial order in a poset. Even though working with grevlex order is a bit cumbersome at first, it is popularly used in applications since it is computationally efficient.

This allows us to define a *reverse lexicographic*, or *pulling*, triangulation of a lattice polytope P, which is any triangulation whose Stanley-Reisner ideal is $\mathrm{rad}\left(\mathrm{in}_{\prec_{\text{grevlex}}}(I_P)\right)$. Thus, a reverse lexicographic triangulation of P may be described as the triangulation whose maximal simplices are the projections of the appropriate facets of P_w where w is a weight vector for $\prec_{\text{grevlex}}$. See [17], for example, for a recursive geometric description of how to create reverse lexicographic triangulations.

Before we prove the main theorem of this section, we will need two more lemmas. Recall that a poset is *graded* if all of its maximal chains have the same length.

Lemma 3.13. Let M_σ denote the matrix corresponding to a permutation σ. For any σ, σ' that are both in $Q_n(132, 312)$ or in $\widetilde{Q}_n(123)$, we have

$$M_\sigma + M_{\sigma'} = M_{\sigma \wedge \sigma'} + M_{\sigma \vee \sigma'} \tag{5}$$

Proof. Our lattices are distributive and so graded in that all maximal chains have the same length. Let r and r' be the lengths of maximal chains in the intervals $[\sigma \wedge \sigma', \sigma]$ and $[\sigma \wedge \sigma', \sigma']$, respectively. Without loss of generality, we can assume $r \geq r'$. We induct on the pairs (r, r') in lexicographic order. The case when $r' = 0$ in trivial, and the case $(r, r') = (1, 1)$ is covered by equation (4). So assume $r \geq 2$ and take a permutation τ covered by σ in the interval $[\sigma \wedge \sigma', \sigma]$. First compare τ and σ'. By choice of τ, we have $\tau \wedge \sigma' = \sigma \wedge \sigma'$. And since the lattice is graded, the length of a maximal chain in $[\tau, \tau \vee \sigma']$ is r'. Comparing σ and $\tau \vee \sigma'$ we see that, since we are in a distributive lattice,

$$\sigma \wedge (\tau \vee \sigma') = (\sigma \wedge \tau) \vee (\sigma \wedge \sigma') = \tau.$$

Also clearly $\sigma \vee (\tau \vee \sigma') = \sigma \vee \sigma'$. Because of the way we have chosen r and r', we can apply induction to the pair τ, σ' and to the pair $\sigma, \tau \vee \sigma'$, giving

$$M_\tau + M_{\sigma'} = M_{\sigma\wedge\sigma'} + M_{\tau\vee\sigma'} \text{ and } M_\sigma + M_{\tau\vee\sigma'} = M_\tau + M_{\sigma\vee\sigma'}.$$

Adding these two equations and canceling finishes the proof. □

Lemma 3.14. For each permutation $\sigma = a_1 \ldots a_n$ define

$$\mu_i(\sigma) = \min\{a_1, \ldots, a_i\}.$$

Suppose $\sigma, \tau \in Q$ where $Q = Q_n(132, 312)$ or $Q = \widetilde{Q}_n(123)$. If $\sigma < \tau$ in Q, then $\mu_i(\sigma) \leq \mu_i(\tau)$ for all i.

Proof. The proof follows quickly by induction if we can prove it for $\sigma \lessdot \tau$. Clearly for any two sets of integers $A = \{a_1, \ldots, a_i\}$ and $B = \{b_1, \ldots, b_i\}$, if we have $a_j \leq b_j$ for all j then $\min A \leq \min B$. Now suppose $\sigma = a_1 \ldots a_n$ and $\tau = b_1 \ldots b_n$. Then τ was obtained from σ by interchanging two elements a_r and a_s where $r < s$ and $a_r < a_s$. So the prefixes of σ and τ satisfy the condition on integral sets above and we are done. □

We are now ready to prove the main result of this section.

Theorem 3.15. The sets $\mathcal{T}_n(132, 312)$ and $\widetilde{\mathcal{T}}_n(123)$ are regular, flag, unimodular reverse lexicographic triangulations of $B_n(132, 312)$ and $\widetilde{B}_n(123)$, respectively.

Proof. First consider $P = B_n(132, 312)$, and let $\mathcal{A} = P \cap \mathbb{Z}^{n\times n}$, so that $\mathcal{A}_P = \{(l, 1) \mid l \in \mathcal{A}\}$. Our strategy will be to construct the reduced Gröbner basis $\mathcal{G}$ of $I_{\mathcal{A}_P}$ with respect to $\prec = \prec_{\text{grevlex}}$. By Theorem 3.11, the initial complex $\Delta_\prec(I_{\mathcal{A}_P})$ is a regular triangulation $\Upsilon_w(\mathcal{A}_P)$ of $\operatorname{conv}(\mathcal{A}_P) = (P, 1)$, which induces a regular triangulation $\Upsilon_w(P)$ of P. We will see that $\mathcal{G}$ consists of binomials whose initial terms are products of distinct pairs of variables corresponding to incomparable elements of $Q_n(132, 312)$. Thus, by Theorem 3.12 and the comment directly afterwards, since $\text{in}_\prec(I_{\mathcal{A}_P})$ is the Stanley-Reisner ideal for $\Upsilon_w(P)$, the triangulation is flag and unimodular with respect to the affine lattice $\mathbb{Z}(P \cap \mathbb{Z}^{n\times n})$. By our description of the minimal non-faces of this triangulation, we will know that the simplices in $\Upsilon_w(P)$ are exactly the elements of $\mathcal{T}_n(132, 312)$. Since we saw in Proposition 3.7 that each $\Gamma \in \mathcal{T}_n(132, 312)$ is unimodular with respect to $(\operatorname{aff} \Gamma) \cap \mathbb{Z}^{n\times n}$, we have that $\mathcal{T}_n(132, 312)$ is actually a triangulation of P with respect to the lattice $(\operatorname{aff} P) \cap \mathbb{Z}^{n\times n}$. Because of how we defined $\prec$, the triangulation $\mathcal{T}_n(132, 312)$ is reverse lexicographic as well.

Consider the set of monomials $t_\sigma t_{\sigma'}$ in $T_{\mathcal{A}_P}$ such that σ and σ' are incomparable in $Q_n(132, 312)$. Because of equation (5), we know that $t_\sigma t_{\sigma'} - t_{\sigma\wedge\sigma'} t_{\sigma\vee\sigma'} \in I_{\mathcal{A}_P}$. By the way we defined $\prec$, the smaller of the two terms is the one containing $t_{\sigma\wedge\sigma'}$. Thus $t_\sigma t_{\sigma'}$ is the initial term of the binomial. Since this monomial is quadratic, it must be the initial term of some binomial in $\mathcal{G}$. It quickly follows from the definition of a reduced Gröbner basis that there can be no binomial in $\mathcal{G}$ of degree 3 or greater whose initial term contains a pair of variables $t_\rho, t_{\rho'}$ corresponding to incomparable elements ρ, ρ' in $Q_n(132, 312)$. Otherwise, this initial term would be divisible by $t_\rho t_{\rho'}$, which is itself an initial term of a binomial in $\mathcal{G}$.

Now we will show that there are no binomials of degree 2 or greater in $\mathcal{G}$ with initial term $t_{\sigma_1}^{u_1} \dots t_{\sigma_r}^{u_r}$ such that $\sigma_1 < \cdots < \sigma_r$ in $Q_n(132, 312)$. If we assume there is such a binomial, let $t_{\sigma'_1}^{v_1} \dots t_{\sigma'_r}^{v_r}$ the other term in the binomial. Because this term is not initial, there is some variable, which we may take to be $t_{\sigma'_1}$, such that that $t_{\sigma'_1} \succ_{\text{grevlex}} t_{\sigma_i}$ for all i. So, by definition of this monomial order, $\sigma'_1 <_{\text{lex}} \sigma_i$ for all i. Letting $\sigma_1 = a_1 \dots a_n$ and $\sigma'_1 = c_1 \dots c_n$, denote by j the smallest index for which $c_j < a_j$. Since we know

$$\sum_{i=1}^{r} u_i M_{\sigma_i} = \sum_{i=1}^{r} v_i M_{\sigma'_i},$$

there is some other $\sigma_p = b_1 \dots b_n$ for which $b_j = c_j$ and $\sigma_1 < \sigma_p$.

We will show c_j is equal to some element in $c_1 \dots c_{j-1} = a_1 \dots a_{j-1}$ and so σ'_1 is not a permutation, the desired contradiction. Using Lemma 3.14 and the definition of j we have

$$\mu_j(\sigma_1) \leq \mu_j(\sigma_p) \leq b_j = c_j < a_j.$$

But from the grid class description of $\mathrm{Av}_n(132, 312)$ it is clear that any prefix of σ_1 forms an interval. So the above inequalities show that $c_j \in \{a_1, \dots, a_{j-1}\}$ as promised. We have shown that the binomials in $\mathcal{G}$ have initial terms that are products of variables that correspond to pairwise incomparable elements in $Q_n(132, 312)$. So, the initial ideal of $I_{\mathcal{A}_P}$ is radical and therefore, by Theorem 3.12, is the Stanley-Reisner ideal of a regular triangulation of $\mathrm{conv}(\mathcal{A}_P)$ which induces a triangulation of $\mathrm{conv}(\mathcal{A}) = P$.

Since the minimal non-edges of the triangulation are pairs of incomparable elements, any chain $\sigma_1 < \cdots < \sigma_r$ in $Q_n(132, 312)$ induces a face $\{M_{\sigma_1}, \dots, M_{\sigma_r}\}$ of the triangulation. The set of all such faces is exactly $\mathcal{T}_n(132, 312)$, so $\mathcal{T}_n(132, 312)$ is actually a regular triangulation of $B_n(132, 312)$. By Proposition 3.7, this triangulation is unimodular with respect to $(\mathrm{aff}\, P) \cap \mathbb{Z}^{n\times n}$, and since the minimal non-faces are edges, this triangulation is flag. Because this triangulation was the result of taking an initial ideal with respect to a grevlex order, the triangulation is reverse lexicographic.

The same proof will work in the case of $\widetilde{B}_n(123)$ except during the demonstration that σ'_1 is not a permutation were we used the grid class structure of $\mathrm{Av}_n(132, 312)$. Instead, we show that there is no such σ'_1 in $\widetilde{Q}_n(123)$ as follows. If c_j occurs among $a_1, \dots, a_{j-1}$ then we are done as before. Otherwise, a_j must occur to the right of c_j in σ'_1. Recall that applying a simple transposition s_i to an element of $\widetilde{Q}_n(123)$ interchanges i which is in odd position with $i+1$ which is in an even position. It follows that elements in odd positions increase with the partial order while those in even positions decrease. Since $a_j > b_j$, we must have j even. If a_j occurs in an even position to the right of c_j in σ'_1, then we have a contradiction since $c_j < a_j$ are the elements in even positions form a decreasing sequence. If a_j is in an odd position, then $c_{j-1} > a_j$ since the elements in odd positions are also decreasing. But then $c_{j-1} > a_j > c_j$ which contradicts the fact that σ'_1 is alternating. This final contradiction finishes the proof. □

Because the triangulations above were obtained using the grevlex order, Corollary 2.5 of [28] gives us

$$h^*(B_n(132, 312)) = h(\mathcal{T}_n(132, 312)) = h(\Delta(Q_n(132, 312))),$$

and likewise for $\widetilde{B}_n(123)$. This fact will come into play in the final section when making statements about the components of h^*-vectors for our polytopes.

It is worth commenting that the work done in proving Proposition 3.7 and Theorem 3.15 is unnecessary if one can show that $B_n(\Pi)$ is (unimodularly equivalent to) an order polytope for some graded poset. In this case, it was shown in [15] that the toric ideal of the polytope has quadratic Gröbner bases with squarefree initial terms. It follows that the initial ideals are squarefree, and $B_n(\Pi)$ therefore has a regular, unimodular triangulation. However, it is currently unclear whether $B_n(132, 312)$ and $\widetilde{B}_n(123)$ fall into this situation.

4. The Ehrhart Theory of $B_n(132, 312)$ and $\widetilde{B}_n(123)$

The previous section identified shellable, regular, unimodular triangulations of $B_n(132, 312)$ and $\widetilde{B}_n(123)$ which arose from order complexes of certain distributive lattices; in this section, we use the EL-labelings of the lattices to study the h^*-vectors of the polytopes. To do so, we require some more definitions and background.

Suppose $P \subseteq \mathbb{R}^n$ is a lattice polytope containing the origin in its interior. We say that P is *reflexive* if its polar dual

$$P^\vee := \{x \in \mathbb{R}^n \mid x^T y \leq 1 \text{ for all } y \in P\}$$

is also a lattice polytope. Any lattice translate of a reflexive polytope is also called reflexive. A lattice polytope P is said to be *Gorenstein* if kP is reflexive for some k, called the *index*. A theorem, due to Stanley, describes exactly the behavior of h^*-vectors for Gorenstein polytopes.

Theorem 4.1 (Theorem 4.4, [27])**.** A lattice polytope is Gorenstein if and only if its h^*-vector is palindromic.

We can use this result together with the following facts about h^*-vectors to determine necessary conditions for P to be Gorenstein. Let $h^*(P) = (h^*_0, \ldots, h^*_d)$ where P is any lattice polytope. We always have $h^*_0 = 1$. Additionally, as a consequence of Ehrhart-Macdonald reciprocity, the first scaling of P containing an interior lattice point is $(\dim P - d + 1)P$, and the number of interior lattice points in this scaling is h^*_d. Since a Gorenstein polytope has a palindromic h^*-vector, then in order to be Gorenstein, the first scaling of P with an interior lattice point must have exactly one such point.

Note that not every set of permutations Π will produce a Gorenstein $B_n(\Pi)$. Take, for example, $\Pi = \{123, 132\}$ and $n = 5$. One may verify that the first nonnegative integer scaling $mB_n(123, 132)$ containing an interior lattice point occurs when $m = 8$, but this scaling contains four interior lattice points rather than the one needed to be Gorenstein.

The main goal of this section will be to prove the following theorem.

Theorem 4.2. For all n, $B_n(132, 312)$ and $\widetilde{B}_n(123)$ are Gorenstein.

If the hyperplane description of a lattice polytope is known, then proving whether it is Gorenstein is often a straightforward task. Such a description of $B_n(132, 312)$ and $\widetilde{B}_n(123)$ has been elusive, though, so we must approach the proof of Theorem 4.2 by showing that their h^* vectors are palindromic and then appealing to Theorem 4.1.

One benefit of going through the work of the previous section is that once a Gorenstein polytope is known to have a regular, unimodular triangulation, it follows that the h^*-vector of the polytope is unimodal in addition to being palindromic [8]. Thus, using Theorem 4.2, the regular unimodular triangulations $\mathcal{T}_n(132,312)$ and $\widetilde{\mathcal{T}}_n(123)$, as well as the EL-labelings of $Q_n(132,312)$ and $\widetilde{Q}_n(123)$, we will be able to establish that the h^*-vectors of these two polytopes are palindromic and unimodal.

We first need to recall some results about shellable triangulations. In such a triangulation with shelling order $F_1,\dots,F_s$, the *restriction* of face F_j is the set $\mathcal{R}(F_j)$ of vertices $v \in F_j$ such that the facet $F_j - v$ is contained in $F_1 \cup \cdots \cup F_{j-1}$. The *shelling number* of F_j is $r(F_j) = |\mathcal{R}(F_j)|$. The following result of Stanley shows that the entries of the h^*-vector of the polytope being shelled can be computed using shelling numbers.

Proposition 4.3 (Corollary 2.6, [28])**.** Suppose that $T_1,\dots,T_k$ is a shelling order of a unimodular triangulation of a lattice polytope P. Then the component h_i^* of $h^*(P)$ is equal to the number of simplices T_j such that $r(T_j) = i$. $\square$

When using EL-shellings, there is an easy way to determine the shelling number of a facet, that is, of a maximal chain c, from its labeling. In particular, if

$$\lambda(c) = (\lambda(q_0,q_1), \lambda(q_1,q_2), \dots, \lambda(q_{k-1},q_k))$$

then $q_m \in \mathcal{R}(c)$ if and only if we have a descent $\lambda(q_{m-1},q_m) > \lambda(q_m,q_{m+1})$ in $\lambda(c)$. This is the content of the following lemma of Björner.

Lemma 4.4 (Lemma 2.6, [6])**.** Let c be a maximal chain of the poset P admitting an EL-labeling λ. Then

$$r(c) = \operatorname{des} \lambda(c)$$

where des is the number of descents.

The last link in our chain will come from a result in the theory of (Q,ω)-partitions as developed by Stanley. A fuller exposition can be found in Chapter 3 of his book [30]. Let Q be a poset with $|Q| = n$, and let $\omega : Q \to [n]$ be a bijection, called a *labeling* of Q. We say $f : Q \to \mathbb{Z}_{\geq 1}$ is a *(dual)* (Q,ω)*-partition* if

(i) f is order preserving, and
(ii) if $s < t$ and $\omega(s) > \omega(t)$, then $f(s) < f(t)$.

In a sense one may think of ω as indicating where strict inequalities of f occur, rather than weak inequalities. If ω itself is order-preserving then, as we have already seen, it is called a *natural* labeling of Q. We call ω *dual natural* if its *dual labeling* $\overline{\omega} : Q \to [n]$, defined by the complementation $\overline{\omega}(q) = n+1-q$, is natural.

We will be concerned with the *order polynomial* of (Q,ω), denoted $\Omega_{Q,\omega}(m)$, which is the number of maps $f : Q \to [m]$ which satisfy conditions (i) and (ii) above. It can be shown that $\Omega_{Q,\omega}(m)$ is a polynomial in m of degree $n = |Q|$. Equivalently, the generating function for the order polynomial must be in the form

$$\sum_{m\geq 0} \Omega_{Q,\omega}(m) t^m = \frac{A_{Q,\omega}(t)}{(1-t)^{n+1}}$$

where $A_{Q,\omega}(t)$ is a polynomial of degree at most n called the *Eulerian polynomial* of (Q,ω). In fact, one can give an explicit description of $A_{Q,\omega}(t)$ as follows. Define the *Jordan-Hölder set* of (Q,ω), denoted $\mathcal{L}(Q,\omega)$, to be the set of all permutations of the form $w = \omega(q_1)\omega(q_2)\dots\omega(q_n)$ as $q_1,q_2,\dots,q_n$ runs over all linear extensions of Q, that is, total orders on Q such that if $q_i < q_j$ in Q then $i < j$.

Theorem 4.5 (Theorem 3.15.8, [30])**.** We have

$$\sum_{m\geq 0}\Omega_{Q,\omega}(m)t^m = \frac{\sum_{w\in\mathcal{L}(Q,\omega)} t^{1+\operatorname{des} w}}{(1-t)^{n+1}}$$

where $n = |Q|$. □

Our next goal is to show that under certain conditions $A_{Q,\omega}(t)$ is palindromic. To do this, we will need a trio of results. Since $\Omega_{Q,\omega}(m)$ is a polynomial it makes sense to talk about its value at a negative argument. Also, there are many properties of the order polynomial which are true for all natural labelings ω. In this case, we shorten $\Omega_{Q,\omega}$ to Ω_Q and similarly for other notation.

Theorem 4.6 (Corollaries 3.15.12 and 3.15.18, [30])**.** Let Q be a poset with $|Q| = n$ and longest chain of length l.

(A) (Reciprocity theorem for order polynomials) For all $m \in \mathbb{Z}$

$$\Omega_{Q,\overline{\omega}}(m) = (-1)^n\Omega_{Q,\omega}(-m).$$

(B) If ω is natural then

$$\Omega_Q(0) = \Omega_Q(-1) = \cdots = \Omega_Q(-l) = 0.$$

(C) Suppose ω is natural. The poset Q is graded if and only if

$$\Omega_Q(m) = (-1)^n\Omega_Q(-m-l)$$

for all $m \in \mathbb{Z}$. □

Theorem 4.7. Let Q be a poset and let ω be a natural labeling of Q. Then the Eulerian polynomial $A_Q(t)$ is palindromic if and only if Q is graded.

Proof. We will prove the backwards direction as going forwards is similar. We will use $\overline{Q}$ as an abbreviation for $(Q,\overline{\omega})$. We also conserve the notation of the previous result. Using Theorem 4.6 (A), Theorem 4.5, and the definition of $\overline{\omega}$ in turn we get

$$(-1)^n\sum_{m\geq 0}\Omega_Q(-m)t^m = \sum_{m\geq 0}\Omega_{\overline{Q}}(m)t^m = \frac{\sum_{w\in\mathcal{L}(\overline{Q})} t^{1+\operatorname{des} w}}{(1-t)^{n+1}} = \frac{\sum_{w\in\mathcal{L}(Q)} t^{n-\operatorname{des} w}}{(1-t)^{n+1}}.$$

On the other hand, using parts (B) and (C) of the previous result and then Theorem 4.5 again gives

$$(-1)^n\sum_{m\geq 0}\Omega_Q(-m)t^m = \sum_{m\geq 0}\Omega_Q(m-l)t^m = t^l\sum_{m\geq 0}\Omega_Q(m)t^m = \frac{\sum_{w\in\mathcal{L}(Q)} t^{l+1+\operatorname{des} w}}{(1-t)^{n+1}}.$$

Comparison of the final numerators in the last two series of displayed equalities implies that $A_Q(t)$ is a palindrome, as desired. □

We now have all our tools in place. The following result, together with Theorem 4.1 proves Theorem 4.2.

Theorem 4.8. The vectors $h^*(B_n(132,312))$ and $h^*(\widetilde{B}_n(123))$ are palindromic for all n.

Proof. We will only deal with the case of $B_n(132, 312)$ as $\widetilde{B}_n(123)$ is similar. Let $Q = \mathrm{Irr}(Q_n(132, 312))$ and note that Q is graded. Let ω be the natural labeling of Q used in the EL-shelling of $Q_n(132, 312)$. Combining Proposition 4.3, Lemma 4.4, and Theorem 4.7, we see that it suffices to show that

$$\mathcal{L}(Q) = \{\lambda(c) \mid c \text{ a maximal chain in } Q_n(132, 312)\}.$$

But this follows since $Q_n(132, 312) = J(Q)$ so that linear extensions $q_0, q_1, q_2, \ldots$ of Q are in bijective correspondence with maximal chains $q_0 \lessdot q_0 \vee q_1 \lessdot q_0 \vee q_1 \vee q_2 \lessdot \ldots$ of $Q_n(132, 312)$, and we are using the same function ω to label both the elements of Q and the covers in the chain. □

Corollary 4.9. The vectors $h^*(B_n(132, 312))$ and $h^*(\widetilde{B}_n(123))$ are unimodal.

Proof. For each n, $B_n(132, 312)$ and $\widetilde{B}_n(123)$ have regular, unimodular triangulations by Theorem 3.15 and are Gorenstein by Theorem 4.2. By the main result of [8], the h^*-vectors for each polytope are h-vectors for boundaries of simplicial polytopes, that is, they are unimodal. □

Corollary 4.10. The normalized volume of $B_n(132, 312)$ is

$$\operatorname{Vol} B_n(132, 312) = \binom{n}{2}! \frac{\prod_{i=1}^{n-1} (i-1)!}{\prod_{i=1}^{n-1} (2i-1)!}$$

The normalized volume of $\widetilde{B}_n(123)$ is

$$\operatorname{Vol} \widetilde{B}_n(123) = \binom{k}{2}! \frac{1}{\prod_{i=1}^{k-1} (2i-1)^{k-i}},$$

where $k = \lceil n/2 \rceil$.

Proof. Since $\mathcal{T}_n(132, 312)$ and $\widetilde{\mathcal{T}}_n(123)$ are unimodular triangulations of $B_n(132, 312)$ and $\widetilde{B}_n(123)$, the normalized volumes of the polytopes are the total number of maximal simplices in the respective triangulations. These are enumerated by counting the maximal chains in $Q_n(132, 312)$ and $\widetilde{Q}_n(123)$, which are in bijection with shifted SYT of shape $(n-1, \ldots, 1)$ and left-justified SYT of shape $(k-1, \ldots, 1)$. Such tableaux are counted by the well-known hook formulas, established in [33] and [14]. □

Acknowledgement. We thank Richard Stanley for pointing out that the theory of (P, ω)-partitions could be used to obtain Theorem 4.7.

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