STOCHASTIC HOMOGENIZATION OF HAMILTON-JACOBI EQUATIONS IN STATIONARY ERGODIC SPATIO-TEMPORAL MEDIA

RUSSELL SCHWAB

ABSTRACT. This paper considers the problem of homogenization of a class of convex Hamilton-Jacobi equations in spatio-temporal stationary ergodic environments. Special attention is placed on the interplay between the use of the Subadditive Ergodic Theorem and continuity estimates for the solutions that are independent of the oscillations in the equation. Moreover, an inf-sup formula for the effective Hamiltonian is provided.

1. Introduction

In this paper we analyze the behavior, as $\varepsilon \to 0$, of the family of initial value problems

$$\begin{cases} u^\varepsilon_t + H(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, Du^\varepsilon, \omega) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u^\varepsilon = u^\varepsilon_0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.1}$$

Here we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and for each $\omega \in \Omega$, $u^\varepsilon(\cdot,\cdot,\omega) \in BUC(\mathbb{R}^n \times [0,T])$ is the unique viscosity solution of (1.1). We show that for a set of full measure, $\tilde{\Omega} \subset \Omega$, solutions of (1.1) converge locally uniformly to the solution of the "averaged" equation

$$\begin{cases} u_t + \bar{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = u_0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.2}$$

Until recently, stochastic homogenization results for such equations had been limited to settings in which there was only oscillatory dependence in space, not in time and space simultaneously. New results were proved in [18] for a "viscous", second order version of (1.1). For (1.1), the main obstruction to proving homogenization results with respect to time and space variables had been the lack of a priori regularity of the solutions. These difficulties are overcome in this work.

We prove this homogenization result for general Hamiltonians that are convex and superlinear in the gradient argument. The following key assumptions are required on the Hamiltonian for proving the above homogenization. They are relevant to extracting some sort of "averaging" behavior of u^ε and for utilizing the Hopf-Lax formula.

- (H1) (Stationarity and Ergodicity) There exists a group of measure preserving transformations on Ω, $\tau_{(x,t)} : \Omega \to \Omega$ for all $(x,t) \in \mathbb{R}^{n+1}$, such that τ is ergodic. Moreover, H is stationary with respect to τ:

$$H(x+y, s+t, p, \omega) = H(x, s, p, \tau_{(y,t)}\omega). \tag{1.3}$$

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The transformation, τ, is said to be ergodic if all subsets of Ω that are invariant under its action have either measure 0 or 1. That is

$$\text{if } \tau_{x,t}A = A \text{ for all } (x,t) \in \mathbb{R}^{n+1}, \text{ then } \mathbb{P}(A) = 1 \text{ or } \mathbb{P}(A) = 0.$$

- (H2) (Convexity) For each ω, x, and t, $H(x,t,p,\omega)$ is convex in its p argument.
- (H3) (Superlinear Growth) There exist constants $C_1 > 0$, $C_2 > 0$, $\alpha_1 > 1$ and $\alpha_2 \geq \alpha_1$ such that for all x, t, p, ω
$$C_1(|p|^{\alpha_1} - 1) \leq H(x,t,p,\omega) \leq C_2(|p|^{\alpha_2} + 1). \tag{1.4}$$
Equivalently, this can be stated in terms of the Legendre transform:
- (L3) There exist constants $B_1 > 0$, $B_2 > 0$, $\beta_2 = \alpha_2/(\alpha_2 - 1)$, $\beta_1 = \alpha_1/(\alpha_1 - 1)$ (the conjugate exponents of α_2 and α_1) such that
$$B_2(|p|^{\beta_2} - 1) \leq H^*(x,t,p,\omega) \leq B_1(|p|^{\beta_1} + 1). \tag{1.5}$$

We also require a technical assumption which comes from [18]; it is used for interchanging the function H with a mollification of the gradient argument (stated for H and its Legendre transform).

- (H4) There exists a modulus, m, and a positive constant, $C > 0$, such that for $|(x,t)| \leq \gamma$
$$H(x,t,p,\omega) \geq (1 + m(\gamma))H\Big(0, 0, \frac{p}{1 + m(\gamma)}, \omega\Big) - Cm(\gamma), \tag{1.6}$$
- (L4)
$$H^*(x,t,p,\omega) \leq (1 + m(\gamma))H^*(0,0,p,\omega) + Cm(\gamma). \tag{1.7}$$

Assumption (H4) is a more general form of an infinitesimal assumption which has been used in other works. If H were Lipschitz, assumptions of this form have appeared, for example, in [4] and [20]. They state that for some $C > 0$,

$$|H_x| + |H_t| + H_p(p) \cdot p - H + C \geq 0.$$

In addition to the assumptions that are directly related to homogenization, we require the standard assumptions that give existence and uniqueness of solutions of (1.1). Uniqueness results for (1.1) in which the Hamiltonian is not globally Lipschitz in the gradient argument can be found in [3]. We refer to there for a list of such assumptions. Above that, we also require for each ω fixed,

- (H5) (Coercivity) Uniformly in x and t, $\lim_{|p|\to\infty} H(x,t,p,\omega) = +\infty$.

In order to state the main result, we need one more piece of information. To identify $\bar{H}$, we need the class of test functions,

$$\mathcal{S} := \Big\{\Phi \Big| \; a.s. \omega \lim_{|(x,t)|\to\infty} \frac{\Phi(x,t,\omega)}{|(x,t)|} = 0; \; D\Phi, \Phi_t \in L^\infty_{loc}(\mathbb{R}^{n+1});$$
$$D\Phi, \Phi_t \text{ are stationary and mean zero}\Big\}.$$

Under these assumptions, we have the following theorem which answers the question of the above convergence of u^ε and "averaging" of (1.1).

Theorem 1.1. *Assume that H satisfies (H1) -(H5) above. Then there exists a coercive, convex function, $\bar{H}$, satisfying the bounds in (1.4) and a set, $\tilde{\Omega} \subset \Omega$, with $\mathbb{P}(\tilde{\Omega}) = 1$, such that:*

(i) *$u^\varepsilon(\cdot,\cdot,\omega) \to u$ locally uniformly as $\varepsilon \to 0$ for all $\omega \in \tilde{\Omega}$, where $u^\varepsilon(\cdot,\cdot,\omega), u \in BUC(\mathbb{R}^n \times [0,T])$ are the unique viscosity solutions of (1.1) and (1.2) respectively, with initial data $u_0^\varepsilon, u_0 \in BUC(\mathbb{R}^n)$, and $u_0^\varepsilon(\cdot,\omega) \to u_0$ locally uniformly, for each ω.*

(ii) $\bar{H}$ *is given by the formula:*

$$\bar{H}(p) = \inf_{\Phi \in \mathcal{S}} \sup_{x,t \in \mathbb{R}^{n+1}} \Big[\Phi_t(x,t,\omega) + H(x,t,p + D\Phi(x,t,\omega),\omega) \Big].$$

A major ingredient in the proof of Theorem 1.1 is a priori continuity estimates for the solutions of (1.1). We prove such estimates hold for specific perturbations of (1.1), and we believe they are of independent interest. They are stated here as an independent proposition.

Proposition 1.2. *Suppose that H is convex and the Legendre transform of H has the form:*

$$H^*(x,t,p,\omega) = f(x,t,p,\omega) + \delta |p|^\beta, \tag{1.8}$$

where f is convex in p, and there are constants A_1, A_2, and $\alpha > 0$ such that

$$-A_1 \le f(x,t,p,\omega) \le A_2(1 + |p|^\alpha),$$

$\alpha < \beta$, and $\beta > 1$. Suppose further that $u_0^\varepsilon \in C^{0,1}(\mathbb{R}^n)$. Then there exists a modulus of continuity, M, which depends only on A_1, A_2, δ, α, β, and $\|u_0^\varepsilon\|_{C^{0,1}(\mathbb{R}^n)}$ such that for all ε and x, t, y, s

$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \le M(|x-y| + |t-s|).$$

For definitions and standard results regarding viscosity solutions of (1.1) and (1.2), the reader should consult [8], [9], [20], and [2]. For a good introduction to homogenization in the context of linear equations, the reader should consult the books, [5] and [16].

The strategy of proving Theorem 1.1 is to analyze the solutions, u^ε, through their Hopf-Lax formulas:

$$u^\varepsilon(x,t,\omega) = \inf_{y \in \mathbb{R}^n} \Big\{ u_0(y) + L^\varepsilon(x,t,y,0,\omega) \Big\}.$$

Here L^ε is thought of as the "fundamental solution" of (1.1) (see [20]). It can be viewed as either the solution of (for y and s fixed)

$$\begin{cases} L^\varepsilon_t + H(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, D_x L^\varepsilon, \omega) = 0 & \text{in } \mathbb{R}^n \times (s,T) \\ L^\varepsilon(x,t,y,t,\omega) = +\infty \text{ if } x \ne y \text{ and } 0 \text{ if } x = y & \text{on } \mathbb{R}^n \times \{s\}, \end{cases} \tag{1.9}$$

or alternatively as the optimal pointwise cost to travel from y to x on the interval $[s,t]$ through the environment described by H^*:

$$L^\varepsilon(x,t,y,s,\omega) = \inf \left\{ \int_s^t H^*(\frac{\xi(r)}{\varepsilon}, \frac{r}{\varepsilon}, \dot{\xi}(r), \omega) dr : \xi \in W^{1,\infty}([s,t];\mathbb{R}^n), \xi(s) = y, \xi(t) = x \right\}.$$

In the case of (1.2), this formula reduces to

$$\bar{L}(x,t,y,s) = (t-s)\bar{H}^*\Big(\frac{x-y}{t-s}\Big).$$

As pointed out in [13] for the case of a periodic Hamiltonian, the Hopf-Lax formula gives a connection between the homogenization of (1.1) and the Gamma-convergence of the functionals on $W^{1,\infty}([0,t];\mathbb{R}^n)$:

$$F_\varepsilon(\xi) = \int_0^t H^*(\frac{\xi(r)}{\varepsilon}, \frac{r}{\varepsilon}, \dot{\xi}(r), \omega) dr.$$

These functionals in the almost periodic setting are considered in [6]. Similar functionals and their Gamma-convergence for the stochastic setting are treated in [11] and [12].

Our strategy will be similar in that we wish to extract pointwise convergence of the "fundamental solutions",

$$L^\varepsilon(x,y,t,\omega) \to t\bar{L}(\frac{x-y}{t}).$$

We then sufficiently strengthen this convergence to be able to conclude that

$$u^\varepsilon(x,t,\omega) \to \bar{u}(x,t) = \inf_y \left\{ u_0(y) + t\bar{L}(\frac{x-y}{t}) \right\} ,$$

in $C(\mathbb{R}^n \times [0,T])$. This yields the theorem because this is the unique solution of (1.2) when $\bar{H}$ is taken to be the Legendre transform of $\bar{L}$. The first convergence question will be answered using the Subadditive Ergodic Theorem of [1]. In order to conclude that the limiting "fundamental solution" is independent of ω, one must have some sort of continuity of L^ε in ε. Uniform continuity of L^ε in x, y, t that is independent of ε and ω is sufficient. It is a very interesting question as to whether it is also necessary. The second convergence will follow from the first in the presence of a priori continuity estimates on L^ε.

An application of the Subadditive Ergodic Theorem in similar contexts typically requires some sort of additional information for both concluding the limit is independent of ω and also for extending the domain of definition of the limiting function from a countably dense subset of $\mathbb{R}^n$ to all of $\mathbb{R}^n$ (we will see later that the limit naturally only holds simultaneously for a.e. ω on a countable domain of spacial inputs). These problems were overcome, for example, in works of [17] and [27] by using the fact that the relevant "L^ε" is actually a distance on $\mathbb{R}^n$. This is not the case for our "fundamental solutions". In [24], [26], [22], the same difficulties were overcome via uniform continuity of L^ε.

Finally in order to illustrate the use of such a result, we briefly mention the possible application of Theorem 1.1 to the the effective propagation of fronts in reaction-diffusion equations with a spatio-temporal stationary ergodic turbulent convection. Previously the results on effective front dynamics were known for a convecting velocity that was periodic in space-time (see [23]) or stationary ergodic in space (see [26]). Once Proposition 1.2 (see also Remark 3.12) has been proved, it is possible to show that v^ε solving

$$v^\varepsilon_t - \varepsilon^\alpha \Delta v^\varepsilon + H(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, Dv^\varepsilon)$$

will satisfy $\|u^\varepsilon - v^\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$. These techniques only require that the solution, u^ε, is Hölder continuous, and if its Hölder exponent is γ then the above convergence will hold for all $\alpha > \alpha_0 > 1$ where α_0 depends on γ. The proof of the closeness of v^ε and u^ε can be adapted directly from [25]. Then the proof of the effective front dynamics Using Theorem 1.1 follows the lines of [23] and [26]. We do not present the details here, but merely mention the possible application for the interest of the reader.

2. Discussion and Outline

2.1. Discussion. The problem of homogenizing Hamilton-Jacobi equations and fully nonlinear elliptic equations has been considered for a relatively long time now. The basis of most of the existing analysis begins with the goal of representing the true solutions, u^ε, as an expansion with non-oscillatory terms (given by u) and oscillatory terms (given by $\varepsilon v(\cdot/\varepsilon)$):

$$u^\varepsilon(x,t) = u(x,t) + \varepsilon v(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}) + \ldots .$$

One expects that $\varepsilon v(\cdot/\varepsilon) \to 0$ locally uniformly as $\varepsilon \to 0$. Thus if an equation governing u can be identified, the homogenization problem is solved.

After a substitution of u^ε back into the PDE, one must make the resulting equation independent of ε. In the case of (1.1) for a space-time periodic Hamiltonian, one tries to identify the effective Hamiltonian, $\bar{H}(P)$, for each $P \in \mathbb{R}^n$, as the *unique* constant such that there is a solution of the global equation on $\mathbb{R}^{n+1}$:

$$w_t + H(x,t,P+Dw) = \bar{H}(P) .$$

This equation is called the "corrector" equation, and w is the "corrector" to the function u in the regions where $Du = P$. A necessary and sufficient condition for the uniqueness of $\bar{H}$ is the existence of a solution, w, that is *strictly sublinear* at infinity (typically in the periodic setting, w is periodic, hence bounded and strictly sublinear). Once this can be done, proving that u satisfies the effective equation has been made standard with the perturbed test function method of [14], [15].

As mentioned in [21], in order to have a *unique* $\bar{H}(P)$, the necessity of the strict sublinearity of the corrector, w, can be seen from the trivial example given by $H(x, Du) = |Du|$. Here the corrector equation is:

$$|Dw + P| = \bar{H}(P).$$

For any choice of $q \in \mathbb{R}^n$, the linear function $w(x) = q \cdot x$ is a solution of the corrector problem with the corresponding $\bar{H}(P) = |P + q|$. However, since there is no x dependence in the original equation, there is no homogenization, and one should recover $\bar{H}(P) = |P|$. The correct $\bar{H}(P)$ is attained for the corrector $w(x) = 0$, which is the only strictly sublinear choice. The sufficiency of a strictly sublinear w can be seen by pursuing the typical viscosity solutions approach to arrive at a contradiction if there are different w_1, w_2 and $\bar{H}_1 > \bar{H}_2$. One considers the points of the maximum:

$$\sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} w_1(x) - w_2(y) - \alpha|x - y|,$$

which thanks to the strict sublinearity of w_1 and w_2 is always finite. The respective corrector equations can then be evaluated at the maximum points to arrive at a contradiction.

The strategy of solving the corrector equation can unfortunately be difficult to carry out in the stochastic setting. The issue of solving the corrector equation is addressed for time independent Hamiltonians in [21]. Moreover, counter examples to the solvability of the corrector problem with a strictly sublinear w were given in [21] and [22].

Thus, we have abandoned this approach for the present work. Instead, we pursue the strategy mentioned in section 1. This was first used for stochastic homogenization of Hamilton-Jacobi equations in [26] (a similar result was also proved in [24]). At first glance, there does not seem to be any reason why the result of [26], which does not try to solve the corrector problem, should not immediately translate to equations with time dependence. The difficulties are hidden in the fact that the method requires uniform continuity of the subadditive quantity (in this context taken to be the "fundamental solutions", denoted as L^ε). This uniform continuity is basically equivalent to showing a-priori uniform continuity of the solutions of (1.1), independent of ε and ω (which as far as the author is aware, is an open problem).

When (1.1) does not have dependence on time, it is an immediate consequence of the uniform coercivity of H that both $|u^\varepsilon_t|$ and $|Du^\varepsilon|$ are bounded depending only on $\|Du_0\|_\infty$. Specifically, the bound on $|u^\varepsilon_t|$ and $|Du^\varepsilon|$ does not depend on T, $\|u_0\|_\infty$, D_xH, or H_t! Unfortunately, this nice estimate does not work however when H depends simultaneously on time and space. Finding a priori uniform continuity of solutions of (1.1) is one of the main contributions of this work that allows the method of [26] to be applied to time dependent homogenization.

In this context, the same methods as in [26] cannot be completely applied. This is due to the lack of uniform Lipschitz estimates on the solutions, $u^\varepsilon(\omega)$. These estimates, combined with other nice properties of the solutions of (1.1), are the main feature that allows one to conclude the homogenization based simply on the behavior of solutions with linear data. This program is related to identifying when a semigroup on $BUC(\mathbb{R}^n \times [0, T])$ corresponds to solutions of an equation such as (1.2); see [25] for these techniques. Instead, we must use a priori continuity estimates combined with the Hopf-Lax solution formulas for (1.1) and (1.2) to prove the result.

We would like to point out that in [18], similar results were proved for time dependent viscous Hamilton-Jacobi equations. The methods of [18] are slightly different and do not apply directly to equations such as (1.1) which do not have a second order term of the form "$\varepsilon\Delta u^\varepsilon$". However, we take significant inspiration from [18], where the main ideas may be different but some of the techniques are directly applicable. In that paper, homogenization was proved using the usual Ergodic Theorem, instead of the Subadditive Ergodic Theorem. Moreover the result was obtained without using a priori uniform continuity of the solutions, u^ε. It is also interesting to note that the presence of a time dependent Hamiltonian caused additional difficulties in [18] which were not present in the time independent case treated in [19].

For a more detailed description of the background of homogenization of Hamilton-Jacobi equations using the Subadditive Ergodic Theorem, the reader should consult [21] and [22].

2.2. **Outline of the Paper.** In light of the aforementioned continuity requirement on our "fundamental solutions", L^ε, when using the Subadditive Theorem, we cannot prove Theorem 1.1 directly by the existing methods. Instead, we must approximate the problem with an appropriate Hamiltonian that satisfies the assumptions of Proposition 1.2. We can then use existing methods for the approximate problem to at least prove the existence of an approximate $\bar{H}$ and the convergence of the approximate u^ε. Significant new ideas are needed to prove the inf-sup formula for the approximate effective Hamiltonian, and this is where some techniques have been borrowed and adapted from [18]. Finally we must prove the convergence of the approximate effective Hamiltonians and the approximate effective solutions to the appropriate limit.

Now we provide the specific notation for this program. Given a generic Hamiltonian, H, satisfying the conditions of Theorem 1.1, we make a perturbation defined through its Legendre transform as:

$$(H^\delta)^*(x,t,p,\omega) = H^*(x,t,p,\omega) + \delta|p|^\beta, \tag{2.10}$$

where $\beta > \beta_1$, given in (L3). This perturbation, H^δ, now satisfies the assumptions of Proposition 1.2. We will refer to the solutions of (1.1) with the Hamiltonian, H^δ, as the functions $u^{\varepsilon,\delta}$, and we refer to the corresponding "fundamental solutions" as $L^{\varepsilon,\delta}$.

We begin the analysis in Section 3 with many preliminary results about u^ε and L^ε, culminating with the proof of Proposition 1.2. Then section 4 is dedicated to the use of the Subadditive Theorem to identify the correct limit for $L^{\varepsilon,\delta}$ as $\varepsilon \to 0$, which is equivalent to identifying the correct $\bar{H}^\delta$. For ease of presentation, we break up the proof of Theorem 1.1 for H^δ into two distinct parts. Section 5 will contain the proof of the convergence statement (part i) of Theorem 1.1 for $u^{\varepsilon,\delta}$, and Section 6 will contain the proof of the inf-sup formula (part ii) for $\bar{H}^\delta$. Finally, in Section 7, we prove the general case of Theorem 1.1 by using the properties of the approximate problems given by H^δ.

2.3. **Notational Comments.** Before we move on to the next section, we briefly include some notation and additional equations that will be helpful later on.

For a convex (and superlinear) function, $F : \mathbb{R}^n \to \mathbb{R}$, we denote its Legendre transform as

$$F^*(q) = \sup_{z\in\mathbb{R}^n} \big(z\cdot q - F(z)\big).$$

We will occasionally use the notion of "half relaxed limits" which are now standard in viscosity solutions theory. Unfortunately, this notation clashes with the above notation for the Legendre transform of a convex function. We believe that the meaning should be clear from the context of each usage. Here we have the "upper lim sup" and the "lower lim inf" of a family of functions, u^ε, given respectively as

$$(u^\varepsilon)^*(x) = \lim_{\varepsilon\to 0} \sup_{\{\delta\le\varepsilon,\ |x-y|\le\varepsilon\}} u^\delta(y); \quad (u^\varepsilon)_*(x) = \lim_{\varepsilon\to 0} \inf_{\{\delta\le\varepsilon,\ |x-y|\le\varepsilon\}} u^\delta(y).$$

We use the shorthand notations for various function spaces:

$$BUC(\mathbb{R}^n \times [0,T]) = \big\{f : \mathbb{R}^n \times [0,T] \to \mathbb{R} | f \text{ is bounded and uniformly continuous } \big\},$$
$$C^{0,1}(\mathbb{R}^n \times [0,T]) = \big\{f : \mathbb{R}^n \times [0,T] \to \mathbb{R} | \|f\|_\infty + \|Df\|_\infty < \infty \text{ and } f \text{ is continuous } \big\},$$
$$W^{1,\infty}([s,t];\mathbb{R}^n) = \big\{f : [s,t] \to \mathbb{R}^n | \|f\|_\infty + \|Df\|_\infty < \infty\big\} .$$

For the optimal control interpretation of the solutions, for $s \le t$ we introduce the space of admissible paths:

$$\mathcal{A}^{s,t}_{y,x} :=\{\xi \in W^{1,\infty}([s,t];\mathbb{R}^n) : \xi(s) = y, \xi(t) = x\}.$$

We will not write the explicit dependence of many equations and functions on ω in what follows, but the implicit dependence should be kept in mind and should be clear from the context.

3. Preliminaries regarding solutions of (1.1) when H satisfies (H2)

In this section, we list many different properties of solutions of (1.1) which will be crucial for proving Theorem 1.1. This section culminates with the proof of Proposition 1.2. For the following results we assume that $u^\varepsilon(\omega)$ is the unique solution of equation (1.1).

For the next three sections of this note, it will be convenient to consider solutions of the terminal value problems instead of the initial value problems given in 1.1. This is due to the fact that the proof of Theorem 1.1 heavily relies on the Hopf-Lax formula for solutions given by the optimal control context of the problem. These formulas are naturally posed for terminal value problems. In the case of a time independent equation, this is not a difficulty since the Hamiltonian is not sensitive to a time reversal. However, when the equation does depend on time, it is easiest to work with the terminal value problems. Thus we consider a new Hamiltonian, given by

$$G(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, p, \omega) = H(\frac{x}{\varepsilon}, \frac{T-t}{\varepsilon}, p, \omega).$$

It then follows that if we let $w^\varepsilon(x,t,\omega) = u^\varepsilon(x, T-t, \omega)$ and $w_T = u_0$, then w^ε solves

$$\begin{cases} w^\varepsilon_t = G(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, Dw^\varepsilon, \omega) & \text{in } \mathbb{R}^n \times (0,T) \\ w^\varepsilon = w_T & \text{on } \mathbb{R}^n \times \{T\}. \end{cases} \tag{3.1}$$

For the remainder of this section and section 5, we will use $w^\varepsilon(\omega)$ to denote the solution of (3.1). We Define the pointwise travel cost as

$$L^\varepsilon(x,t;y,s,\omega) = \inf_{\mathcal{A}^{y,x}_{s,t}} \Big\{ \int_s^t G^*(\frac{\xi(r)}{\varepsilon}, \frac{r}{\varepsilon}, -\dot{\xi}(r), \omega) dr \Big\}. \tag{3.2}$$

In many results to follow, we work with minimizing paths in the definition of L^ε. Given the assumptions on F, it may or may not be true that optimal paths exist in the $W^{1,\infty}$ class. However in all arguments that follow, any path that is within δ of achieving the infimum works just as well as the optimal one. Hence we will work with such paths as though they are optimal, even though we must actually let them approximate the infimum within δ of its value and then take $\delta \to 0$.

The following results, Lemma 3.1 and Proposition 3.2, are classical facts about solutions of (3.1). Proofs can be found in [20] (and also [2]).

Lemma 3.1. *For each (y,s), $L^\varepsilon(x,t;y,s,\omega)$ is a "solution" of $L^\varepsilon_t + G(x/\varepsilon, t/\varepsilon, DL, \omega) = 0$ in $\mathbb{R}^n \times (s,T]$ with the initial data condition*

$$\lim_{t\to s+} L^\varepsilon(x,t;y,s,\omega) = \begin{cases} +\infty & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Proposition 3.2. *Let $w_T \in BUC(\mathbb{R}^n)$. Then the solution, w^ε, of (3.1), is given as*

$$w^\varepsilon(x,t,\omega) = \inf_{y\in\mathbb{R}^n}\left\{w_T(y) + L^\varepsilon(y,T;x,t,\omega)\right\} \tag{3.3}$$

which is equivalent to

$$w^\varepsilon(x,t,\omega) = \inf_{\substack{\xi(t)=x \\ \xi\in W^{1,\infty}([0,t];\mathbb{R}^n)}} \left\{w_T(\xi(T)) + \int_t^T G^*(\frac{\xi(r)}{\varepsilon}, \frac{r}{\varepsilon}, -\dot{\xi}(r))dr\right\}. \tag{3.4}$$

At this point, it will be useful to remark the connection between L^ε for $\varepsilon > 0$ and $\varepsilon = 1$. It is straightforward to check that we have

$$L^\varepsilon(x,t;y,s,\omega) = \varepsilon L^1(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}; \frac{y}{\varepsilon}, \frac{s}{\varepsilon}, \omega). \tag{3.5}$$

From this point forward, we will use $L(x,t;y,x,\omega)$ to denote the corresponding pointwise cost for $\varepsilon = 1$, $L^1(x,t;y,s,\omega)$. The next Lemma describes how the stationarity of F affects the solutions.

Lemma 3.3 (Space-Time Translations)**.** *For all $x,y,z \in \mathbb{R}^n$, $s < t$, and $r > 0$, we have the translation property:*

$$L(x+z,t+r;y,s,\omega) = L(x,t;y-z,s-r,\tau_{(z,r)}\omega). \tag{3.6}$$

Proof of Lemma 3.3. Let us define for z,r,y,s,ω fixed,

$$v(x,t) = L(x+z,t+r;y,s,\omega).$$

Thus v solves the problem:

$$\begin{cases} v_t + G(x+z,t+r,Dv,\omega) = 0 & \text{in } \mathbb{R}^n \times (s,T) \\ v(x,s-r) = \begin{cases} +\infty & \text{if } x+z \neq y, \\ 0 & \text{if } x+z = y. \end{cases} & \text{on } \mathbb{R}^n \times \{s-r\}\ . \end{cases}$$

It follows by the stationarity of G that v also solves

$$\begin{cases} v_t + G(x,t,Dv,\tau_(z,r)\omega) = 0 & \text{in } \mathbb{R}^n \times (s,T) \\ v(x,s-r) = \begin{cases} +\infty & \text{if } x+z \neq y, \\ 0 & \text{if } x+z = y. \end{cases} & \text{on } \mathbb{R}^n \times \{s-r\}\ . \end{cases}$$

Uniqueness for such equations with infinite initial data can be proved with the methods of [10]. It is a result of the strong regularizing effects of the solution operator semigroup (L has infinite initial data, but it is Lipschitz at any $t > s$) and uniqueness of "maximal solutions". Once this is established, we then conclude that since v and $L(x,t;y-z,s-r,\tau_{(z,r)}\omega)$ solve the same equation (as functions of x and t) and they agree at time $t = s-r$, we know that they agree at all later times by uniqueness. A proof of such a uniqueness result is not provided here.

An alternative proof follows from the definition of L, the translation of paths, and stationarity of G. □

The PDE proof above applies directly to w^ε. Rewriting it or simply applying Lemma 3.3 with Proposition 3.2 gives us the next result.

Corollary 3.4. *If we define $v(x,t,\omega) := w^\varepsilon(x+z,t+s,\omega)$, then v solves the equation:*

$$\begin{cases} v_t = G(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, Dv, \tau_{z/\varepsilon, s/\varepsilon}\omega) & \text{in } \mathbb{R}^n \times (-s, T-s) \\ v(x, T-s) = w_T(x+z) & \text{on } \mathbb{R}^n \times \{T-s\}. \end{cases}$$

Another crucial property of L is its subadditivity with respect to the cost of stopping at an intermediate point at an intermediate time.

Lemma 3.5 (Subadditive Property)**.** *For all z and for $s \le \tau \le t$ we have*

$$L(x,t;y,s,\omega) \le L(z,\tau;y,s,\omega) + L(x,t;z,\tau,\omega).$$

Proof of Lemma 3.5. Let us take ψ_1 to optimize in the definition of $L(z,\tau;y,s)$ and ψ_2 to optimize in the definition of $L(x,t;z,\tau)$. We then note that if ξ is the path made by concatenating the two paths ψ_1 and ψ_2, then ξ is admissible for $L(x,t;y,s)$. The inequality follows from the definition of L. □

Due to the uniform growth conditions imposed by (L3), the functions, L^ε have uniform bounds on their growth. This will be very useful in proving many subsequent results, especially the uniform continuity of solutions.

Lemma 3.6 (Uniform Bounds)**.** *For each x, y, s, t we have the bounds on L^ε:*

$$-C(t-s) + (t-s)^{1-\beta_2}|x-y|^{\beta_2} \le L^\varepsilon(x,t;y,s,\omega) \le C(t-s) + (t-s)^{1-\beta_1}|x-y|^{\beta_1},$$

where β_1, β_2 appear in (1.5).

Proof of Lemma 3.6. The left inequality comes from using the lower boundedness of G^*:

$$\begin{aligned} &- C(t-s) + (t-s)^{1-\beta_2}|x-y|^{\beta_2} = \\ &= -C(t-s) + \inf_{\mathcal{A}^{y,x}_{s,t}} \{ \int_s^t |\dot\xi(r)|^{\beta_2} dr \} \\ &\le \inf_{\mathcal{A}^{y,x}_{s,t}} \{ \int_s^t G^*(\frac{\xi(r)}{\varepsilon}, \frac{r}{\varepsilon}, -\dot\xi(r), \omega) dr \} \\ &= L^\varepsilon(x,t;y,s,\omega). \end{aligned}$$

The right inequality is a consequence of plugging in the straight line path from y to x, denoted as ξ, in the definition of L^ε:

$$\begin{aligned} &L^\varepsilon(x,t;y,s,\omega) \le \int_s^t G^*(\frac{\xi(r)}{\varepsilon}, \frac{r}{\varepsilon}, -\dot\xi(r), \omega) dr \\ &\le C(t-s) + (t-s)|\frac{x-y}{t-s}|^{\beta_1}. \end{aligned}$$

□

For a technical reason later on, it will be necessary to have the following result. Due to the strong regularizing nature of the equation (3.1) and hence (1.1), it is possible to state it for only lower semicontinuous, sublinear functions.

Lemma 3.7 (Domain of Dependence)**.** *Suppose that w_T and v_T are strictly sublinear, lower semicontinuous functions on $\mathbb{R}^n$ and for convenience that $|w_T(x)| \le \phi(x)$ and $|v_T(x)| \le \phi(x)$, for some sublinear ϕ (not necessarily strictly sublinear). If w^ε and v^ε are solutions of (3.1) with the corresponding terminal data, then for each fixed $R > 0$, there exists K depending only on ϕ and the growth of L^ε (see Lemma 3.6) such that for all $t > 0$ and x_0*

$$\|w^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{\infty, B_R(x_0)} \le \|w_T - v_T\|_{\infty, B_K(x_0)}.$$

Proof of Lemma 3.7. The proof is very straightforward, using the formula for w^ε and v^ε. To this end, we fix x, t. Let $z(w^\varepsilon, x, t)$ and $z(v^\varepsilon, x, t)$ be the points which achieve the infimum (see Proposition 3.2) for $w^\varepsilon(x,t)$ and $v^\varepsilon(x,t)$ respectively. The growth of L^ε from Lemma 3.6 plus the boundedness of $|w_T|$, $|v_T|$ by ϕ imply $z(w^\varepsilon, x, t)$ and $z(v^\varepsilon, x, t)$ are bounded depending only on ϕ and the bounds in Lemma 3.6. Let

$$K = \sup_{x,t \in B_R \times [0,T]} \{|z(v^\varepsilon, x, t)|, |z(w^\varepsilon, x, t)|\} \ .$$

We then have

$$\begin{aligned} w^\varepsilon(x,t) - v^\varepsilon(x,t) &\leq w_T(z(v^\varepsilon)) + L^\varepsilon(z(v^\varepsilon), T; x, t) - v_T(z(v^\varepsilon)) - L^\varepsilon(z(v^\varepsilon), T; x, t) \\ &\leq w_T(z(v)) - v_T(z(v)) \leq \|w_T - v_T\|_{\infty, B_K}. \end{aligned}$$

The opposite inequality is proved similarly. $\square$

The next lemma asserts that the solutions, w^ε, attain their terminal conditions in a uniform manner. This is necessary for having uniform control on the continuity of w^ε.

Lemma 3.8 (Uniform Separation From Terminal Conditions)**.** *For all $w_T \in C^{0,1}(\mathbb{R}^n)$ with $\|Dw_T\|_\infty \leq K$, there is a constant C_K such that*

$$|w^\varepsilon(x,t,\omega) - w_T(x)| \leq C_K(T-t).$$

Proof of Lemma 3.8. Let $w_T \in C^{0,1}(\mathbb{R}^n)$ be given. Define C_K as:

$$C_K = \sup_{t\in\mathbb{R},\ |x|\in\mathbb{R}^n,\ |p|\leq K} |G\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, p, \omega\Big)|.$$

We note that $C_K(\omega)$ is invariant with respect to τ, and hence constant in ω by ergodicity. Then it follows that we have respectively sub and super solutions of (1.1) given by (recall that the roles of sub and super supersolutions are reversed for a terminal problem)

$$w_T(x) - C_K(T-t) \ \text{ and } \ w_T(x) + C_K(T-t).$$

The claim follows by comparison. $\square$

The last piece of information we will need about L^ε before proving uniform continuity is the properties of its optimal trajectories.

Lemma 3.9 (Regularity of Paths)**.** *There exists $\gamma \in (0,1)$, such that any minizing path, ψ^*, of $L^\varepsilon(x,t;y,s,\omega)$ satisfies*

$$\|\psi^*\|_{C^{0,\gamma}([s,t])} \leq C\big((t-s) + (t-s)^{1-\beta_1}|x-y|^{\beta_1}\big)^{1/\beta_2}.$$

Proof of Lemma 3.9. This Lemma is a direct consequence of Lemma 3.6 and Hölder's inequality. Indeed, by Lemma 3.6, ψ^* satisfies

$$\begin{aligned} &\int_s^t C + |\dot\psi^*(r)|^{\beta_2} dr \leq \\ &\leq \int_s^t G^*(\frac{\psi^*(r)}{\varepsilon}, \frac{r}{\varepsilon}, -\dot\psi^*(r), \omega) dr \\ &\leq C(t-s) + (t-s)^{1-\beta_1}|x-y|^{\beta_1}. \end{aligned}$$

The claim follows using the absolute continuity of ψ^* and Hölder's inequality. $\square$

A consequence of Lemma 3.9, is that we may further restrict the class of paths in the definition of $L(x,t;y,s,\omega)$ to those ξ that not only satisfy $\xi \in W^{1,\infty}$, but also satisfy

$$\|\xi\|_{C^{0,\gamma}([s,t])} \leq C(1+(t-s)^{(1-\beta_2)/\beta_2}|x-y|).$$

The most crucial requirement to carry out the homogenization of (1.1) when using the Subadditive Ergodic Theorem is to have uniform continuity of L^ε which is uniform in both ε and ω. It is not too difficult to see that each $L^\varepsilon(\omega)$ is Lipschitz, but depending on ε. Specifically, we will have $|L^\varepsilon_x|, |L^\varepsilon_t| \leq C/\varepsilon$, which is not good enough. The uniform continuity of L^ε is the point of the next proposition.

Proposition 3.10 (Uniform Continuity)**.** *Assume that H (and hence G) satisfies (1.8). Then for all $R>0$ and $\rho>0$ fixed, $L^\varepsilon(x,t;y,s,\omega)$ is uniformly continuous as a function of x, t, y, s on the set where $|x|\leq R$, $|y|\leq R$ and $t>s+\rho$ independent of ε and ω.*

Proof of Proposition 3.10. For notational purposes, since H satisfies (1.8), we can write the Legendre transform of G as

$$G^*(x,t,p,\omega)=\delta|p|^\beta+F(x,t,p,\omega),$$

where $F(x,t,p,\omega)=f(x,T-t,p,\omega)$. Without loss of generality, we take $\delta=1$ to simplify the notation. We first note that it will suffice to prove uniform continuity in each variable separately, with the other variables fixed. Secondly, we note that the translation properties of L^ε allow us to only consider the cases of $L^\varepsilon(x_1,t_1;0,0,\omega)-L^\varepsilon(x_2,t_2;0,0,\omega)$ and $L^\varepsilon(0,T;y_1,s_1,\omega)-L^\varepsilon(0,T;y_2,s_2,\omega)$. This is because we have

$$\begin{aligned}
&L^\varepsilon(x_1,t_1;y,s,\omega)-L^\varepsilon(x_2,t_2;y,s,\omega)=\\
&=L^\varepsilon(x_1-y,t_1-s;0,0,\tau_{y/\varepsilon,s/\varepsilon}\omega)-L^\varepsilon(x_2-y,t_2-s;0,0,\tau_{y/\varepsilon,s/\varepsilon}\omega), \quad \text{and}\\
&L^\varepsilon(x,t;y_1,s_1,\omega)-L^\varepsilon(x,t;y_2,s_2,\omega)=\\
&=L^\varepsilon(0,T;y_1-x,s-(t-T),\tau_{x/\varepsilon,(t-T)/\varepsilon}\omega)-L^\varepsilon(0,T;y_2-x,s-(t-T),\tau_{x/\varepsilon,(t-T)/\varepsilon}\omega),
\end{aligned}$$

where the size of this difference will be independent of ω. Finally, we will not write the dependence on ε or ω in the calculations to follow. This is justified by the fact that we will use only the growth bounds on $|F|$ (which are independent of ω) and neither $\|D_xF\|_\infty$ nor $\|F_t\|_\infty$. We will proceed in three steps:

i) $L(x_1,t;0,0)-L(x_2,t;0,0)$
ii) $L(x,t_1;0,0)-L(x,t_2;0,0)$
iii) $L(0,T;y_1,s_1)-L(0,T;y_2,s_2)$.

We would like to point out that the idea for modifying the optimal paths below is from [18]. Similar manipulations of such paths also appear in [20].

Step i): Let $|x_2-x_1|=r$. Because we assume $t>\rho$, it suffices to consider $r\leq\rho$. The goal will be to take an optimal path for $L(x_2,t;0,0)$ and modify it to create an admissible path for $L(x_1,t;0,0)$. Thus let ξ_2 be a minimizing path for $L(x_2,t;0,0)$. Define an intermediate time, $\tau<t$ by the formula

$$t-\tau=|x_2-x_1|=|\xi_2(t)-x_1|.$$

We now let ξ_1 be an admissible path for $L(x_1,t;0,0)$ by keeping it identical to ξ_2 for most of the trajectory, and then changing it by a small amount after time τ:

$$\xi_1(s)=\begin{cases}\xi_2(s) & \text{if } 0\leq s\leq\tau\\ \xi_2(s)+x_1-\xi_2(t)+(t-s)\dfrac{\xi_2(t)-x_1}{|\xi_2(t)-x_1|} & \text{if } \tau\leq s\leq t.\end{cases}$$

It is immediate that ξ_1 is admissible. Moreover, we note that on the interval $[\tau, t]$, $\dot{\xi}_1 = \dot{\xi}_2 + e$, where e is the unit vector given by

$$e = \frac{\xi_2(t) - x_1}{|\xi_2(t) - x_1|}.$$

Using the assumptions on G and Hölder's inequality, we are now in a position to estimate the difference:

$$\begin{aligned}
&L(x_1, t; 0, 0) - L(x_2, t; 0, 0) \leq \\
&\quad \leq \int_0^t G^*(\xi_1(s), s, -\dot{\xi}_1(s))ds - \int_0^t G^*(\xi_2(s), s, -\dot{\xi}_2(s))ds \\
&\quad = \int_\tau^t F(\xi_1(s), s, -\dot{\xi}_1(s)) - F(\xi_2(s), s, -\dot{\xi}_2(s)) + |\dot{\xi}_1(s)|^\beta - |\dot{\xi}_2(s)|^\beta ds \\
&\quad = \int_\tau^t F(\xi_1(s), s, -(\dot{\xi}_2(s) + e)) - F(\xi_2(s), s, -\dot{\xi}_2(s)) + |\dot{\xi}_2(s) + e|^\beta - |\dot{\xi}_2(s)|^\beta ds \\
&\quad \leq \int_\tau^t A_2(1 + |\dot{\xi}_2(s) + e|^\alpha) + A_1 + C(1 + |\dot{\xi}_2(s)|^{\beta-1})ds \\
&\quad \leq C\big((t-\tau) + (t-\tau)^{1/p^*}\|\dot{\xi}\|_{L^p} + (t-\tau)^{1/q^*}\|\dot{\xi}\|_{L^q}\big) \\
&\quad \leq C|x_1 - x_2| + C(\rho^{(1-\beta)/\beta}R)|x_1 - x_2|^\gamma,
\end{aligned}$$

for some $\gamma < 1$. We used the notation $p = \beta/\alpha$, $q = \beta/(\beta - 1)$ and p^*, q^* are respectively their conjugate exponents. We have used Lemma 3.6 in the last line. We note that the roles of x_1 and x_2 can be interchanged, and so we are done with step (i).

Step ii): Step (ii) will proceed with an almost identical construction. We must first separate two cases: (a) $t_1 > t_2$ and (b) $t_1 < t_2$.

For case (a), we will again start with ξ_2 as an optimal trajectory for $L(x, t_2; 0, 0)$, and we must modify it to be admissible for $L(x, t_1; 0, 0)$. This is easy because we just define the modified path to be constant, x, on $[t_2, t_1]$:

$$\xi_1(s) = \begin{cases} \xi_2(s) & \text{if } 0 \leq s \leq t_2 \\ x & \text{if } t_2 \leq s \leq t_1 \end{cases}.$$

Repeating the calculations from step (i), we have

$$L(x, t_1; 0, 0) - L(x, t_2; 0, 0) \leq C(t_1 - t_2).$$

For case (b) we can use the regularity of the trajectories described in Lemma 3.9. Let ξ_2 be optimal for $L(x, t_2; 0, 0)$. We define

$$\xi_1(s) = \begin{cases} \xi_2(s) & \text{if } 0 \leq s \leq \tau \\ \xi_2(t_1) + (s - \tau)\dfrac{x - \xi_2(t_1)}{|x - \xi_2(t_1)|} & \text{if } \tau \leq s \leq t_1 \end{cases}.$$

We define the intermediate time, τ, as $t_1 - \tau = |x - \xi_2(t_1)|$, and without loss of generality we may assume $\xi_2(t_1) \neq x$. Note that Lemma 3.9 gives the inequality:

$$t_1 - \tau = |x - \xi_2(t_1)| \leq \|\xi\|_{C^{0,\gamma}}|t_1 - t_2|^\gamma.$$

Plugging these paths into the same calculations from step (i), and writing the unit vector,

$$e = \frac{(x - \xi_2(t_1))}{|x - \xi_2(t_1)|},$$

gives

$$\begin{aligned}
&L(x,t_1;0,0) - L(x,t_2;0,0) \leq \\
&\quad \leq \int_\tau^{t_1} G^*(\xi_1(s),s,-\dot{\xi}_1(s)) - G^*(\xi_2(s),s,-\dot{\xi}_2(s))ds - \int_{t_1}^{t_2} G^*(\xi_2(s),s,\dot{\xi}_2(s))ds \\
&\quad \leq C(t-\tau) + (t-\tau)^{1/p^*}\|\dot{\xi}\|_{L^p} + (t-\tau)^{1/q^*}\|\dot{\xi}\|_{L^q} \\
&\quad \leq C(t_2-t_1)^\gamma + ((t_2-t_1)^\gamma)^{1/p^*}\|\dot{\xi}\|_{L^p} + ((t_2-t_1)^\gamma)^{1/q^*}\|\dot{\xi}\|_{L^q}.
\end{aligned}$$

In view of Lemma 3.9, we are finished with step (ii).

Step iii): All the above constructions were not sensitive to whether we were modifying the end of the path or the beginning of the path. Thus, step (iii) follows by repeating steps (i) and (ii) with the appropriate modifications. This completes the proof of the proposition. □

Remark 3.11. It is useful now to remark on the failure of this argument when only (1.5) holds and not (1.8). At the end of step (i), we would have

$$\begin{aligned}
&L(x_1,t;0,0) - L(x_2,t;0,0) \leq \\
&\quad \leq \int_\tau^{t} G^*(\xi_1(s),s,-(\dot{\xi}_2(s)+e)) - \int_\tau^{t} G^*(\xi_2(s),s,-\dot{\xi}_2(s))ds.
\end{aligned}$$

Invariably, one will be forced to estimate a term equivalent to

$$\int_\tau^t |\dot{\xi}_2(s)+e|^{\beta_1} - |\dot{\xi}_2(s)|^{\beta_1}ds.$$

However, we only have control on $\|\dot{\xi}_2\|_{L^{\beta_2}}$. If it happens to be the case that $\beta_2 > \beta_1 - 1$, then the same proof will work. But this an undesirably restrictive assumption, and we do not pursue an estimate in such cases. Section 7 explains how this difficulty is circumvented.

Remark 3.12. If we let $m(r)$ be the uniform modulus of continuity proven in the previous proposition, then we explicitly have for $|x_1-x_2|, |t_1-t_2| \leq r \leq \rho$, for some $0<\gamma<1$,

$$|L(x_1,t_1;0,0) - L(x_2,t_2;0,0)| \leq m(r) = Cr^\gamma.$$

Thanks to the formula for w^ε, Proposition 3.10 is an immediate consequence. We provide the main outline here, and the remaining details are standard.

Proof of Proposition 1.2. The proof will only be written for w^ε. In light of the connection between w^ε and u^ε, this is sufficient. We first note that the presence of a bound on $\|w_T\|_\infty$ is used in conjunction with the coercivity of L^ε, given in Lemma 3.6, to get a bound on the set over which the infimum is achieved in (3.3). This is necessary to invoke Proposition 3.10 since the continuity of L^ε is not globally uniform in x, y.

Let $r > 0$ be arbitrary. We examine two separate cases depending on whether $\min(T - s, T-t) \leq r$ or $\min(T-s, T-t) > r$. In the first case we appeal to Lemma 3.8; we can then pass the continuity onto the data, w_T, and use $\|Dw_T\|_\infty$. In the second case, we use the representation of w^ε given by Proposition 3.2 along with Proposition 3.10. We note that Lemma 3.6 allows, for some $C>0$, the restriction of $|y| \leq C\|w_T\|_\infty$ of the infimum in the formula from Proposition 3.2. These conclusions in these two cases imply uniform continuity of w^ε. We omit the remaining details. □

Remark 3.13. It is now clear that this method of searching for uniform continuity is far from being sharp. If H is independent of t, then the standard viscosity methods easily give $|w^\varepsilon_t|$ and $|Dw^\varepsilon|$ bounded by $C(H, \|Dw_T\|_\infty)$, depending only on the growth of H and the size of $\|Dw_T\|$. This is of course much better than what is provided by Proposition 1.2.

4. Identifying the Limit

We must now try to appeal to the stationarity and ergodicity of F in order to extract a limit for the "fundamental" solutions, L^ε. Following the ideas of [26], and using the notation in the context of [1], we will define a random map on the collection of intervals in $[0,\infty)$. The result is:

Lemma 4.1. *For $z \in \mathbb{R}^n$ fixed and $\omega \in \Omega$, define a map, $\mu^z(\cdot,\omega)$, from the collection of intervals of $[0,\infty)$ to $\mathbb{R}$ as*

$$\mu^z([a,b),\omega) := L(bz,b;az,a,\omega).$$

Also define the transformation, $\gamma^z_\alpha : \Omega \to \Omega$, by the formula $\gamma^z_\alpha\omega := \tau_{\alpha z,\alpha}\omega$. Then in the sense of [1], μ^z is stationary and subadditive with respect to γ^z on Ω (see assumption (1.3)).

Proof of Lemma 4.1. We will not write the z in μ^z or γ^z for the remainder of this proof. We will first show that μ is stationary and subadditive. Referring to [1], assuming I, I_k are disjoint half open intervals (for a finite number of I_k) with $I = \cup I_k$ we must show:

$$\mu([a,b),\gamma_\alpha(\cdot)) = \mu(\alpha+[a,b),\cdot), \tag{4.1}$$

$$\mu(I,\cdot) \le \sum_k \mu(I_k,\cdot), \tag{4.2}$$

$$\mu([a,b),\cdot) \in L^1(\Omega,\mathbb{P}), \tag{4.3}$$

$$\inf_{a<b}\left\{\frac{1}{|b-a|}\int_\Omega \mu([a,b),\omega)d\mathbb{P}(\omega)\right\} > -\infty. \tag{4.4}$$

All of the above properties are straightforward to check. Requirement (4.1) follows from the definition of μ and Lemma 3.3. (4.2) follows immediately from Lemma 3.5. (4.3) and (4.4) will follow from the definition of μ and Lemma 3.6. □

Lemma 4.2. *Assume (1.8) holds with $\delta = 1$. Then there exists a function, $\bar{L}$, and a set of full measure, $\tilde{\Omega}$, such that*

$$L^\varepsilon(1,z;0,0,\omega) \to \bar{L}(z)$$

pointwise in z and for each $\omega \in \tilde{\Omega}$ fixed. Moreover, $\bar{L}$ is convex and has the bounds

$$-C + |z|^\beta \le \bar{L}(z) \le C + |z|^\beta.$$

Remark 4.3. The assumption that $\delta = 1$ in (1.8) is not a restriction. It is simply chosen to simplify the presentation. The parameter, δ, will not be used until Section 7, at which point it is the existence of such $\bar{L}$ that we are concerned with, and not the growth bounds.

Proof of Lemma 4.2. To begin, we first note the relationship between μ^z and L^ε:

$$\mu^z([0,\frac{1}{\varepsilon})) = L(\frac{z}{\varepsilon},\frac{1}{\varepsilon};0,0,\omega) = \frac{1}{\varepsilon}L^\varepsilon(z,1;0,0,\omega).$$

Next we will establish that a function, $\phi^z(\omega)$ exists for a.e. ω as the limit of $\varepsilon\mu^z([0,1/\varepsilon),\omega)$. Then we will show that in fact ϕ^z is constant in ω. To finish we will show that ϕ^z is convex as a function of z that has the same growth as L^ε. Finally we take $\bar{L}(z) = \phi^z$. Thus, we start by fixing $z \in \mathbb{Q}^n$.

Claim (i): there exists a set, $\tilde{\Omega}_z$, and a limiting function, ϕ^z, such that $\phi^z(\omega)$ is the limit for a.e. $\omega \in \Omega_z$. In order to satisfy the hypothesis of Theorem 2.5 in [1], we verify that

$$\int_\Omega \sup_{0<a<b<1}\{|\mu([a,b),\omega)|\}d\mathbb{P}(\omega) < \infty.$$

This follows from the definition of μ, L, and Lemma 3.6. Applying Theorem 2.5 from [1], we can define ϕ^z almost everywhere as the pointwise limit:

$$\phi^z(\omega) = \lim_{\varepsilon\to 0} \varepsilon\mu([0, \frac{1}{\varepsilon}), \omega),$$

for ω restricted to an appropriate set of full measure, Ω_z.

Claim (ii): ϕ^z exists for all $z \in \mathbb{R}^n$ on a set of full measure, $\tilde{\Omega}$. The previous claim only asserts that for each z, there is a subset of Ω for which the convergence occurs. However, this is much weaker than what is needed. It is required to find one single set of full measure, $\tilde{\Omega}$, such that convergence occurs *simultaneously for all* z. This step can indeed prove to be difficult in the absence of uniform continuity of L^ε. By countability,

$$\tilde{\Omega} := \bigcap_{z\in\mathbb{Q}^n} \Omega_z$$

has full measure, and ϕ^z can be defined by density for $z \in \mathbb{R}^n$ using Proposition 3.10.

Claim (iii): ϕ^z is constant in ω on $\tilde{\Omega}$. We aim to show that ϕ^z is actually invariant with respect to τ. We will again be using the uniform continuity given by Proposition 3.10. Thus we fix y, s and check:

$$\begin{aligned}
\phi^z(\tau_{s,y}\omega) &= \lim_{\varepsilon\to 0} \varepsilon L(\frac{z}{\varepsilon}, \frac{1}{\varepsilon}; 0, 0, \tau_{y,s}\omega) \\
&= \lim_{\varepsilon\to 0} L^\varepsilon(z, 1; 0, 0, \tau_{y,s}\omega) \\
&= \lim_{\varepsilon\to 0} L^\varepsilon(z + \varepsilon y, 1 + \varepsilon s; \varepsilon y, \varepsilon s, \omega) \\
&= \lim_{\varepsilon\to 0} L^\varepsilon(z, 1; 0, 0, \omega) + \lim_{\varepsilon\to 0}\big(L^\varepsilon(z + \varepsilon y, 1 + \varepsilon s; \varepsilon y, \varepsilon s, \omega) - L^\varepsilon(z, 1; 0, 0, \omega)\big) \\
&= \phi^z(\omega).
\end{aligned}$$

Thus since τ is ergodic, we may conclude that ϕ^z is constant on $\tilde{\Omega}$ (see [7]).

Claim (iv): $\bar{L}(z) := \phi^z$ is convex. Here we can appeal to the fact that the subadditive theorem also implies convergence of the expectations:

$$\mathbb{E}(\varepsilon\mu^z\big([0, 1/\varepsilon), \cdot\big)) \to \mathbb{E}(\bar{L}(z, \cdot)).$$

In order to see what we need to show convexity, lets fix x, y, ω and check what we have from Lemma 3.5:

$$L^\varepsilon(\alpha x + (1-\alpha)y, 1; 0, 0, \omega) \le L^\varepsilon(\alpha x, \alpha; 0, 0, \omega) + L^\varepsilon(\alpha x + (1-\alpha)y, 1; \alpha x, \alpha, \omega).$$

Looking at the definition of $\bar{L}(x)$, we see that

$$\alpha\bar{L}(x) = \lim_{\varepsilon\to 0} L^\varepsilon(\alpha x, \alpha; 0, 0, \omega) \quad \text{and} \quad (1-\alpha)\bar{L}(y) = \lim_{\varepsilon\to 0} L^\varepsilon((1-\alpha)y, 1-\alpha; 0, 0, \omega).$$

Taking expectations above, we have:

$$\mathbb{E}\big(L^\varepsilon(\alpha x + (1-\alpha)y, 1; 0, 0, \omega)\big) \le \mathbb{E}\big(L^\varepsilon(\alpha x, \alpha; 0, 0, \omega)\big) + \mathbb{E}\big(L^\varepsilon(\alpha x + (1-\alpha)y, 1; \alpha x, \alpha, \omega)\big)$$

We now use the translation and measure preserving properties of L^ε and τ, respectively, followed by taking limits in the previous line to recover

$$\mathbb{E}\big(\bar{L}(\alpha x + (1-\alpha)y)\big) \le \mathbb{E}\big(\alpha\bar{L}(x)\big) + \mathbb{E}\big((1-\alpha)\bar{L}(y)\big).$$

Since $\bar{L}$ is constant in ω, we obtain

$$\bar{L}(\alpha x + (1-\alpha)y) \le \alpha\bar{L}(x) + (1-\alpha)\bar{L}(y).$$

Claim (v): $\bar{L}$ satisfies the bounds promised in statement of the lemma. This is an immediate consequence of Lemma 3.6 under pointwise limits of L^ε. □

Remark 4.4. The uniform continuity of L^ε was used in a substantial way in the proof of Lemma 4.2 (specifically in claims (ii) and (iii)). Without knowing that $\bar{L}$ is constant in ω, we would have difficulty concluding many aspects of the convergence that will follow in the next section. It would be very interesting to see exactly how far this method can be pushed in the absence of uniform continuity of L^ε.

Using a very similar proof, we can show:

Corollary 4.5 (Pointwise limits for L^ε)**.** *For each $t > 0$ and z fixed there exists a set of full measure $\tilde{\Omega}$ such that for $\omega \in \Omega$*

$$L^\varepsilon(z, t; 0, 0, \omega) \to t\bar{L}(\frac{z}{t}) \ ,$$

where $\bar{L}$ is the function given by the previous Lemma.

Proof of Corollary 4.5. Since the proof will be similar to the previous one, we will simply indicate the ways to modify it for the present proof. Using the new subadditive process given by

$$\mu^{z,t}([a, b), \omega) = L(bz, bt; az, at, \omega),$$

all the same steps can be carried out. In claim (ii) we take z, t rational, and define

$$\tilde{\Omega} := \bigcap_{z,t \in \mathbb{Q}^{n+1}} \Omega_{z,t}.$$

To see that the limit is actually $t\bar{L}(z/t)$, we note that

$$\mu^{z,t}([0, \frac{1}{\varepsilon}), \omega) = \mu^{z/t}([0, \frac{t}{\varepsilon}), \omega),$$

and hence

$$\frac{\varepsilon}{t}\mu^{z,t}([0, \frac{t}{\varepsilon}), \omega) = \frac{\varepsilon}{t}\mu^{z/t}([0, \frac{t}{\varepsilon}), \omega) \to \bar{L}(\frac{z}{t}).$$

In other words,

$$L^\varepsilon(z, t; 0, 0, \omega) \to t\bar{L}(\frac{z}{t}).$$

□

With the conclusion of the previous lemma and the formula from Proposition 3.2, we are now in a position to conjecture the correct limit for the functions $w^\varepsilon(\omega)$. Given the pointwise behavior of L^ε, it is reasonable to try to prove the convergence stated in the next proposition.

Proposition 4.6. *Assume that H (and hence G) satisfies (1.8). Let $w_T \in C^{0,1}(\mathbb{R}^n)$ be given. Then $w^\varepsilon(\omega) \to w$ locally uniformly, where w is defined as*

$$w(x, t) := \inf_{y \in \mathbb{R}^n} \left[w_T(y) + (T - t)\bar{L}\Big(\frac{y - x}{T - t}\Big) \right].$$

The proof of Proposition 4.6 will be the subject of the next section.

5. Proof of Theorem 1.1, (i), Assuming (1.8)

Under the assumption that H satisfies (1.8), Theorem 1.1, (i) is an immediate consequence of Proposition 4.6. The bulk of this section will be dedicated to proving Proposition 4.6, and then the proof of Theorem 1.1 will follow as a straightforward consequence. The proof of Proposition 4.6 will come in two parts. First we will show that the convergence happens for $|x|$, $|t|$ small. Then we will show that this will be sufficient for any x and t due to the ergodic nature of the transformation τ. This strategy follows very similarly to the techniques of [18].

At first glance, the necessity of this section may seem minor. We already have a priori uniform continuity of L^ε, and we have identified the limits of $L^\varepsilon(z,t;0,0,\omega)$. Therefore this should be enough to pass the convergence inside of the infimum in the definition of w^ε. However, the difficulty arises in trying to use $L^\varepsilon(x-y,t-s;0,0,\omega)$ to obtain convergence for $L^\varepsilon(x,t;y,s,\omega)$. We pick up a translation on ω which changes with ε:

$$L^\varepsilon(x,t;y,s,\omega) = L^\varepsilon(x-y,t-s;0,0,\tau_{(y/\varepsilon,s/\varepsilon)}\omega).$$

Thus, more work must be done to show that we do indeed get the correct limit in the end, despite the presence of the translations, $\tau_{(y/\varepsilon,s/\varepsilon)}\omega$.

Lemma 5.1. *Let $\eta > 0$ be given and let t, z be fixed. Then there exists a set, $G_\eta^{z,t}$, such that $\mathbb{P}(G_\eta^{z,t}) \geq 1-\eta$ and*

$$\lim_{\varepsilon\to 0} \sup_{\omega\in G_\eta^{z,t}} |L^\varepsilon(z,t;0,0,\omega) - t\bar{L}(\frac{z}{t})| = 0.$$

Proof of Lemma 5.1. Lemma 4.2 gives us the convergence for a.e.ω, thus this is exactly the statement of Egoroff's theorem. □

Lemma 5.2 (Uniform Convergence of Subsets of Ω)**.** *Let $\rho > 0$ and $K > 0$ be given, then there exists a set G_η such that $\mathbb{P}(G_\eta) \geq 1-\eta$ and*

$$\lim_{\varepsilon\to 0} \sup_{\substack{|z|\leq K\\ t\geq\rho}} \sup_{\omega\in G_\eta} |L^\varepsilon(z,t;0,0,\omega) - t\bar{L}(\frac{z}{t})| = 0.$$

Proof. We must appropriately define the set G_η. Once that has been done, the statement of the lemma is exactly the statement that pointwise convergence of a uniformly equicontinuous family to a uniformly continuous limit is in fact uniform convergence. We accordingly note that the family $L^\varepsilon(z,t,0,0,\omega)$ is uniformly equicontinuous in z,t, due to the restriction that $t\geq\rho$ and $|z|\leq K$. Let z_n, t_n, for $n\geq 1$, be an enumeration of $\mathbb{Q}^{n+1}$. Then take the good sets, $G^{z_n,t_n}_{\eta\times 2^{-n}}$, and construct G_η as

$$G_\eta = \bigcap_{n\geq 1} G^{z_n,t_n}_{\eta\times 2^{-n}}\ .$$

We estimate the measure of G_η^c:

$$\mathbb{P}(G_\eta^c) = \mathbb{P}(\bigcup_{n\geq 1}(G^{z_n,t_n}_{\eta\times 2^{-n}})^c) \leq \sum_{n\geq 1}\eta 2^{-n} = \eta.$$

□

We will now be using the local uniform convergence of the solutions near the point $(0,0)$ in order to establish the convergence for all other points. This starts out with the next proposition.

Proposition 5.3 (Uniform Convergence at $x=0$, $t=0$)**.** *Let η be fixed with G_η as above. Then*

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\sup_{\omega\in G_\eta}\sup_{|x|,|t|\leq r} |w^\varepsilon(x,t,\omega) - w(0,0)| = 0.$$

Moreover, this holds uniformly for any choice of $w_T \in BUC(\mathbb{R}^n)$ *with* $\|w_T\|_\infty \le M$.

Proof of Proposition 5.3. Let x, t be fixed. This result will follow directly from the uniform continuity of the "fundamental" solutions, L^ε. Let m denote this modulus for L^ε.

We will show one inequality first. To this end, suppose that z is a point which optimizes the expression in the definition of w (from Proposition 4.6). The uniform coercivity of $\bar{L}$ ensures that z is uniformly bounded depending only on the bound on x and the bound on $\|w_T\|_\infty$, but not on the particular choice of x and w_T. Then we have:

$$\begin{aligned} &w^\varepsilon(x,t,\omega) - w(0,0)\\ &\le w_T(z) + L^\varepsilon(z,T;x,t,\omega) - w_T(z) - T\bar{L}(\frac{z}{T})\\ &\le L^\varepsilon(z,T;0,0,\omega) - T\bar{L}(\frac{z}{T}) + m(r)\\ &\le \sup_{|z|\le K}\sup_{\omega\in G_\eta} |L^\varepsilon(z,T;0,0,\omega) - T\bar{L}(\frac{z}{T})| + m(r). \end{aligned}$$

Now we treat the reverse inequality. Let $z^\varepsilon(\omega)$ be the point that achieves the infimum for $w^\varepsilon(x,t,\omega)$ (using Proposition 3.2). We note that there is $\tilde{K}$, only depending on the uniform coercivity of L^ε (from Lemma 3.6), such that $|z^\varepsilon(\omega)| \le \tilde{K}$. We thus see

$$\begin{aligned} &w(0,0) - w^\varepsilon(x,t,\omega)\\ &\le w_T(z^\varepsilon(\omega)) + T\bar{L}(\frac{z^\varepsilon(\omega)}{T}) - w_T(z^\varepsilon(\omega)) - L^\varepsilon(z^\varepsilon(\omega),T;x,t,\omega)\\ &\le T\bar{L}(\frac{z^\varepsilon(\omega)}{T}) - L^\varepsilon(z^\varepsilon(\omega),T;0,0,\omega) + m(r)\\ &\le \sup_{|z|\le \tilde{K}}\sup_{\omega\in G_\eta} |T\bar{L}(\frac{z}{T}) - L^\varepsilon(z,T;0,0,\omega)| + m(r). \end{aligned}$$

Therefore, so long as ω is in one of the "good" sets, G_η we have by Lemma 5.2:

$$\lim_{\varepsilon\to 0}\sup_{\omega\in G_\eta} |w^\varepsilon(x,t) - w(0,0)| \le m(r),$$

and we conclude the proof of the lemma. □

Remark 5.4. Suppose that there is a sublinear function, ϕ, such that $|w_T^\varepsilon| \le \phi$ and $|w_T| \le \phi$. Suppose also that $w_T^\varepsilon \to w_T$ locally uniformly as $\varepsilon \to 0$. Then the assertion of Proposition 5.3 still remains true. The proof of this statement goes in almost the same way as above. It just utilizes the note from the proof of Lemma 3.7 regarding the boundedness of the points $z^\varepsilon(\omega)$ and z.

In the upcoming proof of Proposition 4.6, we will want to use a slight variation on the previous lemma. It will be useful to incorporate the fact that the estimate is unchanged with respect to translations in the terminal data. This will be useful in order to utilize the fact that translations in space correspond to translations on Ω, via the transformation, τ.

Remark 5.5. For notational purposes we will define $v_{\hat{x},\hat{t}}$ to be the solution of (1.1) with a terminal time of $T - \hat{t}$ and terminal data given by $v_{\hat{x},\hat{t}}(x, T-\hat{t}) = w_T(x+\hat{x})$.

We thus have with the same techniques as the previous lemma:

Proposition 5.6. *For $K > 0$ and $\rho > 0$, for any translation (see Corollary 3.4 and Remark 5.5) of w^ε:*

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\sup_{\substack{|\hat{x}|\le K\\ T-\hat{t}\ge\rho}}\sup_{|x|,|t|\le r}\sup_{\omega\in G_\eta} |w^\varepsilon_{\hat{x},\hat{t}}(x,t,\omega) - w_{\hat{x},\hat{t}}(0,0)| = 0.$$

Proof. The proof proceeds very similarly to the previous one after we adjust three things:

$$w^{\varepsilon}_{\hat{x},\hat{t}}(x,t,\omega) = \inf_{z\in\mathbb{R}^n}\big\{w_T(z+\hat{x}) + L^{\varepsilon}(z, T-\hat{t}, x, t, \omega)\big\},$$
$$w_{\hat{x},\hat{t}}(x,t) = \inf_{z\in\mathbb{R}^n}\big\{w_T(z+\hat{x}) + (T-\hat{t}-t)\bar{L}(\frac{z}{T-\hat{t}-t})\big\},$$

and we apply lemma 5.2 to the quantity

$$\sup_{\substack{|z|\le K \\ T-\hat{t}\ge\delta}} \sup_{\omega\in G_\eta} |L^{\varepsilon}(z, T-\hat{t}, 0, 0, \omega) - (T-\hat{t})\bar{L}(\frac{z}{T-\hat{t}})|.$$

□

Before we continue to the proof of Proposition 4.6, we will need one more small lemma which comes from [18]:

Lemma 5.7. *Let G_η be such that $\mathbb{P}(G_\eta) \to 1$ as $\eta \to 0$. Then there exist a function $m(\eta)$ and a set of full measure Ω_η, such that for $\varepsilon > 0$ chosen small enough:*

$$\begin{aligned} &m(\eta) \to 0 \text{ as } \eta \to 0,\\ &\forall\omega\in\Omega_\eta \text{ and } \forall(x,t)\in B_{1/\varepsilon}, \text{ there is } (\hat{x},\hat{t}) \text{ such that}\\ &(\hat{x},\hat{t}) \in \{x,t : \tau_{(x,t)}\omega \in G_\eta\} \cap B_{1/\varepsilon}\\ &\text{and } |\hat{t}-t| + |x-\hat{x}| \le \frac{m(\eta)}{\varepsilon}. \end{aligned}$$

Proof of Lemma 5.7. This proof will be a consequence of the Ergodic Theorem combined with the regularity of Lebesgue measure on $\mathbb{R}^n$. For ease of notation, we will use B_r to represent the ball centered at $(0,0)$ in $\mathbb{R}^{n+1}$.

We begin by applying the ergodic theorem to the function F_η, defined as:

$$F_\eta(\omega) = \begin{cases} 1 & \text{if } \omega \in G_\eta \\ 0 & \text{otherwise .}\end{cases}$$

The ergodic theorem says there exists Ω_η with $P(\Omega_\eta) = 1$ and $\forall\omega\in\Omega_\eta$

$$\lim_{r\to\infty}\frac{1}{|B_r|}\int_{B_r} F_\eta(\tau_{x,s}\omega)dxds = \int_\Omega F_\eta(\omega)d\mathbb{P}(\omega).$$

Specifically, for ε small enough and $\forall\omega\in\Omega_\eta$:

$$\frac{|\{x,t : \tau_{x,t}\omega \in G_\eta\} \cap B_{1/\varepsilon}|}{|B_{1/\varepsilon}|} \ge P(G_\eta) - \eta \ge 1 - 2\eta.$$

In other words,

$$|\{x,t : \tau_{x,t}\omega \in G_\eta\} \cap B_{1/\varepsilon}| \ge (1-2\eta)|B_{1/\varepsilon}|.$$

In order to find the function $m(\eta)$, we will use the regularity property of Lebesgue measure. Let us call the good set $G = \{x,t : \tau_{x,t}\omega \in G_\eta\} \cap B_{1/\varepsilon}$ and the bad set will be $G^c \cap B_{1/\varepsilon}$. The outer regularity of Lebesgue measure says that there is a basic set (a finite union of balls),

$$E = \bigcup_{i=1}^{M} B_{r_i},$$

such that $G^c \cap B_{1/\varepsilon} \subset E$ and

$$|E| - \eta \le |G^c \cap B_{1/\varepsilon}|.$$

We also know from above that

$$|G^c \cap B_{1/\varepsilon}| = |B_{1/\varepsilon}| - |G| \leq 2\eta |B_{1/\varepsilon}|.$$

Hence

$$|E| \leq 2\eta |B_{1/\varepsilon}| + \eta.$$

The worst case scenario regarding the distance from $(x,t) \in E$ to $(\hat{x},\hat{t}) \in E^c \cap B_{1/\varepsilon}$ is when E is one ball, and x is at its center. Thus (the $n+1$ comes from $(x,t) \in \mathbb{R}^{n+1}$)

$$\begin{aligned}
|(x,t) - (\hat{x},\hat{t})| &\leq \big(\frac{2\eta C_{n+1}}{\varepsilon^{n+1}} + \eta\big)^{1/(n+1)} \\
&= \big(\frac{(2C_{n+1}\eta + \varepsilon^{n+1}\eta)}{\varepsilon^{n+1}}\big)^{1/(n+1)} \leq \frac{(3C_{n+1}\eta)^{1/(n+1)}}{\varepsilon} \\
&:= \frac{m(\eta)}{\varepsilon}.
\end{aligned}$$

Which completes the proof of the lemma. □

We are now in a position to complete the proof of uniform convergence which was stated in Proposition 4.6.

Proof of Proposition 4.6. We will prove the existence of a set of full measure, $\tilde{\Omega}_1$, such that for $w_T \in C^{0,1}(\mathbb{R}^n)$ and for any $K > 0$, for a.e. $\omega \in \tilde{\Omega}_1$:

$$\lim_{\varepsilon \to 0} \sup_{|x| \leq K,\ t \in [0,T]} |w^\varepsilon(x,t,\omega) - w(x,t)| = 0.$$

First of all, we must recall that previously the set $\tilde{\Omega}$ came from Lemma 4.2. We must further modify it to conclude our result. We will assume that $\eta \in \mathbb{Q}$, and we will call the set Ω_η the set of full measure corresponding to the proof of Lemma 5.7. We take the new $\tilde{\Omega}_1$ as

$$\tilde{\Omega}_1 := \big(\bigcap_{0<\eta<1} \Omega_\eta\big) \cap \tilde{\Omega}. \tag{5.1}$$

Then $\mathbb{P}(\tilde{\Omega}_1) = 1$ and the ergodic theorem used in Lemma 5.7 holds for all F_η.

We begin by noting that in light of Lemma 3.8, we may assume that t is bounded away from T. So we assume $T - t > \rho$ for some $\rho > 0$. First, we let η be fixed, small enough. Now for each $\varepsilon \leq \varepsilon_0$, take $(\hat{x},\hat{t})$, depending on ε, such that

$$|\frac{x}{\varepsilon} - \frac{\hat{x}}{\varepsilon}| + |\frac{t}{\varepsilon} - \frac{\hat{t}}{\varepsilon}| \leq \frac{m(\eta)}{\varepsilon} \quad \text{and} \quad (\frac{\hat{x}}{\varepsilon}, \frac{\hat{t}}{\varepsilon}) \in \{x,t : \tau_{(x,t)}\omega \in G_\eta\}.$$

We will use the translation property of solutions, uniform continuity of w^ε, and uniform continuity of w (which is a classical fact). Suppose that M is a uniform modulus of continuity for w^ε and w. We may then conclude as follows:

$$\begin{aligned}
&|w^\varepsilon(x,t,\omega) - w(x,t)| \\
&\leq |w^\varepsilon(\hat{x},\hat{t},\omega) - w(\hat{x},\hat{t})| + M(|\hat{x} - x| + |\hat{t} - t|) \\
&= |w^\varepsilon_{\hat{x},\hat{t}}(0,0,\tau_{(\hat{x}/\varepsilon,\hat{t}/\varepsilon)}\omega) - w_{\hat{x},\hat{t}}(0,0)| + M(m(\eta)) \\
&\leq \sup_{\substack{|\hat{x}| \leq 2K \\ T-\hat{t} \geq \rho}} \sup_{|(y,s)| \leq m(\eta)} \sup_{\omega \in G_\eta} |w^\varepsilon_{\hat{x},\hat{t}}(y,s,\omega) - w_{\hat{x},\hat{t}}(0,0)| + M(m(\eta)).
\end{aligned}$$

Letting first $\varepsilon \to 0$ and then $\eta \to 0$, we conclude by appealing to Proposition 5.6. □

This completes the proof of Proposition 4.6. We now finish this section with a few other small propositions that will be useful later. Then we will prove the special case of Theorem 1.1 mentioned in the title of this section.

Remark 5.8. Using Lemma 5.7, and the technique of Proposition 4.6, it can be proved that on $\tilde{\Omega}_1$, as $\varepsilon \to 0$, locally uniformly in x, y and for $T - t \geq \rho > 0$,

$$L^\varepsilon(y, T; x, t, \omega) \to (T-t)\bar{L}(\frac{y-x}{T-t}).$$

With the help of Remark 5.4, a slight reworking of the proof of Proposition 4.6 yields the following proposition.

Proposition 5.9. *Suppose that* $w_T^\varepsilon, w_T \in C^{0,1}(\mathbb{R}^n)$ *and there is a sublinear function,* ϕ*, such that* $|w_T^\varepsilon| \leq \phi$ *and* $|w_T| \leq \phi$*. Suppose also that* $w_T^\varepsilon \to w_T$ *locally uniformly as* $\varepsilon \to 0$*. Then the assertion of Proposition 4.6 still remains true.*

There is one more piece of information that will be useful in section 6. It is a one sided outcome of the previous proposition if more general terminal data are allowed.

Proposition 5.10. *Let* $w_T^\varepsilon \in BUC(\mathbb{R}^n)$*,* $\|w_T^\varepsilon\| \leq C$ *uniformly in* ε*, and let* w^ε *be the solution of (3.1) with terminal data* w_T^ε*. Then for* $T - t \geq \delta > 0$*,*

$$(w^\varepsilon)_*(x,t) \geq \inf_{y\in\mathbb{R}^n}\{(w_T^\varepsilon)_*(y) + (T-t)\bar{L}(\frac{y-x}{T-t})\}.$$

Proof of Proposition 5.10. We first note that the above formula is well defined since $(w^\varepsilon)_*$ is bounded below and lower semicontinuous. Let $\omega \in \tilde{\Omega}_1$ (from the proof of Proposition 4.6) be fixed.

We start with the local infimum of w^ε, and suppose $z_\varepsilon, s_\varepsilon$ are points where the infimum is attained,

$$\inf_{\{|z-x|+|t-s|\leq\varepsilon\}} w^\varepsilon(z,s) = w^\varepsilon(z_\varepsilon, s_\varepsilon).$$

Suppose that y_ε is a point achieving the infimum in the definition of $w^\varepsilon(z_\varepsilon, s_\varepsilon)$. We note that y_ε are uniformly bounded by the fact that w_T^ε are uniformly bounded and L^ε enjoy the bounds in Lemma 3.6. Let $\bar{y}$ be any possible limit point of y_ε its corresponding subsequence still denoted by ε. Let m be the modulus of continuity of L^ε. Then we have

$$\begin{aligned}
w^\varepsilon(z_\varepsilon, s_\varepsilon, \omega) &= w_T^\varepsilon(y_\varepsilon, \omega) + L^\varepsilon(y_\varepsilon, T; z_\varepsilon, s_\varepsilon, \omega) \\
&\geq w_T^\varepsilon(y_\varepsilon, \omega) + L^\varepsilon(y_\varepsilon, T; x, t, \omega) - m(\varepsilon) \\
&\geq w_T^\varepsilon(y_\varepsilon, \omega) + L^\varepsilon(\bar{y}, T; x, t, \omega) - m(|y_\varepsilon - \bar{y}|) - m(\varepsilon).
\end{aligned}$$

Now taking $\liminf$ on both sides, we have

$$\begin{aligned}
(w^\varepsilon)_*(x,t) &= \liminf_{\varepsilon\to 0} w^\varepsilon(z_\varepsilon, s_\varepsilon) \\
&\geq \liminf_{\varepsilon\to 0} w_T^\varepsilon(y_\varepsilon, \omega) + \liminf_{\varepsilon\to 0} L^\varepsilon(\bar{y}, T; x, t, \omega) \\
&\geq (w_T^\varepsilon)_*(\bar{y}, \omega) + \liminf_{\varepsilon\to 0} L^\varepsilon(\bar{y}, T; x, t, \omega) \\
&= (w_T^\varepsilon)_*(\bar{y}, \omega) + (T-t)\bar{L}(\frac{\bar{y}-x}{T-t}) \\
&\geq \inf_{y\in\mathbb{R}^n}\{(w_T^\varepsilon)_*(y, \omega) + (T-t)\bar{L}(\frac{y-x}{T-t})\}.
\end{aligned}$$

This concludes the proof of the proposition. $\square$

We are now in a position to conclude this section with the proof of Theorem 1.1, part (i) for the special case that H satisfies (1.8).

Proof of Theorem 1.1, (i) assuming (1.8). First of all, assume that the initial conditions do not depend on ε or ω. That is $u_0^\varepsilon(\omega) \equiv u_0$. We will remove this assumption at the end of the proof. In order to conclude the validity of Theorem 1.1, we shall appeal to the Hopf-Lax-Oleinik formula for the solutions of

$$\begin{cases} v_t = \bar{G}(Dv) \text{ in } \mathbb{R}^n \times [0,T] \\ v(x,T) = w_T(x) \text{ on } \mathbb{R}^n \times \{T\} \ . \end{cases} \tag{5.2}$$

Then we will use Proposition 4.6. For this, we must define $\bar{G}(p) = \bar{L}^*(-p)$.

For $w_T \in BUC(\mathbb{R}^n)$, w as defined in Proposition 4.6 is precisely the solution of (5.2). Thus if $w_T \in C^{0,1}(\mathbb{R}^n)$, Proposition 4.6 gives the proof of Theorem 1.1. If $w_T \in BUC(\mathbb{R}^n)$, the result holds by the density of $C^{0,1}(\mathbb{R}^n)$ in $BUC(\mathbb{R}^n)$ with respect to the norm, $\|\cdot\|_\infty$, and the comparison property of solutions of (1.1).

We now note that with the change of variables from section 3 and equation (3.1), we have proved

$$u^\varepsilon(x,t,\omega) = w^\varepsilon(x,T-t,\omega) \to w(x,T-t).$$

Moreover, under the change of time $t \mapsto T-t$, $w(x,T-t)$ is the unique solution of (1.2) where $\bar{H}^*(p) := \bar{L}(-p)$. Hence we have proved that $u^\varepsilon(\omega) \to u$ locally uniformly for all $\omega \in \tilde{\Omega}$.

Now it is straightforward to remove the restriction on the initial data. Assume that for each ω, $u_0^\varepsilon(\omega) \to u_0$ locally uniformly. We now combine the comparison properties of (1.1) with the work done in the first part of this proof. Suppose that $U^\varepsilon(\omega)$ is the solution of (1.1) with the initial data given by $U^\varepsilon(\cdot,0,\omega) = u_0$. Let K be a compact set. We then have

$$\|u^\varepsilon(\omega) - u\|_{\infty,K} \leq \|u^\varepsilon(\omega) - U^\varepsilon(\omega)\|_{\infty,K} + \|U^\varepsilon(\omega) - u\|_{\infty,K}.$$

For $a.e.\omega$ the first term goes to zero by the assumption on $u_0^\varepsilon(\omega)$ and the comparison property of (1.1). For $a.e.\omega$, the second term goes to zero by the homogenization already proved.

We finally note that the bounds on $\bar{H}$ are an immediate consequence of the bounds on $\bar{L}$ given in Lemma 4.2. This completes the proof of Theorem 1.1, (i). $\square$

Remark 5.11. Proposition 5.9 implies that this result holds for u_0^ε, u_0 strictly sublinear functions on $\mathbb{R}^n$.

6. Proof of Theorem 1.1, (ii) Assuming (1.8)

This section is dedicated to the proof of Theorem 1.1, (ii), the Inf-Sup formula for the effective Hamiltonian. At this point, the notation will become difficult. There are now two representatives for the effective Hamiltonian, $\bar{H}$. The first is the result of the Subadditive Theorem, proved in the previous section. The other is the one given by the inf-sup formula in Theorem 1.1 part (ii). We fix the notation that $\bar{H}(p)$ refers to the first, and $\hat{H}(p)$ refers to the second. The goal, of course, will be to show they are the same.

We proceed with the proof in a number of smaller steps. The main tool that allows us to conclude the validity of Theorem 1.1, (ii) is the analysis found in [18], which was used on the "viscous" version of (1.1). We first recall the definition of the class, $\mathcal{S}$:

$$\mathcal{S} := \big\{\Phi \big| \ a.s.\omega \lim_{|(x,t)|\to\infty} \frac{\Phi(x,t,\omega)}{|(x,t)|} = 0; \ D\Phi, \Phi_t \in L^\infty_{loc}(\mathbb{R}^{n+1});$$
$$D\Phi, \Phi_t \text{ are stationary and mean zero}\big\}.$$

The methods of [18] allow for the construction of approximate subcorrectors which are in the class $\mathcal{S}$. These will be functions, $W^\gamma \in \mathcal{S}$, such that

$$W_t^\gamma + H(x,t,p+DW^\gamma) \leq \bar{H}(p) + m(\gamma),$$

where $m(\gamma) \to 0$ as $\gamma \to 0$. This will be the content of Proposition 6.3.

The proof of Theorem 1.1, (ii) will be broken into a few smaller steps. We begin with those here. The first lemma can be found in the results of [3], and so we state it without proof.

Lemma 6.1. *For each ω, p, and α, there exist unique solutions, $V^\alpha(\omega) \in C^{0,1}(\mathbb{R}^{n+1})$, of the equation*

$$\alpha V^\alpha + V_t^\alpha + H(x,t,p+DV^\alpha,\omega) = 0 \text{ in } \mathbb{R}^{n+1}. \tag{6.1}$$

Moreover, V^α is stationary with respect to τ.

The stationarity of V^α is a consequence of the stationarity of H combined with the uniqueness for (6.1). Moreover, because of the structure of H, there are constant sub and super solutions, $-M/\alpha$ and M/α (here we have $M = \|H(\cdot,\cdot,p,\omega)\|_\infty$). These constants provide bounds on V^α. We note that since V^α is a stationary function, both V_t^α and DV^α are stationary and have mean zero.

Let p be fixed. The function $C(p,\omega) = \|H(\cdot,\cdot,p,\omega)\|_\infty$ is invariant under τ. Hence by ergodicity, $C(p,\omega)$ is a constant in ω. Now consider the new Hamiltonian given by

$$\tilde{H}(x,t,Dv,\omega) = H(x,t,Dv+p,\omega) - C(p),$$

and the function v^α solving

$$\alpha v^\alpha + v_t^\alpha + \tilde{H}(x,t,Dv^\alpha,\omega) = 0 \text{ in } \mathbb{R}^{n+1}. \tag{6.2}$$

Notice that this gives for all x, t, and ω, $\tilde{H}(x,t,0,\omega) \leq 0$. Also, v^α enjoys the same properties just listed above of V^α. Changing the equation by a constant does not change the boundedness or stationarity properties of v^α. Moreover, because $\tilde{H}(x,t,0,\omega) \leq 0$, the constant 0 is a subsolution, and hence $v^\alpha \geq 0$ by comparison.

Lemma 6.2. *Assume that H satisfies (1.8), $\tilde{H}$ is given as above, and that v^α solves the equation corresponding to $\tilde{H}$, (6.2). Then for a.e.ω, we have*

$$\big[\alpha v^\alpha(\frac{\cdot}{\alpha},\frac{\cdot}{\alpha})\big]_*(x,t) \geq -\bar{H}(p) + C(p),$$

where $\bar{H}$ comes from Theorem 1.1, (i).

Proof of Lemma 6.2. The main point of this proof will be to show a one-sided dynamic programming relationship for the function $w = (w^\alpha)_*$, where

$$w^\alpha(x,t) := \alpha v^\alpha(\frac{x}{\alpha},\frac{t}{\alpha}).$$

We claim that w satisfies for each s fixed and $t \geq s$

$$w(x,t) \geq e^{-(t-s)}\big[\inf_y\{w(y,s) + (t-s)[\bar{H}(\cdot+p)]^*(\frac{x-y}{t-s})\} + (t-s)C(p)\big]. \tag{6.3}$$

Once (6.3) has been shown, it follows by classical methods for viscosity solutions that w solves

$$w + w_t + \bar{H}(Dw+p) - C(p) \geq 0 \text{ on } \mathbb{R}^{n+1}.$$

Comparison with the exact solution (which is a constant) gives

$$w \geq -\bar{H}(p) + C(p).$$

To prove (6.3), we begin with the change of variables $w^\alpha = e^{-(t-s)} z^\alpha$, where z^α solves

$$\begin{cases} z_t^\alpha + e^{(t-s)} \tilde{H}(\frac{x}{\alpha}, \frac{t}{\alpha}, e^{-(t-s)} Dz^\alpha, \omega) = 0 & \text{in } \mathbb{R}^n \times [s, s+T] \\ z^\alpha(x,s) = w^\alpha(x,s) & \text{on } \mathbb{R}^n \times \{s\} \ . \end{cases}$$

We note that $w^\alpha \geq 0$ from because $v^\alpha \geq 0$, and hence $w \geq 0$. Also the assumption that $\tilde{H}(x,t,0,\omega) \leq 0$ tells us by convexity for $t \in [s, \infty)$,

$$e^{(t-s)} \tilde{H}(x,t,e^{-(t-s)}p,\omega) \leq \tilde{H}(x,t,p,\omega).$$

Thus z^α is a supersolution of the equation

$$\begin{cases} u_t^\alpha + \tilde{H}(\frac{x}{\alpha}, \frac{t}{\alpha}, Du^\alpha, \omega) = 0 & \text{in } \mathbb{R}^n \times [s, s+T] \\ u^\alpha(x,s) = w^\alpha(x,s) & \text{on } \mathbb{R}^n \times \{s\} \end{cases}$$

and hence $z^\alpha \geq u^\alpha$. However, we now make use of the fact that $\tilde{H}(x,t,q) = H(x,t,p+q) - C(p)$. Thus if U^α solves

$$\begin{cases} U_t^\alpha + H(\frac{x}{\alpha}, \frac{t}{\alpha}, DU^\alpha + p, \omega) = 0 & \text{in } \mathbb{R}^n \times [s, s+T] \\ U^\alpha(x,s) = w^\alpha(x,s) & \text{on } \mathbb{R}^n \times \{s\} \end{cases}$$

we have that $u^\alpha(x,t) = U^\alpha(x,t) + (t-s)C(p)$. Proposition 5.10 implies that

$$(u^\alpha)_*(x,t) \geq \inf_y \{w(y,s) + (t-s)\tilde{\bar{H}}^*(\frac{x-y}{t-s})\} + (t-s)C(p),$$

where we use the notation $\tilde{\bar{H}}(q) = \bar{H}(q+p) - C(p)$. Unravelling the comparison with w^α and u^α, we get

$$w^\alpha \geq e^{-(t-s)} u^\alpha,$$

hence

$$(w^\alpha)_*(x,t) \geq e^{-(t-s)} (u^\alpha)_*(x,t),$$

and (6.3) follows. □

Proposition 6.3 (Kosygina-Varadhan). *Assume that H satisfies (H4), (1.8), and that $\bar{H}(p)$ is given by Theorem 1.1, (i). Then for the modulus, m, given in (H4), for each $\gamma > 0$, there is a function $W^\gamma \in \mathcal{S}$, such that*

$$W_t^\gamma + H(x,t,p+DW^\gamma) \leq \bar{H}(p) + Cm(\gamma), \tag{6.4}$$

where C is fixed and $m(\gamma) \to 0$ as $\gamma \to 0$.

Proof of Proposition 6.3. We will attempt to keep similar notation and terminology used in [18], section 4.

We begin with a discussion of the main ideas so as to keep the overall strategy clear. Then the technical details will be provided. The goal will be to extract a weakly convergent (in $L^1(\Omega)$) subsequence from (v_t^α, Dv^α) and then try to extend these weak limits to a subsolution on $\mathbb{R}^{n+1}$, which will be the function, W^γ. We begin with a few comments. The starting point is the equation satisfied by v^α. We have

$$\alpha v^\alpha + v_t^\alpha + H(x,t,p+Dv^\alpha,\omega) - C(p) = 0 \text{ in } \mathbb{R}^{n+1}, \tag{6.5}$$

and so we would like to extract limits from the functions

$$f^\alpha(\omega) := v_t^\alpha(0,0,\omega) \text{ and } g^\alpha(\omega) := Dv^\alpha(0,0,\omega).$$

However, it may not be possible to do this directly since v^α is only Lipschitz, and not necessarily C^1. This will be made rigorous in what follows. Under this choice for f^α and g^α, the properties of viscosity solutions, and the convexity of H, we have in for each $q \in \mathbb{R}^n$ fixed and $a.e.\omega$:

$$\alpha v^\alpha(0,0,\omega) + f^\alpha(\omega) + q \cdot g^\alpha(\omega) - [H(0,0,p+\cdot,\omega)]^*(q) - C(p) \leq 0.$$

If we can take weak limits of $f^\alpha \rightharpoonup f$ and $g^\alpha \rightharpoonup g$, then for each $q \in \mathbb{R}^n$ fixed, we have by Fatou's Lemma and Lemma 6.2

$$-\bar{H}(p) + C(p) + f(\omega) + q \cdot g(\omega) - [H(0,0,p+\cdot,\omega)]^*(q) - C(p) \leq 0.$$

We note that $(\alpha v^\alpha)_*(0,0) \leq \liminf \alpha v^\alpha(0,0)$. A function W can be constructed as the anti-derivative of $[f,g]$. Then after mollification, we will arrive at the goal of constructing W^γ. It will satisfy as a smooth function on $\mathbb{R}^{n+1}$ for a.e. ω and for q fixed,

$$-\bar{H}(p) + W_t^\gamma + q \cdot DW^\gamma - [H(x,t,p+\cdot,\omega)]^*(q) \leq m(\gamma).$$

and hence

$$W_t^\gamma + H(x,t,p+DW^\gamma,\omega) \leq \bar{H}(p) + m(\gamma). \tag{6.6}$$

In fact, it may not be possible to take weak limits of a subsequence of f^α directly. The actual way of obtaining f will be through the decomposition of f^α into its positive and negative parts, and then through a further decomposition of the negative part, $(f^\alpha)^-$, into good and bad pieces.

With the main ideas of the proof out of the way, we begin with the details. Our first step will be to compensate for the fact that v^α are not C^1. Let ρ_ε be the standard mollifier, and define $w^{\alpha\varepsilon}$ as the convolution

$$w^{\alpha\varepsilon} = v^\alpha * \rho_\varepsilon. \tag{6.7}$$

(We draw attention to the fact that this use of the parameter ε here is not at all related to the one used for the scaling of (1.1) in the previous sections.) Classical properties of viscosity solutions (see [20]), (H4), and convexity of H (see [19] for details) yield that for each q fixed:

$$\alpha w^{\alpha\varepsilon} + w_t^{\alpha\varepsilon} + q \cdot Dw^{\alpha\varepsilon} - [H(x,t,p+\cdot,\omega)]^*(q) - C(p) \leq m(\varepsilon). \tag{6.8}$$

We define

$$f^{\alpha\varepsilon} = w_t^{\alpha\varepsilon}(0,0,\cdot) \text{ and } g^{\alpha\varepsilon} = Dw^{\alpha\varepsilon}(0,0,\cdot). \tag{6.9}$$

We wish to identify a subsequence in α of $f^{\alpha\varepsilon}$ and $g^{\alpha\varepsilon}$ along which we can take weak limits. This will be done in three parts: first we will show uniform integrability in $L^1(\Omega)$ of $(f^{\alpha\varepsilon})^+$, then we will state a decomposition lemma for $(f^{\alpha\varepsilon})^-$ which allows for a subsequence converging to zero, and finally we show uniform integrability of $g^{\alpha\varepsilon}$.

We now will show that $(f^{\alpha\varepsilon})^+$, is uniformly integrable. This will be established with two initial observations:

i) For each $k > 0$, $\sup\Big\{E(f^{\alpha\varepsilon}\Phi) : 0 \leq \Phi \leq k, E(\Phi) = 1\Big\} \leq M$.

ii) $E(|f^{\alpha\varepsilon}|) \leq M$.

The uniform integrability of $(f^{\alpha\varepsilon})^+$ will first be proved assuming (i) and (ii).

For each $l > 0$, we wish to estimate the integral:

$$\int_{\{f^{\alpha\varepsilon} > l\}} f^{\alpha\varepsilon}(\omega)dP(\omega).$$

To prove uniform integrability, we recall that this quantity must converge to 0 as $l \to \infty$, uniformly in α. We introduce the weight function Φ_k^α given by the formula

$$\Phi_k^\alpha(\omega) = \begin{cases} \frac{1}{k} & \text{if } f^{\alpha\varepsilon}(\omega) \leq l \\ k & \text{if } f^{\alpha\varepsilon}(\omega) > l. \end{cases}$$

We now see that we have by (i), using $\Phi^\alpha_k/(\mathbb{E}(\Phi^\alpha_k))$:

$$\frac{1}{k}\int_{\{f^{\alpha\varepsilon}\leq l\}} f^{\alpha\varepsilon}dP + k\int_{\{f^{\alpha\varepsilon}>l\}} f^{\alpha\varepsilon}dP = \\ = E(f^{\alpha\varepsilon}\Phi^\alpha_k) \leq ME(\Phi^\alpha_k).$$

Furthermore, we use the fact that $E(f^{\alpha\varepsilon})=0$, hence

$$\int_{\{f^{\alpha\varepsilon}>l\}} f^{\alpha\varepsilon}dP = -\int_{\{f^{\alpha\varepsilon}\leq l\}} f^{\alpha\varepsilon}dP.$$

(Recall v^α is stationary, and since convolution preserves stationarity, $w^{\alpha\varepsilon}$ is as well. This implies $w^{\alpha\varepsilon}_t$ and $Dw^{\alpha\varepsilon}$ are mean zero.) We thus have

$$\Big(k-\frac{1}{k}\Big)\int_{\{f^{\alpha\varepsilon}>l\}} f^{\alpha\varepsilon}dP \leq ME(\Phi^\alpha_k) = \\ = M\Big[\frac{1}{k}P(f^{\alpha\varepsilon}\leq l) + kP(f^{\alpha\varepsilon}>l)\Big] \\ \leq M\Big[\frac{1}{k} + k(\frac{1}{l}E(|f^{\alpha\varepsilon}|))\Big].$$

Now after applying (ii) and rearranging, we arrive at the inequality

$$\int_{\{f^\alpha>l\}} f^\alpha dP \leq \tilde{M}\frac{(1/k+k/l)}{(k-1/k)}.$$

First taking k large and then $l\to\infty$, we conclude that the integral can be made as small as we like, and hence the claim of uniform integrability. This implies that $(f^{\alpha\varepsilon})^+$ is uniformly integrable, and we are finished with the one of three parts.

To see why (i) holds, we go back to the original equation. By the coercivity of H^* (see (1.8)) we have for all ω and a.e.x,t,

$$w^{\alpha\varepsilon}_t(x,t,\omega) + C|Dw^{\alpha\varepsilon}(x,t,\omega)|^\beta \leq M,$$

where M depends only on H and p; C depends only on H (we have used the boundedness of $\alpha w^{\alpha\varepsilon}$). Hence after evaluating at $(0,0)$, multiplying by Φ, and taking expectations we get

$$\mathbb{E}(f^{\alpha\varepsilon}\Phi) \leq \mathbb{E}(f^{\alpha\varepsilon}\Phi) + \mathbb{E}(C|g^{\alpha\varepsilon}|^\beta\Phi) \leq M.$$

The assertion (ii) follows by noting that since $\mathbb{E}(f^{\alpha\varepsilon})=0$, we have $E([f^{\alpha\varepsilon}]^+)=E([f^{\alpha\varepsilon}]^-)$. The equation (6.8) implies for all ω and x,t that $w^{\alpha\varepsilon}_t(x,t,\omega)\leq M$. Hence $\mathbb{E}([f^{\alpha\varepsilon}]^+)\leq M$.

At this point we include a technical lemma from [18] which will be applied to $(f^{\alpha\varepsilon})^-$:

Lemma 6.4 (Kosygina-Varadhan)**.** *Let $\{h_n\}$ be a sequence of nonnegative functions in $L^1(\Omega)$ and $\sup_n\mathbb{E}(h_n)\leq C$. Then there exists a subsequence, $\{h_{n_j}\}$, such that*

$$h_{n_j} = \hat{h}_{n_1} + r(h_{n_j}) \tag{6.10}$$

with $\hat{h}_{n_j}$ uniformly integrable, and $r(h_{n_j})\to 0$ in probability.

In order to conclude that $g^{\alpha\varepsilon}$ is uniformly integrable, we simply use the fact that $f^{\alpha\varepsilon}$ has mean 0 and the lower bound on H (from (1.8)):

$$\mathbb{E}(C(|g^{\alpha\varepsilon}|^\beta - 1)) \leq \mathbb{E}(f^{\alpha\varepsilon}) + \mathbb{E}(H(g^{\alpha\varepsilon})) \leq M,$$

where M depends only on H and p. Hence, by Hölder's inequality, it follows that $g^{\alpha\varepsilon}$ is uniformly integrable.

We now put together the three parts of the weak limits. There will be multiple subsequences used, and we still refer to all of them with the index α (ε is still fixed, temporarily). Take the subsequence and decomposition $(f^{\alpha\varepsilon})^- = \hat{f}^{\alpha\varepsilon} + r^{\alpha\varepsilon}$, as provided by Lemma 6.4. Both $(f^{\alpha\varepsilon})^+$ and $g^{\alpha\varepsilon}$ have weakly convergent subsequences converging to f^ε and g^ε, respectively due to their uniform integrability. Finally, take a further subsequence for which $r^{\alpha\varepsilon} \to 0$ for $a.e.\omega$. Before taking limits in (6.8), it must be checked that $\alpha w^{\alpha\varepsilon}(0,0,\omega)$ will still provide the correct term in the limit. The definition of $w^{\alpha\varepsilon}$ gives

$$\begin{aligned}\alpha w^{\alpha\varepsilon}(0,0,\omega) &= \int \alpha v^\alpha(-x,-t,\omega)\rho_\varepsilon(x,t)dxdt\\ &= \int \alpha v^\alpha(-\frac{\alpha x}{\alpha}, -\frac{\alpha t}{\alpha},\omega)\rho_\varepsilon(x,t)dxdt.\end{aligned}$$

Due to Fatou's Lemma and Lemma 6.2, the limit gives for $a.e.\omega$

$$\lim_{\alpha\to 0} \alpha w^{\alpha\varepsilon}(0,0,\omega) \geq -\bar{H}(p) + C(p).$$

To take limits of (6.8), we integrate against a test function, $\psi \geq 0$, with $\int \psi d\mathbb{P} = 1$. The first two terms can be controlled with another application of Fatou's Lemma to conclude:

$$-\bar{H}(p) + \int f^\varepsilon \psi d\mathbb{P} + q \cdot \int g^\varepsilon \psi d\mathbb{P} - \int [H(0,0,p+\cdot,\omega)]^*(q)\psi d\mathbb{P} \leq m(\varepsilon).$$

Since this holds for any such ψ, we conclude the inequality for $a.e.\omega$. Finally, we rewrite f^ε as $f^\varepsilon = \tilde{f}^\varepsilon + \mathbb{E}(f^\varepsilon)$ with $\mathbb{E}(\tilde{f}^\varepsilon) = 0$ (recall $\mathbb{E}(f^\varepsilon) \geq 0$) to conclude for $a.e.\omega$

$$\tilde{f}^\varepsilon + q \cdot g^\varepsilon - [H(0,0,p+\cdot,\omega)]^*(q) \leq m(\varepsilon) + \bar{H}(p),$$

where $\mathbb{E}(\tilde{f}^\varepsilon) = \mathbb{E}(g^\varepsilon) = 0$.

The functions $\tilde{f}^\varepsilon$ and g^ε are almost ready to be used to construct the subcorrectors, W^γ. All that is needed is one more mollification to smooth $\tilde{f}^\varepsilon$ and g^ε. Take $F(x,t,\omega) = \tilde{f}^\varepsilon(\tau_{x,t}\omega)$, $G(x,t,\omega) = g^\varepsilon(\tau_{x,t},\omega)$, $F^{\varepsilon\gamma} = F * \rho_\gamma$, and $G^{\varepsilon\gamma} = G * \rho_\gamma$. Again using the convexity of H and assumption (1.6), see [19], we have for $a.e.\omega$:

$$F^{\varepsilon\gamma}(x,t,\omega) + q \cdot G^{\varepsilon\gamma}(x,t,\omega) - [H(x,t,p+\cdot,\omega)]^*(q) \leq \bar{H}(p) + m(\varepsilon) + m(\gamma).$$

This holds for all ε, and so we take $\varepsilon < \gamma$, and drop the superscripts from the resulting functions.

Due to the fact that F^γ and G^γ weakly have gradient structure, it is possible to antidifferentiate them to a function, W^γ, such that $(W^\gamma_t, DW^\gamma) = (F^\gamma, G^\gamma)$. Moreover, due to the fact that F^γ and G^γ are stationary and mean zero by construction, it follows that $W^\gamma \in \mathcal{S}$. Details of these last statements can be found in the appendix of [18]. □

We are now in a position to prove part (ii) of Theorem 1.1 in the case of (1.8).

Proof of Theorem 1.1 assuming (1.8). Let $\bar{H}(p)$ be given from part (i) of the theorem, as proved in section 5, and let $\hat{H}(p)$ be given by the formula in part (ii). Without loss of generality, assume that $p = 0$; denote $\hat{H}(0) = \hat{H}$ and $\bar{H}(0) = \bar{H}$. From Proposition 6.3, we obtain one inequality:

$$\hat{H} \leq \bar{H}.$$

Now to prove the reverse inequality, we argue in a very similar fashion to [22]. We fix γ and start with a function, Φ, such that

$$\Phi_t + H(x,t,D\Phi,\omega) \leq \hat{H} + \gamma.$$

We note that $\Phi^\varepsilon(x,t) := \varepsilon\Phi(x/\varepsilon, t/\varepsilon)$ solves

$$\Phi^\varepsilon_t + H(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, D\Phi^\varepsilon, \omega) \leq \hat{H} + \gamma,$$

and by the strict sublinearity of Φ, $\Phi^\varepsilon(\cdot,0,\omega)$ is strictly sublinear as well, with a uniform in ε bound on its growth. Consider now the function w^ε as the solution of

$$\begin{cases} w^\varepsilon_t + H(\frac{x}{\varepsilon},\frac{t}{\varepsilon},Dw^\varepsilon,\omega) = 0 & \text{in } \mathbb{R}^n\times[0,T] \\ w^\varepsilon(x,0) = \Phi^\varepsilon(x,0) & \text{on } \mathbb{R}^n\times\{0\}\ . \end{cases}$$

Taking $u^\varepsilon(x,t) = w^\varepsilon(x,t) + t(\hat{H}+\gamma)$, we then have u^ε solves

$$\begin{cases} u^\varepsilon_t + H(\frac{x}{\varepsilon},\frac{t}{\varepsilon},Du^\varepsilon,\omega) = \hat{H}+\gamma & \text{in } \mathbb{R}^n\times[0,T] \\ u^\varepsilon(x,0) = \Phi^\varepsilon(x,0) & \text{on } \mathbb{R}^n\times\{0\}\ . \end{cases}$$

Thus by comparison, $u^\varepsilon \geq \Phi^\varepsilon$, and hence

$$w^\varepsilon(x,t) \geq \Phi^\varepsilon(x,t) - t(\hat{H}+\gamma).$$

Also, the sublinearity of Φ gives that $\Phi^\varepsilon(\cdot,0)\to 0$ locally uniformly on $\mathbb{R}^n$. Hence we may apply Remark 5.11. We note that $\bar{w}(x,t) = -t\bar{H}(0)$ is the unique solution of

$$\begin{cases} \bar{w}_t + \bar{H}(D\bar{w}) = 0 & \text{in } \mathbb{R}^n\times[0,T] \\ \bar{w}(x,0) = 0 & \text{on } \mathbb{R}^n\times\{0\}. \end{cases}$$

Hence Theorem 1.1 tells us that $w^\varepsilon(0,1)\to -\bar{H}$. Thus

$$-\bar{H} = \lim_{\varepsilon\to 0} w^\varepsilon(0,1) \geq \lim_{\varepsilon\to 0}\Phi^\varepsilon(0,1) - (\hat{H}+\gamma).$$

Now again by the strict sublinearity of Φ, we conclude that $\lim_{\varepsilon\to 0}\Phi^\varepsilon(0,1) = 0$. Thus we have shown that $-\bar{H}\geq -(\hat{H}+\gamma)$, which concludes the second half of the theorem. □

7. Proof of Theorem 1.1 for General H

We will now prove the main homogenization result for general convex, superlinear Hamiltonians. The main idea of this section is to show that the approximation given by

$$(H^\delta)^*(x,t,p) := H^*(x,t,p) + \delta|p|^\beta, \tag{7.1}$$

provides sufficient control of u^ε to conclude that homogenization takes place. Here β is chosen so that the assumptions of Proposition 1.2 hold. We will then let $\delta\to 0$.

Proposition 7.1. *Let $\bar{H}^\delta$ be given by Theorem 1.1, (ii), applied to H^δ. Let $\bar{H}$ be defined via the inf-sup formula as in Theorem 1.1, using H. Then locally uniformly in p as $\delta\to 0$,*

$$\bar{H}^\delta(p)\to\bar{H}(p).$$

Proof of Proposition 7.1. Let us assume the pointwise convergence of $\bar{H}^\delta$, which will be proved in Lemma 7.2. Then we conclude this proof with two observations. The first is that $\bar{H}^\delta$ is an increasing sequence of uniformly bounded convex functions with a finite limit, and the second is that its limit must also be convex. Hence the uniform bound on the $\bar{H}^\delta$ gives a uniform Lipschitz bound by convexity. By Arzela-Ascoli, the sequence has at least one local uniform limit point, which must be unique by the assumed pointwise convergence. □

Lemma 7.2. *Assume that H satisfies (H4). As $\delta\to 0$, we have the pointwise convergence,*

$$\bar{H}^\delta(p)\to\bar{H}(p)\ .$$

Proof of Lemma 7.2. We first note that without loss of generality, we may assume $p = 0$. We will denote $\bar{H}^\delta(0) = \bar{H}^\delta$ and $\bar{H}(0) = \bar{H}$. To start, we observe that $\bar{H}^\delta$ is increasing as $\delta \to 0$, and $\bar{H}^\delta \le \bar{H}$. Thus $\bar{H}^\delta$ has a limit, and we denote it as $\tilde{H} := \lim_{\delta\to 0} \bar{H}^\delta$, and we automatically have $\tilde{H} \le \bar{H}$. We must now show that $\tilde{H} \ge \bar{H}$.

The strategy will be to use the same construction from Proposition 6.3. If we take Φ^δ to be functions that achieve the infimum in $\hat{H}^\delta$, we will then extract weak limits of $D\Phi^\delta$ and Φ^δ_t to construct subsolutions of

$$\Phi_t + H(x,t,D\Phi,\omega) \le \tilde{H}.$$

Now included is a sketch of the details which lead us to the same construction as in Proposition 6.3. To this end, let $\alpha > 0$ be fixed. Take $\Phi^{\delta,\alpha}$ to be functions from the inf-sup formula of $\bar{H}^\delta$ such that

$$\Phi^{\delta,\alpha}_t + H^\delta(x,t,D\Phi^{\delta,\alpha},\omega) \le \bar{H}^\delta + \alpha.$$

We will then extract subsequences from $D\Phi^{\delta,\alpha}$ and $\Phi^{\delta,\alpha}_t$ as $\delta \to 0$. The previous inequality also implies the same for $\tilde{H}$:

$$\Phi^{\delta,\alpha}_t + H^\delta(x,t,D\Phi^{\delta,\alpha},\omega) \le \tilde{H} + \alpha.$$

Hence for each q fixed,

$$\Phi^{\delta,\alpha}_t + q\cdot D\Phi^{\delta,\alpha} - (H^\delta)^*(x,t,q,\omega) \le \tilde{H} + \alpha.$$

From the definition of $(H^\delta)^*$ this says

$$\Phi^{\delta,\alpha}_t + q\cdot D\Phi^{\delta,\alpha} - H^*(x,t,q,\omega) - \delta|q|^\beta \le \tilde{H} + \alpha.$$

Arguing as in Proposition 6.3 above, it is possible to extract a weakly convergent subsequence, as $\delta \to 0$, to obtain functions $W^\gamma \in \mathcal{S}$ that satisfy for each $\gamma > 0$:

$$W^\gamma_t + H(x,t,DW^\gamma,\omega) \le \tilde{H} + m(\gamma) + \alpha.$$

We thus conclude, after taking the infimum over $\mathcal{S}$, that

$$\bar{H} \le \tilde{H} + \alpha.$$

Since α was arbitrary, we conclude the proof of Lemma 7.2. □

Now that the local uniform convergence of $\bar{H}^\delta \to \bar{H}$ has been established, it follows immediately from the stability property of viscosity solutions that u^δ is a solution of the same equation as u. Thus, so long as u^δ and u have the same initial data, they are exactly the same function by uniqueness for (1.2). Hence we have proved:

Lemma 7.3. *Let u^δ and u to be the solutions of (1.2) with the Hamiltonians $\bar{H}^\delta$ and $\bar{H}$ respectively. Then as $\delta \to 0$ we have the local uniform convergence:*

$$u^\delta \to u. \tag{7.2}$$

We now have enough tools to be able to prove Theorem 1.1, which we will do so here.

Proof Theorem 1.1 for general H. We only prove the theorem for a fixed initial data, $u_0 \in C^{0,1}(\mathbb{R}^n)$, for both u^ε and u. The statement as it applies for $u_0^\varepsilon \to u_0$ locally uniformly is exactly that as in the proof of the theorem given in the section 5.

A standard technique to show the local uniform convergence of u^ε is to show that the half relaxed upper and lower limits are the same: $(u^\varepsilon)^* = (u^\varepsilon)_*$. We will control $(u^\varepsilon)^*$ and $(u^\varepsilon)_*$ in two different fashions. One bound will come from a natural ordering imposed by the approximations, H^δ; the other will come from the inf-sup formula in the definition of $\bar{H}$.

We first will prove that $\bar{u} \le (u^\varepsilon)_*$. Note that for each p and ω fixed, the definition of $\bar{H}(p)$ implies the existence of approximate subcorrectors, $V^n \in \mathcal{S}$, such that

$$V_t^n + H(x,t,p+DV^n) \le \bar{H}(p) + \frac{1}{n} \ .$$

These strictly sublinear functions, V^n, allow for an application of the perturbed test function method, as used in [15]. This implies that $(u^\varepsilon)_*$ is indeed a supersolution of (1.2). Hence by comparison, we conclude that $\bar{u} \le (u^\varepsilon)_*$.

The control on $(u^\varepsilon)^*$ comes from our choice of approximations, H^δ, of the original Hamiltonian, H. Since the Legendre transform of H^δ is defined as

$$(H^\delta)^*(x,t,p,\omega) := H^*(x,t,p,\omega) + \delta|p|^\alpha \ge H^*(x,t,p,\omega),$$

the reverse inequality holds for the Hamiltonians:

$$H^\delta(x,t,p,\omega) \le H(x,t,p,\omega).$$

Take the functions, $u^{\varepsilon,\delta}$, to be the solutions of

$$\begin{cases} u_t^{\varepsilon,\delta}(x,t,\omega) + H^\delta(\frac{x}{\varepsilon},\frac{t}{\varepsilon},Du^{\varepsilon,\delta},\omega) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u^{\varepsilon,\delta}(\cdot,0,\omega) = u_0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{7.3}$$

Then $u^{\varepsilon,\delta}$ are supersolutions of (1.1), and hence $u^\varepsilon \le u^{\varepsilon,\delta}$ for all $\varepsilon, \delta > 0$. Thus for each δ, we have by the above comparison and Theorem 1.1

$$(u^\varepsilon)^* \le (u^{\varepsilon,\delta})^* = \bar{u}^\delta.$$

Here, we take $\bar{u}^\delta$ to be the solution of

$$\begin{cases} u_t(x,t) + \bar{H}^\delta(Du) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{7.4}$$

where $\bar{H}^\delta$ is the effective Hamiltonian given by Theorem 1.1. Finally, letting $\delta \to 0$, we conclude $(u^\varepsilon)^* \le \bar{u}$ by Lemma 7.3. It is worth noting that Lemma 3.8 takes care of the assertion that

$$(u^\varepsilon)^*(x,0) = (u^\varepsilon)_*(x,0).$$

We have proved the statement of convergence, and it only remains to remark on the bounds on $\bar{H}$ inherits from H. This is in fact immediate from the definition of $\bar{H}$. We show the upper bound:

$$\begin{aligned} \bar{H}(p) &= \inf_\Phi \sup_{x,t} \{\Phi_t(x,t) + H(x,t,p+D\Phi(x,t))\} \\ &\le \inf_\Phi \sup_{x,t} \{\Phi_t(x,t) + C_2(|p+D\Phi(x,t)|^{\alpha_2} - 1)\} \le C_2(|p|^{\alpha_2} - 1). \end{aligned}$$

The lower bound is similar. We note that since Φ_t has mean zero, there must be some x_0, t_0, ω_0 for which $\Phi_t(x_0,t_0,\omega_0) \ge 0$. Since this is true for all choices of $\Phi \in \mathcal{S}$, it follows that

$$\bar{H}(p) \ge \inf_\Phi \{C_1(|p + D\Phi(x_0,t_0)|^{\alpha_1} - 1)\} = C_1(|p|^{\alpha_1} - 1).$$

This completes the proof. $\square$

Remark 7.4. Since the general version Theorem 1.1 is proved without a direct use of the Subadditive Ergodic Theorem, it is reasonable to ask if $(\bar{H})^*(p)$ coincides with the limit, $\bar{L}(-p)$, which is provided by the Subadditive Theorem applied to the fundamental solutions, $L^\varepsilon(p,1;0,0,\omega)$. Indeed, this will be the case, which we show here.

The function $L^\varepsilon(p,1;y,t,\omega)$ can be approximated by the functions

$$L^\varepsilon_M(y,t) := \inf_{\xi(t)=y} \big\{ M|\xi(t)-p| + \int_t^1 G^*(\frac{\xi(r)}{\varepsilon}, \frac{r}{\varepsilon}, \dot{\xi}(r))dr \big\},$$

where $L^\varepsilon_M \to L^\varepsilon$ as $M \to \infty$. Then L^ε_M are solutions of (3.1) on $\mathbb{R}^n \times [0,1]$ such that the terminal condition is $L^\varepsilon_M(y,1) = M|y-p|$. (Actually, $|y|$ should be truncated to be constant outside a large ball, dictated by the bounds L^ε imposes on the points achieving the infimum. But we will not include these technicalities.)

Theorem 1.1 implies that $L^\varepsilon_M \to \bar{L}_M$, which is the solution of

$$\begin{cases} (\bar{L}_M)_t = \bar{H}(D\bar{L}_M) & \text{in } \mathbb{R}^n \times [0,1] \\ \bar{L}_M(y,1) = M|y-p| & \text{on } \mathbb{R}^n \times \{1\}. \end{cases}$$

After a time change, $\bar{L}_M(y,1-t)$ is a solution of (1.2). We now use existing results about lower semicontinuous solutions of (1.2) with possibly infinite data, which can be found in [10]. There it is proved that $\bar{L}_M$ increase to the "fundamental solution", which in this case is given by $t\bar{H}^*(\frac{y-p}{t})$ (see [20]). Hence, on one hand, the Subadditive Ergodic Theorem gives a limit for $L^\varepsilon(p,1;0,0,\omega)$ and on the other hand Theorem 1.1 also gives a limit, via $L^\varepsilon_M(p,1;0,0,\omega)$. Thus by uniqueness of limits, they are the same.

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Department of Mathematics, University of Texas at Austin, 1 University Station C1200, Austin, TX 78712-0257

E-mail address: rschwab@math.utexas.edu