Integral Closures of Ideals in Completions of Regular Local Domains

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1 Abstract

In this paper we exhibit an example of a three-dimensional regular local domain \((A, \mathfrak{n})\) having a height-two prime ideal \(P\) with the property that the extension \(P\hat{A}\) of \(P\) to the \(\mathfrak{n}\)-adic completion \(\hat{A}\) of \(A\) is not integrally closed. We use a construction we have studied in earlier papers: For \(R = k[x, y, z]\), where \(k\) is a field of characteristic zero and \(x, y, z\) are indeterminates over \(k\), the example \(A\) is an intersection of the localization of the power series ring \(k[y, z][[x]]\) at the maximal ideal \((x, y, z)\) with the field \(k(x, y, z, f, g)\), where \(f, g\) are elements of \((x, y, z)k[y, z][[x]]\) that are algebraically independent over \(k(x, y, z)\). The elements \(f, g\) are chosen in such a way that using results from our earlier papers \(A\) is Noetherian and it is possible to describe \(A\) as a nested union of rings associated to \(A\) that are localized polynomial rings over \(k\) in five variables.
2 Introduction and Background

We are interested in the general question: What can happen in the completion of a ‘nice’ Noetherian ring? We are examining this question as part of a project of constructing Noetherian and non-Noetherian integral domains using power series rings. In this paper as a continuation of that project we display an example of a three-dimensional regular local domain \((A, \mathfrak{n})\) having a height-two prime ideal \(P\) with the property that the extension \(P\hat{A}\) of \(P\) to the \(\mathfrak{n}\)-adic completion \(\hat{A}\) of \(A\) is not integrally closed. The ring \(A\) in the example is a nested union of regular local domains of dimension five.

Let \(I\) be an ideal of a commutative ring \(R\) with identity. We recall that an element \(r \in R\) is integral over \(I\) if there exists a monic polynomial \(f(x) \in R[x], f(x) = x^n + \sum_{i=1}^{n} a_i x^{n-i}\), where \(a_i \in I^i\) for each \(i, 1 \leq i \leq n\) and \(f(r) = 0\). Thus \(r \in R\) is integral over \(I\) if and only if \(IJ^{n-1} = J^n\), where \(J = (I, r)R\) and \(n\) is some positive integer. (Notice that \(f(r) = 0\) implies \(r^n = -\sum_{i=1}^{n} a_i r^{n-i} \in IJ^{n-1}\) and this implies \(J^n \subseteq IJ^{n-1}\).) If \(I \subseteq J\) are ideals and \(IJ^{n-1} = J^n\), then \(I\) is said to be a reduction of \(J\). The integral closure \(\overline{I}\) of an ideal \(I\) is the set of elements of \(R\) integral over \(I\). If \(I = \overline{I}\), then \(I\) is said to be integrally closed. It is well known that \(\overline{I}\) is an integrally closed ideal. An ideal is integrally closed if and only if it is not a reduction of a properly bigger ideal. A prime ideal is always integrally closed. An ideal is said to be normal if all the powers of the ideal are integrally closed.

We were motivated to construct the example given in this paper by a question asked by Craig Huneke as to whether there exists an analytically unramified Noetherian local ring \((A, \mathfrak{n})\) having an integrally closed ideal \(I\) for which \(I\hat{A}\) is not integrally closed, where \(\hat{A}\) is the \(\mathfrak{n}\)-adic completion of \(A\). In Example 2.2, the ring \(A\) is a 3-dimensional regular local domain and \(I = P = (f, g)A\) is a prime ideal of height two. Sam Huckaba asked if the ideal of our example is a normal ideal. The answer is ‘yes’. Since \(f, g\) form a regular sequence and \(A\) is Cohen-Macaulay, the powers \(P^n\) of \(P\) have no embedded associated primes and therefore are \(P\)-primary \([?, (16.F), p. 112], [?, Ex. 17.4, p. 139]\). Since the powers of the maximal ideal of a regular local domain are integrally closed, the powers of \(P\) are integrally closed. Thus the Rees algebra \(A[Pt] = A[ft, gt]\) is a normal domain while the Rees algebra \(\hat{A}[ft, gt]\) is not integrally closed.

A problem analogous to that considered here in the sense that it also deals with the behavior of ideals under extension to completion is addressed by Loepp and Rotthaus in [?]. They construct nonexcellent local Noetherian domains to demonstrate that tight closure need not commute with completion.

Remark 2.1 Without the assumption that \(A\) is analytically unramified, there exist examples even in dimension one where an integrally closed ideal
of \( A \) fails to extend to an integrally closed ideal in \( \hat{A} \). If \( A \) is reduced but analytically ramified, then the zero ideal of \( A \) is integrally closed, but its extension to \( \hat{A} \) is not integrally closed. An example in characteristic zero of a one-dimensional Noetherian local domain that is analytically ramified is given by Akizuki in his 1935 paper [?]. An example in positive characteristic is given by F.K. Schmidt [?, pp. 445-447]. Another example due to Nagata is given in [?, Example 3, pp. 205-207]. (See also [?, (32.2), p. 114].)

**Remark 2.2** Let \( R \) be a commutative ring and let \( R' \) be an \( R \)-algebra. We list cases where extensions to \( R' \) of integrally closed ideals of \( R \) are again integrally closed. The \( R \)-algebra \( R' \) is said to be quasi-normal if \( R' \) is flat over \( R \) and the following condition \((N_{R,R'})\) holds: If \( C \) is any \( R \)-algebra and \( D \) is a \( C \)-algebra in which \( C \) is integrally closed, then also \( C \otimes_R R' \) is integrally closed in \( D \otimes_R R' \).

1. By [?, Lemma 2.4], if \( R' \) is an \( R \)-algebra satisfying \((N_{R,R'})\) and \( I \) is an integrally closed ideal of \( R \), then \( IR' \) is integrally closed in \( R' \).

2. Let \((A, n)\) be a Noetherian local ring and let \( \hat{A} \) be the \( n \)-adic completion of \( A \). Since \( A/\mathfrak{q} \cong \hat{A}/\mathfrak{q} \hat{A} \) for every \( n \)-primary ideal \( \mathfrak{q} \) of \( A \), the \( n \)-primary ideals of \( A \) are in one-to-one inclusion preserving correspondence with the \( \hat{n} \)-primary ideals of \( \hat{A} \). It follows that an \( n \)-primary ideal \( I \) of \( A \) is a reduction of a properly larger ideal of \( A \) if and only if \( I \hat{A} \) is a reduction of a properly larger ideal of \( \hat{A} \). Therefore an \( n \)-primary ideal \( I \) of \( A \) is integrally closed if and only if \( I \hat{A} \) is integrally closed.

3. If \( A \) is excellent, then the map \( A \to \hat{A} \) is quasi-normal by [?, (7.4.6) and (6.14.5)], and in this case every integrally closed ideal of \( A \) extends to an integrally closed ideal of \( \hat{A} \).

4. If \((A, n)\) is a local domain and \( A^h \) is the Henselization of \( A \), then every integrally closed ideal of \( A \) extends to an integrally closed ideal of \( A^h \). This follows because \( A^h \) is a filtered direct limit of étale \( A \)-algebras [?, (iii), (i), (vii) and (ix), pp. 800- 801].

5. In general, integral closedness of ideals is a local condition. Suppose \( R' \) is an \( R \)-algebra that is locally normal in the sense that for every prime ideal \( P' \) of \( R' \), the local ring \( R'_{P'} \) is an integrally closed domain. Since principal ideals of an integrally closed domain are integrally closed, the extension to \( R' \) of every principal ideal of \( R \) is integrally closed. In particular, if \((A, n)\) is an analytically normal Noetherian local domain, then every principal ideal of \( A \) extends to an integrally closed ideal of \( \hat{A} \).

6. If \( R \) is an integrally closed domain, then for every ideal \( I \) and element \( x \) of \( R \) we have \( x \overline{I} = \overline{xI} \). If \((A, n)\) is analytically normal and also a
UFD, then every height-one prime ideal of \(A\) extends to an integrally closed ideal of \(\hat{A}\). In particular if \(A\) is a regular local domain, then \(P\hat{A}\) is integrally closed for every height-one prime \(P\) of \(A\). If \((A, \mathfrak{n})\) is a 2-dimensional regular local domain, then every nonprincipal integrally closed ideal of \(A\) has the form \(x\mathfrak{I}\), where \(I\) is an \(\mathfrak{n}\)-primary integrally closed ideal and \(x \in A\). In view of item 2, every integrally closed ideal of \(A\) extends to an integrally closed ideal of \(\hat{A}\) in the case where \(A\) is a 2-dimensional regular local domain.

7. Suppose \(R\) and \(R'\) are Noetherian rings and assume that \(R'\) is a flat \(R\)-algebra. Let \(I\) be an integrally closed ideal of \(R\). The flatness of \(R'\) over \(R\) implies every \(P' \in \text{Ass } (R'/IR')\) contracts in \(R\) to some \(P \in \text{Ass } (R/I)\) [?, Theorem 23.2]. Since a regular map is quasi-normal, if the map \(R \to R'_{P'}\) is regular for each \(P' \in \text{Ass } (R'/IR')\), then \(IR'\) is integrally closed.

3 A non-integrally closed extension

In the construction of the following example we make use of results from [?]–[?).

CONSTRUCTION OF EXAMPLE 3.1 Let \(k\) be a field of characteristic zero and let \(x, y\) and \(z\) be indeterminates over \(k\). Let \(R := k[x, y, z]_{(x, y, z)}\) and let \(R^*\) be the \((xR)\)-adic completion of \(R\). Thus \(R^* = k[y, z]_{(y, z)}[[x]]\), the formal power series ring in \(x\) over \(k[y, z]_{(y, z)}\).

Let \(\alpha\) and \(\beta\) be elements of \(xk[[x]]\) which are algebraically independent over \(k(x)\). Set

\[f = (y - \alpha)^2, \quad g = (z - \beta)^2, \quad \text{and } A = k(x, y, z, f, g) \cap R^*.\]

Then the \((xA)\)-adic completion \(A^*\) of \(A\) is equal to \(R^*\) [?, Lemma 2.3.2, Prop. 2.4.4].

In order to better understand the structure of \(A\), we recall some of the details of the construction of a nested union \(B\) of localized polynomial rings over \(k\) in 5 variables associated to \(A\). (More details may be found in [?).)

APPROXIMATION TECHNIQUE 3.2 With \(k, x, y, z, f, g, R\) and \(R^*\) as in (3.1), Write

\[f = y^2 + \sum_{j=1}^{\infty} b_j x^j, \quad g = z^2 + \sum_{j=1}^{\infty} c_j x^j,\]
for some \( b_j, c_j \in k[y] \) and \( c_j \in k[z] \). There are natural sequences \( \{ f_r \}_{r=1}^\infty \), \( \{ g_r \}_{r=1}^\infty \) of elements in \( R^* \), called the \( r \)-th endpieces for \( f \) and \( g \) respectively which “approximate” \( f \) and \( g \). These are defined for each \( r \geq 1 \) by:

\[
f_r := \sum_{j=r}^{\infty} (b_j x^j) / x^r, \quad g_r := \sum_{j=r}^{\infty} (c_j x^j) / x^r.
\]

For each \( r \geq 1 \), define \( B_r \) to be \( k[x, y, z, f_r, g_r] \) localized at the maximal ideal generated by \( (x, y, z, f_r - b_r, g_r - c_r) \). Then define \( B = \bigcup_{r=1}^{\infty} B_r \). The endpieces defined here are slightly different from the notation used in [?]. Also we are using here a localized polynomial ring for the base ring \( R \). With minor adjustments, however, [?, Theorem 2.2] applies to our setup.

**Theorem 3.3** Let \( A \) be the ring constructed in (3.1) and let \( P = (f, g) A \), where \( f \) and \( g \) are as in (3.1) and (3.2). Then

1. \( A = B \) is a three-dimensional regular local domain that is a nested union of five-dimensional regular local domains.
2. \( P \) is a height-two prime ideal of \( A \).
3. If \( A^* \) denotes the \((x A)\)-adic completion of \( A \), then \( A^* = k[y, z]_{(y, z)}[[x]] \) and \( PA^* \) is not integrally closed.
4. If \( \hat{A} \) denotes the completion of \( A \) with respect to the powers of the maximal ideal of \( A \), then \( \hat{A} = k[[x, y, z]] \) and \( PA \) is not integrally closed.

**Proof:** Notice that the polynomial ring \( k[x, y, z, \alpha, \beta] = k[x, y, z, y - \alpha, z - \beta] \) is a free module of rank 4 over the polynomial subring \( k[x, y, z, f, g] \) since \( f = (y - \alpha)^2 \) and \( g = (z - \beta)^2 \). Hence the extension

\[
k[x, y, z, f, g] \to k[x, y, z, \alpha, \beta][1/x]
\]

is flat. Thus item (1) follows from [?, Theorem 2.2].

For item (2), it suffices to observe that \( P \) has height two and that, for each positive integer \( r \), \( P_r := (f, g) B_r \) is a prime ideal of \( B_r \). We have \( f = x f_1 + y^2 \) and \( g = x g_1 + z^2 \). It is clear that \( (f, g) k[x, y, z, f, g] \) is a height-two prime ideal. Since \( B_1 \) is a localized polynomial ring over \( k \) in the variables \( x, y, z, f_1 - b_1, g_1 - c_1 \), we see that

\[
P_1 B_1[1/x] = (x f_1 + y^2, x g_1 + z^2) B_1[1/x]
\]

is a height-two prime ideal of \( B_1[1/x] \). Indeed, setting \( f = g = 0 \) is equivalent to setting \( f_1 = -y^2 / x \) and \( g_1 = -z^2 / x \). Therefore the residue class ring \((B_1 / P_1)[1/x]\) is isomorphic to a localization of the integral domain
$k[x, y, z][1/x]$. Since $B_1$ is Cohen-Macaulay and $f, g$ form a regular sequence, and since $(x, f, g)B_1 = (x, y^2, z^2)B_1$ is an ideal of height three, we see that $x$ is in no associated prime of $(f, g)B_1$ (see, for example [?, Theorem 17.6]). Therefore $P_1 = (f, g)B_1$ is a height-two prime ideal.

For $r > 1$, there exist elements $u_r \in k[x, y]$ and $v_r \in k[x, z]$ such that $f = x^r f_r + u_r x + y^2$ and $g = x^r g_r + v_r x + z^2$. An argument similar to that given above shows that $P_r = (f, g)B_r$ is a height-two prime of $B_r$. Therefore $(f, g)B$ is a height-two prime of $B = A$.

For items 3 and 4, $R^* = B^* = A^*$ by Construction ?? and it follows that $\widehat{A} = k[[x, y, z]]$. To see that $PA^* = (f, g)A^*$ and $\widehat{P}A$ are not integrally closed, observe that $\xi := (y - \alpha)(z - \beta)$ is integral over $PA^*$ and $\widehat{P}A$ since $\xi^2 = fg \in \widehat{P}$2. On the other hand, $y - \alpha$ and $z - \beta$ are nonassociate prime elements in the local unique factorization domains $A^*$ and $\widehat{A}$. An easy computation shows that $\xi \notin \widehat{P}A$. Since $PA^* \subseteq \widehat{P}A$, this completes the proof.

**Remark 3.4** In a similar manner it is possible to construct for each integer $d \geq 3$ an example of a $d$-dimensional regular local domain $(A, n)$ having a prime ideal $P$ of height $h := d - 1$ such that $\widehat{P}A$ is not integrally closed. Indeed, let $k$ be a field of characteristic zero and let $x, y_1, \ldots, y_h$ be indeterminates over $k$. Let $\alpha_1, \ldots, \alpha_h \in xk[[x]]$ be algebraically independent over $k(x)$. For each $i$ with $1 \leq i \leq h$, define $f_i = (y_i - \alpha_i)^h$. Proceeding in a manner similar to what is done in (3.1) we obtain a $d$-dimensional regular local domain $A$ and a prime ideal $P = (f_1, \ldots, f_h)A$ of height $h$ such that the $y_i - \alpha_i \in \widehat{A}$. Let $\xi = \prod_{i=1}^{h}(y_i - \alpha_i)$. Then $\xi^h = f_1 \cdots f_h \in P^h$ implies $\xi$ is integral over $\widehat{P}A$, but using that $y_1 - \alpha_1, \ldots, y_h - \alpha_h$ is a regular sequence in $\widehat{A}$, we see that $\xi \notin \widehat{P}A$.

### 4 Comments and Questions

In connection with Theorem ?? it is natural to ask the following question.

**Question 4.1** For $P$ and $A$ as in Theorem ??, is $P$ the only prime of $A$ that does not extend to an integrally closed ideal of $\widehat{A}$?

**Comments 4.2** In relation to the example given in Theorem ?? and to Question ??, we have the following commutative diagram, where all the maps shown are the natural inclusions:
\[ B = A \xrightarrow{\gamma_1} A' := k(x, y, z, \alpha, \beta) \cap R^* \xrightarrow{\gamma_2} R^* = A^* \]

\[ \delta_1 \uparrow \quad \quad \quad \delta_2 \uparrow \quad \quad \quad \psi \uparrow \]

\[ S := k[x, y, z, f, g] \xrightarrow{\varphi} T := k[x, y, z, \alpha, \beta] \]

\[ T \]

Let \( \gamma = \gamma_2 \cdot \gamma_1 \). Referring to the diagram above, we observe the following:

1. The discussion in [?, bottom p. 668 to top p. 669] implies that [?, Thm. 3.2] applies to the setting of Theorem 3.3. By [?, Prop. 4.1 and Thm. 3.2], \( A'[1/x] \) is a localization of \( T \). By Theorem 3.3 and [?, Thm 3.2], \( A[1/x] \) is a localization of \( S \). Furthermore, by [?, Prop. 4.1] \( A' \) is excellent. (Notice, however, that \( A \) is not excellent since there exists a prime ideal \( P \) of \( A \) such that \( \hat{\mathcal{A}} \) is not integrally closed.) The excellence of \( A' \) implies that if \( Q^* \in \text{Spec} \ A^* \) and \( x \notin Q^* \), then the map \( \psi_{Q^*} : T \to A_{Q^*}^* \) is regular [?, (7.8.3 v)].

2. Let \( Q^* \in \text{Spec} \ A^* \) be such that \( x \notin Q^* \) and let \( q' = Q^* \cap T \). By [?, Theorem 32.1] and Item 1 above, if \( \varphi_{q'} : S \to T_{q'} \) is regular, then \( \gamma_{Q^*} : A \to A_{Q^*}^* \) is regular.

3. Let \( I \) be an ideal of \( A \). Since \( A' \) and \( A^* \) are excellent and both have completion \( \hat{\mathcal{A}} \), Remark ???.3 shows that the ideals \( IA' \), \( IA^* \) and \( I\hat{\mathcal{A}} \) are either all integrally closed or all fail to be integrally closed.

4. The Jacobian ideal of the extension \( \varphi : S = k[x, y, z, f, g] \to T = k[x, y, z, \alpha, \beta] \) is the ideal of \( T \) generated by the determinant of the matrix

\[ J := \begin{vmatrix} \frac{\partial f}{\partial \alpha} & \frac{\partial g}{\partial \alpha} \\ \frac{\partial f}{\partial \beta} & \frac{\partial g}{\partial \beta} \end{vmatrix}. \]

Since the characteristic of the field \( k \) is zero, this ideal is \( (y-\alpha)(z-\beta)T \).

In Proposition ??, we relate the behavior of integrally closed ideals in the extension \( \varphi : S \to T \) to the behavior of integrally closed ideals in the extension \( \gamma : A \to A^* \).

**Proposition 4.3** With the setting of Theorem ?? and Comment ???.2, let \( I \) be an integrally closed ideal of \( A \) such that \( x \notin Q \) for each \( Q \in \text{Ass} (A/I) \). Let \( J = I \cap S \). If \( JT \) is integrally closed (resp. a radical ideal) then \( IA^* \) is integrally closed (resp. a radical ideal).

**Proof:** Since the map \( A \to A^* \) is flat, \( x \) is not in any associated prime of \( IA^* \). Therefore \( IA^* \) is contracted from \( A^*[1/x] \) and it suffices to show \( IA^*[1/x] \) is integrally closed (resp. a radical ideal). Our hypothesis implies
\[ I = IA[1/x] \cap A. \] By Comment \ref{1.1}, \( A[1/x] \) is a localization of \( S \). Thus every ideal of \( A[1/x] \) is the extension of its contraction to \( S \). It follows that \( IA[1/x] = JA[1/x] \). Thus \( IA^*[1/x] = JA^*[1/x] \).

Also by Comment \ref{1.1}, the map \( T \to A^*[1/x] \) is regular. If \( JT \) is integrally closed, then Remark \ref{7} implies that \( JA^*[1/x] \) is integrally closed. If \( JT \) is a radical ideal, then the regularity of the map \( T \to A^*[1/x] \) implies the \( JA^*[1/x] \) is a radical ideal.

**Proposition 4.4** With the setting of Theorem \ref{??} and Comment \ref{??}, let \( Q \in \text{Spec } A \) be such that \( QA \) (or equivalently \( QA^* \)) is not integrally closed. Then

1. \( Q \) has height two and \( x \notin Q \).

2. There exists a minimal prime \( Q^* \) of \( QA^* \) such that with \( q' = Q^* \cap T \), the map \( \varphi_q : S \to T_{q'} \) is not regular.

3. \( Q \) contains \( f = (y - \alpha)^2 \) or \( g = (z - \beta)^2 \).

4. \( Q \) contains no element that is a regular parameter of \( A \).

**Proof:** By Remark \ref{6}, the height of \( Q \) is two. Since \( A^*/xA^* = A/xA = R/xR \), we see that \( x \notin Q \). This proves item 1.

By Remark \ref{7}, there exists a minimal prime \( Q^* \) of \( QA^* \) such that \( \gamma_{Q^*} : A \to A_{Q^*} \) is not regular. Thus item 2 follows from Comment \ref{2}.

For item 3, let \( Q^* \) and \( q' \) be as in item 2. Since \( \gamma_{Q^*} \) is not regular it is not essentially smooth \([?, 6.8.1]\). By \([?, (2.7)]\), \( (y - \alpha)(z - \beta) \in q' \). Hence \( f = (y - \alpha)^2 \) or \( g = (z - \beta)^2 \) is in \( q' \) and thus in \( Q \). This proves item 3.

Suppose \( w \in Q \) is a regular parameter for \( A \). Then \( A/wA \) and \( A^*/wA^* \) are two-dimensional regular local domains. By Remark \ref{6}, \( QA^*/wA^* \) is integrally closed, but this implies that \( QA^* \) is integrally closed, which contradicts our hypothesis that \( QA^* \) is not integrally closed. This proves item 4.

**Question 4.5** In the setting of Theorem \ref{??} and Comment \ref{??}, let \( Q \in \text{Spec } A \) with \( x \notin Q \) and let \( q = Q \cap S \). If \( QA^* \) is integrally closed, does it follow that \( qT \) is integrally closed?

**Question 4.6** In the setting of Theorem \ref{??} and Comment \ref{??}, if a prime ideal \( Q \) of \( A \) contains \( f \) or \( g \), but not both, and does not contain a regular parameter of \( A \), does it follow that \( QA^* \) is integrally closed?

In Example \ref{??}, the three-dimensional regular local domain \( A \) contains height-one prime ideals \( P \) such that \( \hat{A}/P\hat{A} \) is not reduced. This motivates us to ask:
Question 4.7 Let \((A, \mathfrak{n})\) be a three-dimensional regular local domain and let \(\widehat{A}\) denote the \(n\)-adic completion of \(A\). If for each height-one prime \(P\) of \(A\), the extension \(PA\) is a radical ideal, i.e., the ring \(\widehat{A}/PA\) is reduced, does it follow that \(PA\) is integrally closed for each \(P \in \text{Spec } A\)?

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