Exotic Cyclic Group Actions on Smooth 4-Manifolds

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Joint work with Ron Stern and Nathan Sunukjian
How exotic is ‘exotic’?

**Exotic smooth structures**
Important consequences of Seiberg-Witten (and Donaldson) theory
- Existence of nondiffeomorphic but homeomorphic smooth 4-manifolds
- Existence of surfaces in a fixed smooth 4-manifold which are topologically but not smoothly equivalent

**Exotic smooth group actions**
- Existence of smooth actions of a group on a smooth 4-manifold which are equivariantly homeomorphic but not equivariantly diffeomorphic.

Example: Exotic involutions on $S^4$, Quotient $= \text{Fake } RP^4$ (F- Stern/ Cappell - Shaneson, Gompf)
- Want orientation-preserving examples
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Ue’s examples

Ue’s Theorem, 1998

For any nontrivial finite group \( G \) there exists a smooth 4-manifold that has infinitely many free \( G \)-actions so that their orbit spaces are homeomorphic but mutually nondiffeomorphic.

The examples

\( Y \): \( \mathbb{Q} \)-homology \( S^4 \) with \( \pi_1(Y) \to G \), onto, s. t. corr. cover is \( \tilde{Y} = S^2 \times S^2 \# \mathbb{Z} \), some \( \mathbb{Z} \). Get \( Y \) by spinning known 3D example.

\( X_0 = E(2)_p, \ X_1 = E(2)_q, \ p \neq q \) odd (log transformed K3’s)

\( X_0 \# Y, \ X_1 \# Y \) homeo not diffeo using Seiberg-Witten

The \( G \)-covers \( Q_i \) come from \( \pi_1(X_i \# Y) \to \pi_1(Y) \to G \)

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• The \( Q_i \) are reducible.
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Exotic cyclic group actions

Theorem (F., Stern, Sunukjian)

Let $Y$ be a simply connected 4-manifold with $b^+ \geq 1$ containing an embedded surface $\Sigma$ of genus $g \geq 1$ of nonnegative self-intersection. Suppose that $\pi_1(Y \setminus \Sigma) = \mathbb{Z}_d$ and that the pair $(Y, \Sigma)$ has a nontrivial relative Seiberg-Witten invariant. Suppose also that $\Sigma$ contains a nonseparating loop which bounds an embedded 2-disk in $Y \setminus \Sigma$. Let $d'$ divide $d$, and let $X$ be the (simply connected) $d'$-fold cover of $Y$ branched over $\Sigma$. Then $X$ admits an infinite family of smoothly distinct but topologically equivalent actions of $\mathbb{Z}_{d'}$. 
Some simple examples

Curves in $\mathbb{CP}^2$

$Y = \mathbb{CP}^2$, $\Sigma = \text{embedded degree } d \text{ curve}$.
$X = \text{degree } d \text{ hypersurface in } \mathbb{CP}^3$

If $d = 3$, $X = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$ $\Rightarrow$ we have infinitely many smoothly inequivalent topologically equivalent $\mathbb{Z}_3$-actions on $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$.

If $d = 4$, $X = K3$, $\Rightarrow$ smoothly inequivalent topologically equivalent $\mathbb{Z}_4$-actions on the K3-surface.
Also theorem $\Rightarrow$ families of $\mathbb{Z}_2$ and $\mathbb{Z}_3$-actions on K3.

$\mathbb{Z}_5$-actions on quintics, etc.
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Knot and rim surgery (F. - Stern)

Knot surgery

$K$: Knot in $S^3$, $T$: square 0 essential torus in $X$

$X_K = X \setminus N_T \cup S^1 \times (S^3 \setminus N_K)$

$S^1 \times (S^3 \setminus N_K)$ has the homology of $T^2 \times D^2$.

Facts

- If $X$ and $X \setminus T$ both simply connected, so is $X_K$. (So $X_K$ homeo to $X$)
- $SW_{X_K} = SW_X \cdot \Delta_K(t^2)$

Rim surgery

$\Sigma \subset X$: embedded orientable surface in simply connected 4-manifold.

$C$: homologically essential loop in $\Sigma$

Rim torus: preimage of $C$ in bdry of normal bundle of $\Sigma$.

Rim surgery = knot surgery on rim torus.

Can change embedding type of $\Sigma$. Get $\Sigma_K \subset X$. 
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Spinning a knot $K$ in $S^3$ gives 2-knot in $S^4$: 
$S^1$-action on $S^4$. Orbit space $B^3$. 
Spun knot = preimage of knotted arc. Preimage of $\partial B^3=\text{twin}$.
Knot surgery replaces $C \times S^1 \times D^2$ with $S^4 \setminus (\text{spun knot} \cup \text{twin})$.
$C \times B^3 = \text{complement of trivial twin in } S^4$. 

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Get smoothly inequivalent embeddings if original SW inv’t is \( \neq 0 \). (E.g. symplectic submanifold.) Relative SW-invariant lives in monopole Floer homology group.

Want to take cyclic branched covers — need \( \pi_1(X \setminus \Sigma) = \mathbb{Z}_d \). Problem: Rim surgery will not preserve this condition.

Solution (Kim - Ruberman) \( k \)-Twist-spun rim surgery does preserve \( \pi_1 = \mathbb{Z}_d \) as long as \( k \) is prime to \( d \).
In fact, they show that the new surface obtained is topologically equivalent to the old one in this case.
Relative SW-invariant is the same as for ordinary rim surgery.
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$K$: knot in $S^3$. Twist-spinning operation due to Zeeman. Get knotted $S^2$ in $S^4$ and circle action.

Twist-spun rim surgery, $\Sigma_{K,k}$

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Determined up to equivariant diffeomorphism by orbit data.

Orbit space: $B^3$ or $S^3$

Fixed point set = $S^0$ or $S^2$. Exceptional orbit image 0, 1 or 2 arcs.

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$k$-twist spin of $K = p^{-1}(\bar{A}) \subset S^4$.

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gets replaced with $S^4 \setminus \text{Nbd}(p^{-1}(E_k))$ where $E_k =$ closed arc labeled ‘$\mathbb{Z}_k$’.
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Twist-spun rim surgery, $\Sigma_{K,k}$
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[Diagram of twist-spinning a knot]
Circle actions on $S^4$ and Twist-spinning

Determined up to equivariant diffeomorphism by orbit data.
Orbit space: $B^3$ or $S^3$
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Relative Seiberg-Witten invariants

By blowing up, assume $\Sigma \cdot \Sigma = 0$.

Seiberg-Witten invariant of $Y \setminus N(\Sigma)$ obtained from spin$^c$-structures $s$ on $Y$ satisfying $\langle c_1(s), \Sigma \rangle = 2g - 2$.

$SW_{(Y|\Sigma)} : H_2(Y \setminus N(\Sigma), \Sigma \times S^1; \mathbb{R}) \to \mathbb{R}$ (Kronheimer/ Mrowka)

Role of basic classes played by $z \in \pi_0(B(Y \setminus N(\Sigma); [a_0]))$, principal homogeneous space for $H^2(Y \setminus N(\Sigma), \partial)$

$z = [(A, \Phi)]$ solving SW eq’ns.

$a_0$: unique spin$^c$-structure on $\Sigma \times S^1$ of degree $2g - 2$

Knot surgery theorem

Basic classes for $Y|\Sigma_K,k$: $z + j\rho$, $\rho = PD$(rim torus), $t^j$ has $\neq 0$ coeff in $\Delta_K(t)$. $\implies$

Given $(Y, \Sigma, C)$ there is an infinite family of knots $K$ and surfaces $\Sigma_K,k$ all topologically equivalent but smoothly inequivalent obtained by $(K, k)$-twist-rim surgery.
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Cyclic group actions

$Y$: simply connected smooth 4-manifold.

$\Sigma$ genus $\geq 1$ surface embedded in $Y$ such that $\pi_1(Y \setminus \Sigma) = \mathbb{Z}_d$.

$C$: nonseparating loop on $\Sigma$, bounds $D^2$ in complement.

$X = d$-fold branched cyclic cover.

Choose $k$ relatively prime to $d$  

$\exists$ family of knots $K_i$ so that $d$-fold branched covers $X_i$ of $(Y, \Sigma_{K_i, k})$ are all topologically equivalent but smoothly distinct covers.

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Topologically equivalent but smoothly distinct actions of $\mathbb{Z}_d$.

Need to see that $X_i$ are diffeomorphic to each other.
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Branched covers of twist-spun knots

\[ S^4(K; k, d) \xrightarrow{p'} \mathbb{Z}_k \]

\[ \mathbb{Z}_d \]

\[ p \]

\[ p^{-1}(A) \]

\[ \mathbb{Z}_k \]

\[ \mathbb{Z}_d \]

A = principal orbits
In cover, replacing $C \times I \times D^2$ with $S^4 \setminus E_k \neq S^1 \times B^3$

$C$ bounds disk, $C \times I \times D^2 \cup \text{Nbd}(\text{disk}) = B^4$ in $X$

After knot surgery in $Y$, $B^4$ in cover becomes $S^4(K; k, d) \setminus B^4$.

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