6.2 # 4, 5, 7, 9, 10, 11, 12, 13, 23

#4 Done in class on Friday 04/29/11

#5 If \( I \triangleleft R \) and \( R \) is an integral domain, is \( R/I \) an integral domain?

No: \( R = \mathbb{Z}, \; I = 4\mathbb{Z} \)

#7 Let \( T = 3\mathbb{Z}, \; I = \langle 6 \rangle \). Show \( I \triangleleft T \).

If differences of multiples of 6 are multiples of 6, a multiple of 6 multiplied by a multiple of 3 is a multiple of 6.

More generally, if \( I \triangleleft T \subseteq R \) and \( I, T \triangleleft R \), then \( I \triangleleft T \).

Proof: We are given that \( I \) is a subring, so we only need check that if \( t \in T \), then for each \( a \in I \), \( t + a \in I \) and \( a + t \in I \). Thus \( t + I \) and so \( \forall t \in I \), \( a \in I \).

#7(b) Write out table for \( T/I \). To see it is a field:

\[
\begin{array}{c|cc}
    & I & 3+I \\
----&----&----
I & I & 3+I \\
3+I & 3+I & 9+I = 3+I \\
    & 6+I & 6+I \\
\end{array}
\]

\( T/I = \{ I, 3+I \} \)

#9 If \( R \) is a ring, then \( R/(0) \cong R \).

Proof: Let \( f: R \to R \) be the identity function, i.e. \( f(a) = a \). Then \( f \) is clearly an isomorphism.

Ker\( f = (0) \). By the first isomorphism theorem \( \cong R/(0) \).
10. Let $R, S$ rings. Show $\pi : R \times S \to R$ defined by $\pi (r, s) = r$ is a surjective homomorphism of $\ker \pi \subseteq S$.

**Proof:** Surjective: Given $r \in R$, then $(r, 0) \in R \times S$. Hence $\pi (r, 0) = r$.

Homomorphism: Suppose $(r_1, s_1), (r_2, s_2) \in R \times S$

\[
\pi (r_1, s_1) + (r_2, s_2) = \pi (r_1 + r_2, s_1 + s_2) = r_1 + r_2 = \pi (r_1, s_1) + \pi (r_2, s_2)
\]

\[
\pi (r_1, s_1)(r_2, s_2) = \pi (r_1 r_2, s_1 s_2) = r_1 r_2 = \pi (r_1, s_1) \circ \pi (r_2, s_2)
\]

$\ker \pi = \{ (0, s) : s \in S \}$.

Define $\psi : \ker \pi \to S$ by $\psi (0, s) = s$.

This is clearly an isomorphism.

11. Let $K \subseteq R$. Show $(I / K) \otimes (R / K)$.

**Proof:** Given $a + K, b + K \in I / K$.

This means $a, b \in I$.

$(a + K) - (b + K) = (a - b) + K \in I / K$ since $a - b \in I$.

If $(a + K) \in R / K$, then $(a + K)(a + K) = a^2 + K$, $a^2 \in I$ since $I \subseteq R$. Similarly $(a + K)(a + K) \in I / K$.

Thus $I / K \otimes R / K$. 
#12 (a) \( f : R \to S \) surjective hom of rings, \( I \triangleleft R \). Show \( f(I) \triangleleft S \).

**Proof.** Given \( a, b \in f(I) \), this means \( \exists c, d \in R \) s.t. \( a = f(c) \), \( b = f(d) \).
\[ a - b = f(c) - f(d) = f(c - d) \quad \text{(since } f \text{ is a hom) \}
\]
\[ = f(\text{an element of } I) \quad \text{since } c - d \in I.
\]
\[ \therefore a - b \in f(I) \]

If \( a \in S \), then \( a = f(x) \) for some \( x \in R \).
\[ \Rightarrow a \cdot a = f(x) \cdot f(c) = f(xc) \]
\( \forall c \in I \) since \( x \in R, xc \in I \triangleleft R \).
\[ \therefore a \cdot a \in f(I) \]

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#12 (b) Give an example to show \( @ \) may be false if \( f \) is not surjective.

Let \( R = \mathbb{Z} \) and \( S = \mathbb{Q} \). Let \( f : \mathbb{Z} \to \mathbb{Q} \)
be the inclusion map: \( f(n) = n \).
Then \( f \) is a hom: \( f(n + n_2) = n + n_2 = f(n) + f(n_2) \)
\[ f(n_1 \cdot n_2) = n_1 \cdot n_2 = f(n_1) \cdot f(n_2) \]

But \( f(2) \) is not an ideal since \( \frac{1}{2} \in \mathbb{Q} \) but \( \frac{1}{2} \cdot f(1) = \frac{1}{2} \cdot \frac{1}{2} \notin f(2) \).
#13 If $R$ is a commutative ring of 1 and $(x)$ is the principal ideal generated by $x \in \mathbb{R}[x]$, prove that $\mathbb{R}[x]/(x) \cong R$.

**Proof:** Let $f : \mathbb{R}[x] \to R$ be

$$f(p(x)) = p(0).$$

Then $f$ is a surjective homomorphism. (Compare with what was done in lecture on 04/29/11)

$\ker f = (x) \implies \mathbb{R}[x]/(x) \cong R$

by the 1st isomorphism theorem.

"We proved this assuming $R$ was a field" (in class)

#23 Use the 1st isomorphism theorem to prove that $\mathbb{Z}_{20}/(5) \cong \mathbb{Z}_5$.

We proved, more generally, in class on 04/29/11, that if $m/n$, then

$$\mathbb{Z}_n/(m) \cong \mathbb{Z}_m.$$ 

We also proved that $(m) \cong \mathbb{Z}_n$, where $n = mk$.