“If ..., then ...” Statements

Definition: Statements of the form “If statement A is true, then statement B is true.” are called implications. Mathematically this is denoted by \( A \Rightarrow B \).

\[ A \Rightarrow B \]

- “If A then B”
- “A implies B”
- “A only if B”
- “B if A”
- “B whenever A”
- “A is sufficient for B”
- “B is necessary for A”

Examples: Determine which statement constitutes the hypothesis (assumption) and which statement is the conclusion.

1. If \( x \in \mathbb{N} \), then \( 2x \) is even.

2. If pigs could fly, then I am on Mars.

3. The value of \( x + y \) is even whenever \( x \) and \( y \) are odd.

4. I am going to carry an umbrella, only if it rains.
   - If I am going to carry an umbrella, then it means it is going to rain.
5. \( x^2 < 1 \) whenever \( x < 1 \). Note that this is a false statement!

When is the statement \( A \Rightarrow B \) true?

Is the following statement true?
*If pigs could fly, then I am on Mars.*

“\( A \Rightarrow B \)” says nothing about whether \( A \) or \( B \) are true or false.

The following cases are possible implications to be true.

- \( A \) - true and \( B \) - true
- \( A \) - false and \( B \) - false
- \( A \) - false and \( B \) - true

If the assumption is false, the conclusion could be anything!

Give an example illustrating each of the above cases.

**Truth table for \( A \Rightarrow B \).**

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<tr>
<th>A</th>
<th>B</th>
<th>( A \Rightarrow B )</th>
<th>( \sim (A \Rightarrow B) )</th>
<th>( \sim B )</th>
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What is the negation of \( A \Rightarrow B \)?

**Negation of if-then statement**

\( A \) : “I do well in college.” \( B \) : “I will get a good job.”

- If \( A \) then \( B \):

- not(If \( A \) then \( B \)):


**Theorem:** The negation of $A \Rightarrow B$ is equivalent to $A$ and (not $B$).

$$(A \Rightarrow B) \equiv (A \land \neg B)$$

Restate in the form of an "if-then" statement and negate the following statements.

1. “The room is quiet, if the door is closed.”

2. “I am productive in the morning, only if I have slept well.”

3. “I am an adult, if I am 30 years old.”

4. “In order to have a driver’s license, it is necessary to be at least 16 years old.”

5. “To pass MTH299, it is sufficient to have 90% on all tests and assignments.”

**Open Sentences**

Let

$$P(x, y) : x^2 + y^2 = 4 \quad \text{and} \quad Q(x, y) : \frac{y}{x} \in \mathbb{Z}$$

be open sentences with domain $A \times B$, where $A = \{1, 2\}$ and $B = \{0, \sqrt{3}\}$. Determine for what elements in the domain the statement $P(x, y) \Rightarrow Q(x, y)$ is true.
**Inverse** of an if-then statement

“If I am 30 years old, then I am an adult.”

The **inverse** of the above statement is:

“If I am not 30 years old, then I am not an adult.”

**Theorem:** The **inverse** of the implication “If A, then B.” is the implication “If not(A), then not(B.).”

Is the inverse, in general, equivalent to the original statement?

Think of an example when a statement and its inverse are equivalent and when they are not.

**Necessary Conditions**

“In order to pass MTH299, it is necessary that a student completes most daily homework assignments.”

What is the assumption and what is the conclusion?

**Definition:** A necessary condition is one that must hold in order for the result to be true. It does not guarantee that the result is true.

\[ A \text{ is necessary for } B \text{ is equivalent to } B \text{ is true only if } A \text{ is true}. \]

which is equivalent to \( B \Rightarrow A \).

\[ x \in (-1, 1) \text{ is necessary for } x^2 - 1 < 0. \]
**Sufficient Conditions**

“To pass MTH299, it is sufficient to have 90% on all tests and assignments.”

What is the assumption and what is the conclusion?

**Definition:** A **sufficient** condition is one such that if it holds, the result is guaranteed to be true. The conclusion may be true even if the condition is not satisfied.

\[ A \implies B \]

\[ x \in (0, 1) \] is sufficient for \( x^2 - 1 < 0. \)

\[ x \in (-1, 1) \] is sufficient for \( x^2 - 1 < 0. \)

**Necessary and Sufficient Conditions**

Fill in the blank with **necessary**, **sufficient** or **necessary and sufficient**.

1. \( x > 1 \) is ______________________ for \( x^2 > 1 \)

2. \( x \in \mathbb{N} \) is ______________________ for \( x \geq 0 \)

3. \( |x| > 1 \) is ______________________ for \( x^2 > 1 \)

4. “Mary earned an A in MTH299.” is ______________________ for “Mary passed MTH299.”

5. “The function \( f \) is continuous at \( x = c \).” is ______________________ for “The function \( f \) has a derivative at \( x = c \).”
Contrapositive

\( A \Rightarrow B \)

“If I am 30 years old, then I am an adult.”

We saw that the inverse of the above statement is:

\( \text{not}(A) \Rightarrow \text{not}(B) \)

“If I am not 30 years old, then I am not an adult.”

and it is not equivalent to the original one.

Can you construct an implication using \( \text{not}(A) \) and \( \text{not}(B) \), which is equivalent to the original one?

The contrapositive of the statement \( A \Rightarrow B \) is \( \text{not}(B) \Rightarrow \text{not}(A) \).

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Theorem: A statement and its contrapositive are equivalent.

Find the inverse and the contrapositive of the following statements.
1. If Jane has grandchildren, then she has children.
2. If \( x = 1 \), then \( x \) is a solution to \( x^2 - 3x + 2 = 0 \).

\( A \Rightarrow B \) is equivalent to \( (\text{not } B) \Rightarrow (\text{not } A) \)

Sometimes it is easier to prove the contrapositive than it is to prove the forward statement.

Example: Prove that \( \emptyset \subseteq A \), for any set \( A \).

• If \( x \in \emptyset \Rightarrow x \in A \)

• Contrapositive: