Sec 2.1: Statements

Mathematics is the business of proving mathematical statements to be true or false. Logic lays the foundation for rigorous mathematical proofs.

**Definition:** A statement is a sentence that is either true or not.

Give examples of some statements.

Give an example of a sentence which is not a statement.

“This statement is false.”

The above is an example of a self-referential sentence.

Determine if the following are statements. Explain.

1. “Assume that the set $A$ is nonempty.”

2. “The set $A$ is nonempty.”
3. “The set $A$, defined by $A = \{x \in \mathbb{R} \mid x^2 + 5 = 0\}$ is nonempty.”

4. $P(x) : x^2 - 8 \geq 0$.

Be pedantic!
Which of these statements are true?

(1) There are 18 students registered for this class.

(2) There are 5 students registered for this class.

(3) There are 50 students registered for this class.

(4) There is a student registered for this class.

(5) There are no students registered for this class.
Sec 2.2 : The Negation of a Statement

**Definition:** The **negation** of statement A is another statement that is interpreted as being false when A is true and true when A is false.

The negation of the statement A is written as ∼A or not (A).

- A: “I like ice cream.”
  ∼A:

  ∼ (∼A):

- B: “All sheep are black.”
  ∼B:

  ∼ (∼B):
Theorem: $\sim (\sim A)$ is equivalent to $A$.

Truth Tables (Sec 2.8 : Logical Equivalence)

<table>
<thead>
<tr>
<th>A</th>
<th>$\sim A$</th>
<th>$\sim (\sim A)$</th>
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**Definition:** Two statements $P$ and $Q$ are called logically equivalent if the two statements have the same truth values for all combinations of truth values of their component statements.

**Notation:** If two statement $P$ and $Q$ are logically equivalent, then this is denoted by $P \equiv Q$

**Remark:** If we can show that $P$ is true, then $Q$ is true as well.

1. **Statements with AND ($A \land B$)**

   “Yesterday I went biking and I saw a fox.”

   If this statement is not true, what must be true?

   (What is the negation of the above?)

**Definition:** $A \land B$ is true only if both $A$ and $B$ are true.
2. **Statements with \textbf{OR}** \((A \lor B)\)

“I have a candy in my left pocket or in my right pocket.”

\(x \in A \cup B\) is equivalent to \(x \in A\) or \(x \in B\).

**Definition:** \(A \lor B\) is true when at least one of \(A\) or \(B\) is true.

The mathematical OR is not exclusive. Unlike the conversational OR, it is not “either - or”!

List some statements with \textbf{OR}.

\[
\begin{array}{cccc}
A & B & A \lor B & \sim (A \lor B) \\
T & T & T & \ \\
T & F & T & \ \\
F & T & T & \ \\
F & F & F & \ \\
\end{array}
\]

3. **Negation of \textbf{and} statement**

- A: “Jesse is tall” B: “Daniel is tall”
- \(A \land B\) :
- \(\sim (A \land B)\) :

**Theorem:** The \textbf{negation} of \(A \land B\) is equivalent to \((\text{not } A) \text{ or } (\text{not } B)\).
4. Negations of or statements

- A: “Rachel’s major is mathematics”  B: “Asia’s major is mathematics”
- \( \sim (A \lor B) : \)

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Theorem: The **negation** of A or B is equivalent to (not A) and (not B).

5. Negation of AND and OR

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<th>\sim (A \land B)</th>
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**Theorem:** not(A \lor B) is equivalent to not(A) and not(B).

\[ \sim (A \lor B) \equiv (\sim A) \land (\sim B) \]

Prove on your own:

**Theorem:** not(A \land B) is equivalent to not(A) or not(B).

\[ \sim (A \land B) \equiv (\sim A) \lor (\sim B) \]