Existence Proofs

Our goal in this section is to prove a statement of the form

There exists $x$ for which $P(x)$. (That is, $\exists x, P(x)$).

I. A constructive proof of existence: The proof is to display a specific value $x = a$ in a given set and verify that $P(a)$ is true.

**EX:** Prove that, for every natural number $x$, there exists a natural number $y$ such that $2x - y = -1$.

II. A nonconstructive proof of existence: Use theorems which imply the existence of an $x$ such that $P(x)$ is true without indicating how to explicitly produce such $x$.

The **Intermediate Value Theorem** and the **Mean Value Theorem** are examples of existence theorems that can be used in this manner.

**EX:** Prove that there exists a real number $x$ in $[-1, 1]$ such that $2x^3 + 1 = 0$.

**Examples: A constructive proof of existence**

1. Prove that there are pairs of irrational numbers $x$ and $y$ such that $x^y$ is rational.

   (This week’s recitation: Prove that there are two distinct irrational numbers $x$ and $y$ such that $x^y$ is rational.)
2. Prove that there exists an integer \( x \) such that \( \frac{8x+2}{3x-1} = 2 \).

3. There exist distinct perfect squares \( x, y, \) and \( z \) such that \( x + y = z \).

4. Prove that for \( \varepsilon = 1 \), there exists a positive real number \( \delta \) such that
\[
|x - 2| < \delta \implies |(2x + 3) - 7| < \varepsilon.
\]
(We will revisit the formal definition of a limit of a function in Chapter 12.)
5. There is a prime number \( p \) such that \( p + 2 \) and \( p + 6 \) are also prime numbers.

6. There exists an even integer \( n \) that can be written in \textbf{two different} ways as a sum of two distinct primes.

\textbf{Examples: A nonconstructive proof of existence}

- **The Intermediate Value Theorem:** If \( f \) is a function that is continuous on the closed interval \([a, b]\) and \( k \) is a number between \( f(a) \) and \( f(b) \), then there \textbf{exists} a number \( c \in (a, b) \) such that \( f(c) = k \).

- **The Mean Value Theorem:** If a function \( f \) is continuous on the closed interval \([a, b]\), and differentiable on the open interval \((a, b)\), then there \textbf{exists} a point \( c \in (a, b) \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

\textbf{Examples}

1. There exists a solution for the equation \( x^3 + 3x - 2 = 0 \) in the interval \((0, 1)\).
2. Let $f(x) = 4x^5 - x + 2$. Prove that there exists a $c \in (0, 1)$ such that $f'(c) = 3$. Note that $f(0) = 2$ and $f(1) = 5$.

Unique Existence. Examples.

1. An equation $x^5 + 4x - 1 = 0$ has exactly one solution.

2. Prove that, for every $x$, there exists a unique $y \in \mathbb{R}$ such that $2x + 1 = 2y - 1$.

   (1) Prove the existence $y$ to $2x + 1 = 2y - 1$.

   (2) Prove the uniqueness by contradiction.
Disproving Existence Statements

\[ \sim (\exists x \in S, P(x)) \equiv \forall x \in S, \sim P(x) \]

If the statement, “\(\exists x \in S, P(x)\)”, is false, every \(x \in S\) satisfies “\(\sim P(x)\)”.

**Examples**: Disprove the statements

1. There is a real number \(x\) for which \(x^4 - 6x^2 + 2 < -7\).

2. There exist odd integers \(a\) and \(b\) such that \(4|(3a^2 + 7b^2)\). (Textbook).