READ THIS FIRST:  This homework assignment is different.  After your solutions are graded and returned, you will be asked to make corrections, revise your solutions, and re-submit the entire assignment.  Your revised solutions will again be graded.  So, effectively, this assignment is worth double points.

The purpose of the “submit, revise, re-submit” process is to provide you with an opportunity to work on improving your mathematical writing.  Your writing will be carefully scrutinized and the grade you earn will reflect this.

Directions:  Read Chapter 7.  Then write a solution to each of the exercises below.  Each solution must begin with a complete and accurate restatement of the exercise.  These exercises are modified versions of some of the exercises in Chapter 7.

1. Two recursively defined sequences \( \{a_n\} \) and \( \{b_n\} \) of positive integers have the same recurrence relation: for each \( n \geq 3 \),

\[
a_n = 2a_{n-1} + a_{n-2} \quad \text{and} \quad b_n = 2b_{n-1} + b_{n-2}.
\]

The initial values for \( \{a_n\} \) are \( a_1 = 1 \) and \( a_2 = 3 \), whereas the initial value for \( \{b_n\} \) are \( b_1 = 1 \) and \( b_2 = 2 \).

Determine whether each of the following conjectures is true or false.

**Conjecture A:** \( a_n = 2^{n-2} \cdot n + 1 \) for every integer \( n \geq 2 \).

**Conjecture B:** \( b_n = \frac{1}{2\sqrt{2}} [(1 + \sqrt{2})^n - (1 - \sqrt{2})^n] \) for every integer \( n \geq 2 \).

2. Express the statement below in symbols (for example, using the symbols \( \exists, \forall, \implies, \lor, \land, \iff, \text{and} \sim \)). Then prove the statement.

For every positive real number \( a \) and positive rational number \( b \), there exist a real number \( c \) and irrational number \( d \) such that \( ac + bd = 1 \).

3. Prove or disprove: for every two sets \( A \) and \( B \), we have that \( (A \cup B) - B = A \).

4. Prove or disprove: for every rational number \( a/b \) such that \( a, b \in \mathbb{N} \), there exists a rational number \( c/d \) such that \( c \) and \( d \) are positive odd integers and \( 0 < c/d < a/b \).

5. Prove or disprove: there exist positive integers \( x \) and \( y \) such that \( x^2 - y^2 = 101 \).