1. Disprove the statement: There is a real root of equation \( \frac{1}{5}x^5 + \frac{2}{3}x^3 + 2x = 0 \) on the interval (1, 2).

2. Prove: there exists a real number \( x \) such that \( \frac{x^2 + 3x - 3}{2x + 3} = 1 \).

3. Let \( f(x) = x^3 - 3x^2 + 2x - 4 \). Prove that there exists a real number \( r \) such that \( 2 < r < 3 \) and \( f(r) = 0 \).

4. Prove that there is no smallest positive rational number.

5. Prove that there is no largest prime.
   
   Hint: Use proof by contradiction and consider \( n = p_1...p_k + 1 \), where \( p_i, i = 1, ..., k \) are all the possible prime numbers, as per your assumption.

6. Let \( x \) be an irrational real number. Prove that either \( x^2 \) or \( x^3 \) is irrational.

7. Prove that \( \sqrt{5} \) is irrational. (You can use the fact that \( 5 \mid x^2 \) if and only if \( 5 \mid x \).)

8. Prove that if \( x, y \in \mathbb{Z} \), then \( x^2 - 4y \neq 2 \).

9. Use induction to prove that
   
   \[
   1 + 3 + 6 + \cdots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}
   \]
   
   for all \( n \in \mathbb{N} \).

10. Prove that
    
    \[
    1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n+1}
    \]
    
    for all \( n \in \mathbb{N} \) with \( n \geq 3 \). (Note that \( 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} > 1 + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{4}} = 2 \).)

11. Use the Strong Principle of Mathematical Induction to prove that for each integer \( n \geq 13 \), there are nonnegative integers \( x \) and \( y \) such that \( n = 3x + 4y \).

12. Let \( R \) be a relation defined on \( \mathbb{N}^2 \) by \((a,b)R(c,d)\) if \( ad = bc \). Prove or disprove that a relation \( R \) is an equivalence relation and describe the elements in the equivalence class \([(1,2)]\).
13. For \((a, b) \in \mathbb{R}^2\), define \((a, b) \sim (c, d)\) if \([a] = [c]\) and \([b] = [d]\), where \([x]\) is the greatest integer less than or equal to \(x\). Prove or disprove that a relation \(\sim\) is an equivalence relation in \(\mathbb{R}^2\).

14. A relation \(R\) is defined on the set of positive rational numbers by \(aRb\) if \(\frac{a}{b} \in \{3^k : k \in \mathbb{Z}\}\). Prove that a relation \(R\) is an equivalence relation and describe the elements in the equivalence class \([2]\).

15. (a) Fill in the following addition and multiplication tables for \(\mathbb{Z}_4\).

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(b) For each of the following modular arithmetic equations, use the tables above to either find all solutions or explain why it has no solution. The coefficients and variables should be taken in \(\mathbb{Z}_4\).

i. \([3]x + [1] = [2]\)

ii. \([2]x + [2] = [3]\)

iii. \([2]x + [1] = [3]\)

16. Let \([a], [b] \in \mathbb{Z}_{13}\) and \([a] \neq [0]\). Prove that the equation \([a]x + [b] = 0\) always has exactly one solution.

17. Let \([a], [b] \in \mathbb{Z}_{13}\), and assume \([a] \cap [b] \neq \emptyset\). Prove that \([a] = [b]\).

18. Let \([a], [b] \in \mathbb{Z}_{11}\) and \([a] \neq [0]\). Prove that the equation \([a]x + [b] = 0\) always has exactly one solution.

19. Let \([a], [b] \in \mathbb{Z}_{11}\), and assume \([a] \cap [b] \neq \emptyset\). Prove that \([a] = [b]\).

20. Fill in the blanks in part(a).

(a) Let \(A\) and \(B\) be sets. A relation \(R \subset A \times B\) defines a function from \(A\) to \(B\) if

(1) \(\forall a \in A\), \(\exists b \in B\) such that \(\) and

(2) \(\forall a \in A\), \(\forall b_1, b_2 \in B\), if \((a, b_1) \in R\) and \((a, b_2) \in R\), then \(\)
(b) Let $A$ be a set. Prove that there exists a unique relation $R$ on $A$ such that $R$ is an equivalence relation on $A$ and $R$ is a function from $A$ to $A$.

21. Let $A$ and $B$ be sets. Suppose that $f : A \rightarrow B$ is a function. Let $C, D \subseteq B$. Prove that

$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D).$$

22. Let $\mathbb{Z}_7 = \{[0], [1], \ldots, [6]\}$ be the set of congruence classes of integers modulo 7 together with the operations of addition and multiplication of congruence classes. Suppose that $f : \mathbb{Z}_7 \rightarrow \mathbb{Z}_7$ is the function defined by the rule

$$f([x]) = [2x + 1] \text{ for each } x \in \mathbb{Z}.$$ 

(a) Prove that $f$ is a bijection.

(b) Prove that there exists integers $a$ and $b$ such that $f^{-1}$ is given by the rule

$$f^{-1}([x]) = [ax + b] \text{ for each } x \in \mathbb{Z}.$$