Solutions to Homework 10

Section 7.2, exercise # 1 (b,d):

(b) Compute the value of $\int_R f\,dV$, where $f(x,y)=y/x$ and $R=[1,3]\times[2,4]$.

Solution: Since f is continuous over R, f is integrable over R. Let $x\in[1,3]$. Since f is continuous over R, the function $g(y)=f(x,y)$ is continuous for each $x\in[1,3]$. (Can you prove this?)

In particular, $g(y)$ is integrable over $[2,4]$ for each $x\in[1,3]$. Let $F(x)=\int_2^4 f(x,y)\,dy$. Thus,

$$F(x)=\int_2^4 \frac{y}{x}\,dy=\frac{1}{x}\int_2^4 y\,dy=\frac{6}{x}.$$

Since $F(x)=6/x$ is continuous on $[1,3]$, it is integrable on $[1,3]$. Therefore, all of the hypotheses of Fubini's theorem are satisfied. Applying Fubini's theorem, one has that

$$\int_R f\,dV=\int_1^3\int_2^4 f(x,y)\,dy\,dx=\int_1^3\frac{6}{x}\,dx=6\ln 3.$$

The above solution gives much more detail than one would normally give. Moreover, one would like an easily applicable test for whether Fubini's theorem can be applied.

One such test is given in exercise #20 of Section 7.2: if $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$ and if $\frac{\partial f}{\partial x}$ is continuous on R, then $F(x)=\int_c^d f(x,y)\,dy$ is not only continuous on $[a,b]$, it is differentiable on $[a,b]$ and $F'(x)=\int_c^d \frac{\partial f}{\partial x}\,dy$.

Another such test is given in Corollary 2.2. The corollary implies that if f is continuous and if $F(x)=\int_c^d f(x,y)\,dy$ and $G(y)=\int_a^b f(x,y)\,dx$ are continuous (on $[a,b]$ and $[c,d]$, respectively), then f is integrable on R and the value of $\int_R f\,dV$ is equal to the value of each of the iterated integrals.

For the function in (b), it is easy to apply either test. For instance, $f(x,y)=y/x$ is continuous on $[1,3]\times[2,4]$. (It is built out of continuous functions using arithmetic operations.) And it is equally easy to see that $\frac{\partial f}{\partial x} = -y/x^2$ is continuous on $[1,3] \times [2,4]$.

(d) Evaluate $\int_R (x+y)z\, dV$ where $R = [-1,1] \times [1,2] \times [2,3]$.

Fubini's theorem applies to the above function since f is continuous and it will be clear from the computation below that the functions which result from computing the iterated integrals are also continuous.

$$\int_2^3 (x+y)z\, dz = (x+y)\frac{z^2}{2}\Big|_2^3 = \frac{5}{2}(x+y)$$

$$\int_1^2 \frac{5}{2}(x+y)\, dy = \frac{5}{2}(xy + \frac{y^2}{2})\Big|_1^2 = \frac{5}{2}x - \frac{15}{4}$$

$$\int_{-1}^1 (\frac{5}{2}x - \frac{15}{4})\, dx = \frac{5}{2}\int_{-1}^1 x\, dx + \frac{15}{4}\int_{-1}^1 1\, dx = 0 + \frac{15}{4} = \frac{15}{4}$$

Section 7.2, exercise # 2 (b,c,d,e):

Interpret the iterated integral as an (double) integral $\int_\Omega f\, dA$ for the appropriate region Ω. Sketch Ω and change the order of integration. You may assume that f is continuous.

Note: The term "double integral" is synonymous with "iterated integral consisting of two integrals". So, I would not use the phrase "double integral" as the author has done. I would simply call $\int_\Omega f\, dA$ the integral of f over Ω.

(b): The result of changing the order of integration is printed in the textbook.

(c): $\displaystyle\int_1^2 \int_{y^2}^4 f(x,y)\, dx\, dy$

Solution: The region $\Omega \subset \mathbb{R}^2$ is the region given by the following inequalities:

$$y^2 \le x \le 4 \quad \text{and} \quad 1 \le y \le 2.$$

The first inequality describes the closed, bounded region in the plane bounded by the parabola $x = y^2$ and the line $x = 4$. The second inequality describes the closed, unbounded region which lies between the lines $y = 1$ and $y = 2$. The set of points which satisfy both inequalities is the intersection of these two regions. Thus, Ω is a closed, bounded region having three "sides". It is bounded below by the line $y = 1$, above (or to the left) by the parabola $x = y^2$ and on the right by the line $x = 4$.

Since the integrand is assumed to be continuous, Fubini's Theorem applies. (This is not entirely clear. How do we know, for instance, that $G(y) = \int_{y^2}^4 f(x, y)\, dx$ is continuous on $[1, 2]$? We will assume that this is the case for purposes of this problem and the other integrals in this exercise.)

Therefore, $\int_\Omega f\, dA$ is equal to the value of the iterated integral above. Moreover, it is also equal to the value of the other iterated integral obtained by changing the order of integration.

To change the order of integration, we need to describe the region Ω by two inequalities. These inequalities need to take the following form:

$$b(x) \le y \le t(x) \quad \text{and} \quad l \le x \le r,$$

where the names of these functions and constants are suggestive of their geometric significance: the curve $y = b(x)$ is the curve which describes the bottom of the region, $y = t(x)$ describes the top of the region, and l and r describes vertical lines which bound the region on the left and on the right.

The following inequalities describe Ω:

$$1 \le y \le \sqrt{x} \quad \text{and} \quad 0 \le x \le 4.$$

This is geometrically clear if one sketches the regions described by each pair of inequalities separately and then considers the intersection of these two regions.

One process for deducing the above description is the following. Choose a generic point p in the region Ω. (Generic means that you should not choose a point which lies on the boundary of Ω.) Now let the y-coordinate of the point p vary. By looking at the sketch of Ω it is clear that a generic point p must lie between the line $y = 1$ and the curve $y = \sqrt{x}$ (equivalently, the curve $x = y^2$) as its y-coordinate varies. Thus, $1 \le y \le \sqrt{x}$ is a condition on the y-coordinate of points which belong to Ω. Next consider the projection of the region Ω onto the x-axis. This is the interval $[0, 4]$. And so, $0 \le x \le 4$ is a condition on the x-coordinates of points in Ω. Thus, every point p in Ω must satisfy these two inequalities. Sketching the region described by these two inequalities results in the same region Ω.

A second process for deducing the above description is to (carefully) solve the inequalities. The goal of such a "solution" is a pair of inequalities of the form $b(x) \le y \le t(x)$ and $l \le x \le r$. Care needs to be taken when applying operations to inequalities. For example, multiplying by a negative number reverses the inequality. Applying an increasing function to both sides of an inequality preserves the inequality. Applying a decreasing function to both sides reverses an inequality.

It the present case, the original inequalities $1 \le y \le 2$ and $y^2 \le x \le 4$ imply that (by applying the increasing function of taking the square root) $y \le \sqrt{x} \le 2$ and that (by rearranging the original inequalities) $1 \le y \le \sqrt{x}$ and that $x \le 4$. Finally, since $1 \le y$, $1 \le y^2 \le x$ and so both $1 \le x \le 4$ and $1 \le y \le \sqrt{x}$ hold true.

(d) $\displaystyle\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$

The region Ω described by the inequalities $-1 \le y \le 1$ and $0 \le x \le \sqrt{1-y^2}$ is right half of the closed unit disk, i.e. the region inside the disk $x^2 + y^2 \le 1$ and inside the half-space $0 \le x$. This can be deduced geometrically or by solving the system of inequalities: squaring implies that $0 \le x^2 \le 1 - y^2$, and the condition $-1 \le y \le 1$ places no further restriction since $x^2 + y^2 \le 1$ implies that $y^2 \le 1$ and so $-1 \le y \le 1$.

Choosing a generic point p, one sees that as the y-coordinate varies, the point is bounded below by the semi-circle $y = -\sqrt{1-x^2}$ and above by the semi-circle $y = \sqrt{1-x^2}$. Projecting the region Ω onto the x-axis results in the interval $[0,1]$. Therefore, $0 \le x \le 1$. A quick sketch verifies that Ω is indeed described by the inequalities $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ and $0 \le x \le 1$.

These inequalities can also be obtained algebraically. Either way, the resulting integral with the order of integration reversed is the following:

$$\int_{0}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

and above by the semicircle $y = \sqrt{1-x^2}$, and so $0 \le y \le \sqrt{1-x^2}$. Projecting onto

(e) $\int_0^1 \int_{x^2}^{x} f(x,y)\,dy\,dx$

Solution: Algebraically manipulating the inequalities, one sees that $x^2 \le y \le x$ and $0 \le x \le 1$ implies that $0 \le x \le \sqrt{y}$ and $y \le x \le 1$ and $0 \le x^2 \le y \le x \le 1$. Therefore, $0 \le y \le 1$ and $y \le x \le \sqrt{y}$.

(I do not recommend this method. The preferred method is to sketch the region and let a generic point vary its position.)

Answer:

$$\int_0^1 \int_y^{\sqrt{y}} f(x,y)\,dx\,dy.$$

(f) The solution appears in the textbook.

Section 7.2, exercise # 3 (a,b):

Solution:

(a) Change the order of integration and evaluate the resulting integral:

$$\int_0^1 \int_0^x (x+y)\,dy\,dx$$

The region $0 \le y \le x$, $0 \le x \le 1$ is a triangular region having vertices $(0,0)$, $(1,1)$, and $(0,1)$. Choosing a point in this region an varying its x coordinate shows that the minimum value of the x coordinate is determined by the line $y = x$ and that the maximum value of x is equal to $x = 1$. So, $y \le x \le 1$, $0 \le y \le 1$.

$$\int_0^1 \int_x^1 (x+y)\,dx\,dy$$

The above integral is straightforward to compute. Answer: 1/2. The original integral is easier to evaluate; so, in practice, there would be no reason to change the order of integration.

(b) Change the order of integration and evaluate the resulting integral:

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y \, dx \, dy$$

The integral describes a double integral over the region enclosed by the semi-circle $y = \sqrt{1-x^2}$ and the line $y = 0$.

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} y \, dy \, dx$$

The inner integral evaluates to $(1/2)(\sqrt{1-x^2})^2 = (1/2)(1-x^2)$. Answer: 2/3. The original integral is not difficult to compute; they are perhaps of equal difficulty. This integral is also easy to compute using polar coordinates.

Section 7.2, exercise # 4: Does the integral

$$\int_0^1 \int_0^1 f(x,y) \, dy \, dx$$

exist, where $f(x,y) = 1$ if $y \in \mathbb{Q}$ and $f(x,y) = 2x$ if $y \notin \mathbb{Q}$?

Solution: No, this integral does not exist. The meaning of the above is that the expression is an iterated integral. But the first integral is not defined:

$$\int_0^1 f(x,y) \, dy.$$

Suppose that $x \in [0,1]$ is chosen and held constant; to emphasize this, let $c = x$. The function $f(c,y) = g(y) = 1$ if $y \in \mathbb{Q}$ and $f(c,y) = g(y) = 2c$. But the function $g(y)$ is no integrable unless $c = 1/2$. This is because the upper sum of $g(y)$ with respect to any partition of $[0,1]$ is always equal to the larger of 1 and $2c$ (since every subinterval must contain both rational and irrational points) and the lower sum of $g(y)$ with respect to any partition is always equal to the smaller of 1 and $2c$ for the same reason. Therefore, the upper and lower sums approach the same value as the partition is refined if and only if $2c = 1$. Thus, the function $F(x) = \int_0^1 f(x,y) \, dy$ is undefined at every value of x except when $x = 1/2$.

Section 7.2, exercise # 8 (b,c): Evaluate the integrals.

Solution:

(b)

$$\int_0^1 \int_{\sqrt[3]{x}}^1 e^{y^4}\, dy\, dx$$

Since there is no apparent anti-derivative with respect to y, change the order of integration. The integral defines a double integral over a region bounded below by $y = \sqrt[3]{x}$, above by $y = 1$, on the left by $x = 0$, and the upper and lower curves meet in a single point, namely $(1, 1)$, on the right-side. Thus,

$$\int_0^1 \int_0^{y^3} e^{y^4}\, dx\, dy$$

The inner integral evaluates to $y^3 e^{y^4}$. And this can be integrated with respect to y by using the substitution $u = y^4$, $du = 4y^3\, dy$. Answer: $(e - 1)/4$.

(c)

$$\int_0^1 \int_{\sqrt{y}}^1 e^{y/x}\, dx\, dy$$

The integral above describes a double integral over the region bounded below by the line $y = 0$, above by the parabola $y = x^2$, on the right by the line $x = 1$, and the upper and lower curves meet in a single point, namely $(0, 0)$ on the left. The integrand $e^{y/x}$ is undefined at $(0, 0)$, but is otherwise continuous on this region. Since a single point has 2-dimensional volume zero, the integrand is integrable. There is no obvious way to evaluate the innermost integral, so change the order of integration:

$$\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx.$$

The inner integral evaluates to $xe^{y/x}\Big|_0^{y^2} = xe^x - x$. Using integration by parts $\int xe^x\, dx = xe^x - e^x + C$. And so,

$$\int_0^1 (xe^x - x)\, dx = (xe^x - e^x - (1/2)x^2)\Big|_0^1 = \frac{1}{2}$$

Section 7.2, exercise # 12 (b,c,e): Sketch the region of integration and change the order of integration so that the innermost integral is with respect to y. Assume that the integrand is continuous.

Solution:

(b)

$$\int_0^1 \int_0^{1-x^2} \int_0^y f\,dz\,dy\,dx$$

The region lies in the octant $x, y, z \geq 0$. It is bounded above by the plane $z = y$ and below by the xy-plane. One side of the region is a triangle in the yz-plane and the other side is part of the parabolic cylinder $y = 1 - x^2$. Choosing a point in this region and varying its y-coordinate reveals that the minimum y-coordinate is determined by the plane $z = y$ and that the maximum y-coordinate is determined by the cylinder $y = 1 - x^2$. This can also be seen from the inequalities: $0 \leq z \leq y$ and $0 \leq y \leq (1 - x^2)$ imply that $z \leq y \leq (1 - x^2)$. Next project the region onto the xz-plane. This region determines the bounds of the outer two integrals. Since the intersection of the plane $z = y$ and the cylinder $y = 1 - x^2$ is the arc whose points have the form $(1 - t^2, t, t)$, where $0 \leq t \leq 1$, this arc projects to the parabola in the xz-plane given by the equation $z = 1 - x^2$. Thus, $0 \leq z \leq 1 - x^2$. So, the new integral is

$$\int_0^1 \int_0^{1-x^2} \int_z^{1-x^2} f\,dy\,dz\,dx.$$

Another answer is

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_z^{1-x^2} f\,dy\,dx\,dz.$$

(c)

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 f\,dz\,dy\,dx$$

The first inequality, $\sqrt{x^2 + y^2} \leq z \leq 1$, say, geometrically, that the region is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. The second two inequalities say, geometrically, that the projection of the region onto the xy-plane is the unit disk centered at the origin. So, the region is the region bounded above by the plane $z = 1$, below by the plane

$z = 1$. (The projection of this region is the unit disk, i.e. the second two inequalities impose no additional restrictions.)

If a point in this region is chosen and its y-coordinate is varied, then the minimum y-coordinate is determined by the cone and, likewise, the maximum y-coordinate is determined by the cone. Solve $z = \sqrt{x^2 + y^2}$ for y: $y^2 = z^2 - x^2$, and so $-\sqrt{z^2 - x^2} \leq y \leq \sqrt{z^2 - x^2}$. The projection of the region onto the xz-plane is a triangle bounded by the lines $z = 1$, $x = z$, and $x = -z$. So, this needs to be divided into two regions if you wish to integrate next with respect to z. Answer:

$$\int_0^1 \int_0^x \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f\, dy\, dz\, dx + \int_0^1 \int_0^{-x} \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f\, dy\, dz\, dx$$

If instead we integrate with respect to x before z, then it is much simpler:

$$\int_0^1 \int_{-z}^{z} \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f\, dy\, dx\, dz$$

(e)

$$\int_0^1 \int_0^{1-x} \int_0^{x+y} f\, dz\, dy\, dx$$

The region of integration is a pyramid with vertex at the origin and with a rectangular base whose vertices are $(1, 0, 0)$, $(1, 0, 1)$, $(0, 1, 0)$, and $(0, 1, 1)$. To see this, sketch the plane $z = x + y$. For example, the points $(0, 0, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ lie on this plane. The region lies between the xy-plane (since $0 \leq z$) and the aforementioned plane. The outer two integrals describe the region which results by projecting onto the xy-plane. This projection is a triangle bounded by the line $y = 1 - x$ and by the line $x = 0$ and $y = 0$.

If a point is chosen in the solid region above and its y-coordinate is varied, then the largest y-value is determined by the plane $y = 1 - x$; but the smallest y-value is determined either by the plane $z = x + y$ or by the plane $y = 0$. Two integrals are required. If $z \geq x$, then $z - x \leq y \leq 1 - x$. If $z \leq x$, then $0 \leq y \leq 1 - x$. The projection onto the xz-plane is a triangle in both cases. In the first case, it is a triangle bounded by the lines $x = z$, $x = 0$, and $z = 1$. In the second case it is a triangle bounded by the lines

$x = z$, $x = 1$, and $z = 0$.

$$\int_0^1 \int_x^1 \int_{z-x}^{1-x} f\,dy\,dz\,dx + \int_0^1 \int_0^x \int_0^{1-x} f\,dy\,dz\,dx$$

Another solution is

$$\int_0^1 \int_0^z \int_{z-x}^{1-x} f\,dy\,dx\,dz + \int_0^1 \int_z^1 \int_0^{1-x} f\,dy\,dx\,dz$$

Section 7.3, exercise # 3: Determine the area of the cardioid $r = 1 + \cos\theta$.

Solution: Use $dA = r\,dr\,d\theta$. From the figure (refer to the textbook) it is clear that if a point is chosen inside the cardioid and if its r-coordinate is varied, then $0 \le r \le 1 + \cos\theta$. There are points in the region for all values of θ, and so $0 \le \theta \le 2\pi$.

$$\int_0^{2\pi} \int_0^{1+\cos\theta} r\,dr\,d\theta =$$

$$\int_0^{2\pi} \frac{(1+\cos\theta)^2}{2} d\theta = \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta,$$

where in the second integral I have doubled the integral and integrated over $[0, \pi]$ rather than $[0, 2\pi]$. To integrate $\cos^2\theta$ use the half-angle formula $\cos^2\theta = (1/2)(1 + \cos 2\theta)$ or use integration by parts (which gives a rather nice and easy to remember formula:

$$\int \cos^2\theta\,d\theta = \frac{1}{2}(x + \cos x \sin x) + C.$$

Let's use the half-angle formula:

$$\int_0^{\pi} (1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta)\,d\theta = \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$$

There is no need to integrate the terms involving cosine: these values are clearly equal to zero over $[0, \pi]$ (think about the graph of $y = \cos\theta$).

Section 7.3, exercise # 10: Determine the volume of the region bounded above by $z = 2y$ and below by $z = x^2 + y^2$.

Solution: The region is bounded above by a plane and below by a paraboloid. The intersection of the two surfaces is given by $2y = x^2 + y^2$ which can be rewritten as $x^2 + (y-1)^2 = 1$, a circle. For this problem, use polar coordinates. The projection of the region onto the xy-plane is the circle centered at $(0, 1)$ with radius 1. In polar coordinates, this circle has equation $r = 2\sin\theta$, where $0 \leq \theta \leq \pi$. The plane $z = 2y$ has equation $z = 2r\sin\theta$ in cylindrical coordinates. The paraboloid has equation $z = r^2$. The volume of the region is computed by the following integral in cylindrical coordinates:

$$\int_0^{\pi} \int_0^{2\sin\theta} \int_{r^2}^{2r\sin\theta} r\, dz\, dr\, d\theta.$$

The inner integral evaluates to $2r^2\sin\theta - r^3$. The next integral evaluates to $(4/3)\sin^4\theta$. To integrate $\sin^4\theta\, d\theta$, use the double angle formula twice. Answer: $3\pi/4$.

Section 7.3, exercise # 13: Determine the volume of the region inside both $x^2 + y^2 = 1$ and $x^2 + y^2 + z^2 = 2$.

Solution: The figure is the region inside a cylinder of radius 1 and inside a sphere of radius $\sqrt{2}$. Cylindrical coordinates are most appropriate. Using symmetry, it suffices to double the volume of the region which lies above the xy-plane. This region is bounded above by the sphere so $0 \leq z \leq \sqrt{2 - x^2 - y^2} = \sqrt{2 - r^2}$. The projection of the region onto the xy-plane is the unit disk centered at the origin. Therefore the volume of the original region is compute by

$$2\int_0^{2\pi} \int_0^{1} \int_0^{\sqrt{2-r^2}} r\, dz\, dr\, d\theta.$$

The inner integral is equal to $r\sqrt{2 - r^2}$. The next integral is equal to $(2\sqrt{2} - 1)/3$ (use $u = 2 - r^2$ and $du = -2r\, dr$). So, the final answer is $(4\pi/3)(2\sqrt{2} - 1)$.

Section 7.3, exercise # 19: Find the volume of the region lying above the plane $z = a$ and inside the sphere $x^2 + y^2 + z^2 = 4a$ by integrating in both cylindrical and spherical coordinates.

Solution: The intersection of the two surfaces is the circle $x^2 + y^2 = 3a^2$ lying in the plane $z = a$. If $a = 0$, then this is just a point. To simplify matters, let's assume that $a > 0$ since we can determine the values when $a \leq 0$ by using geometric arguments and the solution when $a > 0$.

In cylindrical coordinates, the z-coordinate of a point in the region is bounded below by the plane $z = a$ and above by the sphere $z = \sqrt{4a^2 - x^2 - y^2} = \sqrt{4a^2 - r^2}$. The projection of the region onto the xy-plane is the disk $r^2 = x^2 + y^2 \leq 3a^2$. Therefore, if $a > 0$, the following integral computes the volume:

$$\int_0^{2\pi} \int_0^{a\sqrt{3}} \int_a^{\sqrt{4a^2-r^2}} r\,dz\,dr\,d\theta.$$

In spherical coordinates, the plane $z = a$ is expressed as $\rho \cos\phi = a$ and the sphere is simply $\rho = 2a$. Thus, $a \sec\phi \leq 2a$. (This assumes $a > 0$). The value of ϕ ranges from 0 to the angle determined by the triangle having height $z = a$ and base $r = a\sqrt{3}$. Thus, $0 \leq \phi \leq \tan^{-1}\sqrt{3} = \pi/3$. Therefore, if $a > 0$, the following integral computes the volume of the region:

$$\int_0^{2\pi} \int_0^{\pi/3} \int_{a\sec\phi}^{2a} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta.$$

Answer: $5a^3\pi/3$.