1. (3 points) Use factoring to solve the following equations. (Hint: Use a substitution.)

\[5^{2x} - 7 \cdot 5^x - 18\]

Let \( u = 5^x \). Since \( u^2 = (5^x)^2 = 5^{2x} \), the equation above reduces to the quadratic equation \( u^2 - 7u - 18 = 0 \). This can be solved by factoring: \((u - 9)(u + 2) = 0\). Therefore, \( u = 9 \) or \( u = -2 \). In the first case, \( u = 9 \) is equivalent to \( 5^x = 9 \); rewriting this as a logarithmic equation, \( x = \log_5 9 \); alternatively, take a logarithm of both sides of the equation \( 5^x = 9 \) and then solve to obtain \( x = \ln 9 / \ln 5 \). In the second case, \( u = -2 \) is equivalent to \( 5^x = -2 \); since the range of the exponential function \( y = 5^x \) is \( y > 0 \), there is no solution. Hence, the only solution to the original equation is \( x = \log_5 9 = \ln 5 / \ln 9 \).

2. (3 points) Solve the following equation.

\[\ln (x + 6) - \ln (x + 2) = \ln 5\]

Rewrite the left hand side as a single logarithm using the quotient property of logarithms:

\[\ln \left( \frac{x + 6}{x + 2} \right) = \ln 5.\]

By the one-to-one property of logarithms,

\[\frac{x + 6}{x + 2} = 5.\]

Multiply both sides of the equation by \( x + 2 \) to obtain \( x + 6 = 5(x + 2) \). This reduces to \( 4x = -4 \), and so \( x = -1 \). Check if this is a solution by letting \( x = -1 \) in the original equation:

\[\ln (-1 + 6) - \ln (-1 + 2) = \ln 5,\]

which is true since the logarithms above are defined. (We are checking to see if the value of \( x \) is not in the domain of the logarithmic functions in the original equation; in other words, we are checking to see if \( x \) is a restricted value of the equation.) Thus, \( x = -1 \) is the only solution to the original equation.
3. (2 points) Radium-225 has a half-life of 1601 years. If the exponential decay model of \( f(t) = A_0 e^{kt} \) is used, what is the value of \( k \)? Give an exact value.

A half-life of 1601 years means that \( f(1601) = (1/2)A_0 \), i.e. half of the initial amount remains after 1601 years. Using the given equation, we have that

\[
f(1601) = A_0 e^{k(1601)} = (1/2)A_0.
\]

Dividing both sides by \( A_0 \), we see that \( e^{1601k} = 1/2 \). Take a logarithm of both sides of the equation to obtain \( 1601k = \ln(1/2) \). Finally, divide by 1601. Thus,

\[
k = \frac{\ln(1/2)}{1601}.
\]

4. (2 points) Using the model in the problem above, determine how long it will take a sample of Radium-225 to decay to 95% of its initial mass. Round your answer to the nearest whole year.

The question is asking us to solve the equation \( f(t) = 0.95A_0 \) for \( t \). Using the original equation, we have that

\[
A_0 e^{kt} = 0.95A_0.
\]

Divide both sides of the equation by \( A_0 \) and then take a logarithm of both sides of the equation to obtain \( kt = \ln(0.95) \). Divide by \( k \) to see that \( t = (1/k) \ln(0.95) \). Finally, use the above value of \( k \):

\[
t = \frac{1601}{\ln(1/2)} \cdot \ln(0.95) = 118.47 \cdots \approx 118.
\]

So it will take approximately 118 years for a sample to decay to 95% of its original mass.

**A criticism of the above problem:** My wording is not sufficiently precise. The solution above answers the question of how long it takes for the original amount of radium-225 to decay to 95% of the original amount of radium-225. However, if we were to keep track of the total mass of the sample, then the sample after 118 years consists of radium-225 and isotopes of other elements to which the radium has decayed. These other elements contribute to the mass of the sample that remains after 118 years. (The sum of the masses will not equal the original mass since some of the mass has been converted to energy.)