Activity: Rational Exponents and Equations with Radicals

Numbers and Nomenclature: Here are some commonly encountered sets of numbers.

the positive integers: \{1, 2, 3, 4, \ldots \}
the non-negative integers: \{0, 1, 2, 3, 4, \ldots \}
the integers: \{\ldots , -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots \}
the rational numbers: This set consists of all fractions of integers \(m/n\), where \(n \neq 0\). Two fractions \(a/b\) and \(m/n\) represent the same rational number if \(an = bm\). For example, \(2/3\) and \(10/15\) represent the same rational number since \(2(15) = 3(30)\).
the real numbers: This consists of the sets of all lengths (with direction) on line. We choose one point to represent 0. The length of a segment from 0 to a point \(P\) to the right of 0 represents a positive real number. The length of a segment from 0 to a point \(Q\) to the left of 0 represents a negative number
the complex numbers: This is the set of numbers the form \(a + bi\), where \(a\) and \(b\) are real numbers and \(i^2 = -1\).

Definition of Rational Exponents: If \(p/q\) is a rational number, and \(x\) is a real number, then \(x^{p/q}\) is the number \(\sqrt[q]{x^p}\). It is also equal to \((\sqrt[q]{x})^p\). For example,

\[4^{5/2} = \sqrt[2]{4^5} = \sqrt{1024} = 32.\]

If we compute, \((\sqrt[2]{4})^5\), we also get 32.

If \(p\) is even, then there are two \(p\)-th roots. So, for example, to solve \(x^{2/5} = 4\), we compute the \(5/2\)-th power of both sides, but we must also include the negative of this value:

\[x^{2/5} = 4 \implies x = \pm\sqrt[5]{4^2} = \pm 32.\]

We can check that this is correct. If \(x = -32\), then \(x^{2/5}\) equals \(\sqrt[5]{(-32)^2} = 4\).

If \(p\) is odd, then there is only one root.

The \(p\)-th root of a negative number is defined if \(p\) is odd. If \(p\) is even, then a negative number does not have any real roots, however there are complex roots. For example, \(\sqrt[3]{-8} = -2\) since \((-2)^3 = (-2)(-2)(-2) = -8\). But \(\sqrt[4]{-16}\) is not a real number since the fourth power of any real number is a positive number. There are, however, complex roots.

(It’s very difficult to find a 4th root of \(-16\). Perhaps, suprisingly, there are four fourth roots of \(-16\). Here’s one \((\sqrt{2} + \sqrt{2}i)/2\). We’ll only encounter square roots of negative numbers in this class.)
1. Solve $y^{4/3} = 16$.

2. Solve $a^{3/8} = 8$.


An equation with radicals can sometimes be solved by moving a radical to one side of the equation and moving all other terms to the opposite side. Then raise both sides of the equation to the same power so as to eliminate the radical. Here is an example:

\[
\sqrt{3x - 1} + \sqrt{x - 2} = 1
\]
\[
\sqrt{3x - 1} = 1 - \sqrt{x - 2}
\]
\[
(\sqrt{3x - 1})^2 = (1 - \sqrt{x - 2})^2
\]
\[
3x - 1 = 1 - 2\sqrt{x - 2} + (x - 2)
\]
\[
2x = -2\sqrt{x - 2}
\]
\[
-x = \sqrt{x - 2}
\]
\[
(-x)^2 = (\sqrt{x - 2})^2
\]
\[
x^2 = x - 2
\]
\[
x^2 - x + 2 = 0
\]
\[
x = \frac{1 \pm \sqrt{1 - 4(2)}}{2}
\]
\[
x = \frac{1}{2} \pm \frac{\sqrt{3}i}{2}
\]

1. Solve $x - \sqrt{x + 11} = 1$.

2. Solve $\sqrt{x + 5} - \sqrt{x - 3} = 2$.

Equations with radicals can sometimes be solved by making a clever substitution.

1. Solve $x - 8\sqrt{x} - 20 = 0$.

2. Solve $4x^4 = 13x^2 - 9$.

3. Solve $6x^{2/5} + 11x^{1/5} + 3 = 0$. 