LB 220 Homework 6 (due Wednesday, 04/10/13)

Directions. Please solve the problems below. Your solutions must begin with a clear statement (or re-statement in your own words) of the problem. You solutions should be clear, legible, and demonstrate at minimum partial progress towards a complete solution to the problem. Please refer to the syllabus for the policy on grading (communication, completeness, and correctness) and late homework (homework is due at the start of class, late homework is assessed a 20% penalty if submitted within the next 48 hours)

Collaboration. I encourage you to discuss the homework problems with your classmates. However, each student must write and submit his or her own homework solutions.

Calculators. You can use calculators to determine a numerical approximation to an answer to an application question, but you should use exact values until the very last step in the problem. Calculators are not, however, permitted on any quizzes or exams.

1. Sketch the vector field $\mathbf{F}$ given below.

$$\mathbf{F}(x,y) = \frac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2+y^2}}$$

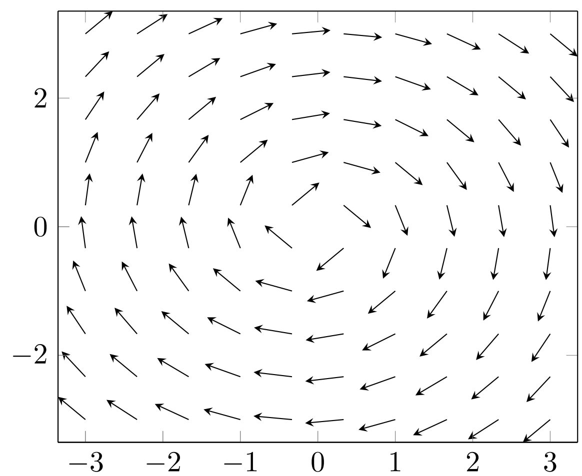
Also, determine an expression for curl $\mathbf{F}$. Finally, answer the following question (with an explantion of your reasoning): is the vector field $\mathbf{F}$ conservative?

Solution: The vector field will look as if it is spinning in a clockwise direction. The field is not conservative since the curl is non-zero as we now show:

$$\frac{\partial M}{\partial y} = \frac{\sqrt{x^2+y^2}\cdot 1 - y\cdot \frac{y}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x^2}{(x^2+y^2)^{3/2}}$$

The other derivative is computed in a simliar fashion. It follows that the curl is equal to

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = \frac{-1}{\sqrt{x^2+y^2}}\,\mathbf{k}.$$



2. Sketch the gradient field of the potential function below, where $a = 3$ and $b = 4$.

$$f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

   Also, describe the shape of a curve which is everywhere tangent to the vector field $\boldsymbol{\nabla} f$. Try to prove your assertion.

   Solution: The vector field is orthogonal to the level curves of f since it is the gradient field. The level curves of f are ellipses and these can be visualized easily. If these are drawn on the page, the vector field $\mathbf{F} = \boldsymbol{\nabla} f$ can be visualized in an equally simple manner: the field lines radiate outward from the concentric ellipses. The curves which are tangent to the field appear to be either lines or arcs of what appear to be hyperbolas.

3. Determine a potential function for the vector field $\mathbf{F}$ given below.

$$\mathbf{F}(x, y) = (3x^2 + 2y^2)\,\mathbf{i} + (4xy + 3)\,\mathbf{j}$$

   Also, give an example of a vector field which does not have a potential function (and explain why this is so).

   Solution: A vector field which does not have a potential is $\mathbf{F} = \langle y, -x \rangle$; this is so because its curl is non-zero. (The vector field in the first problem was also an example.)

To find a potential function for the given vector field, we first check that this might be possible by verifying that the curl is zero. Since $M = 3x^2 + 2y^2$ and $N = 4xy + 3$, it is easy to see that $M_y = 4y = N_x$, and so $(N_x - M_y) = 0$.

Next we attempt to solve the system of partial differential equations: $\langle f_x, f_y \rangle = \langle M, N \rangle$.

Since $f_x = 3x^2 + 2y^2$, we have that $f = x^3 + 2xy^2 + C(y)$, where $C(y)$ is a function which only depends on y and not on x. Similarly, $f_y = 4xy + 3$ implies that $f = 2xy^2 + 3y + D(x)$, where $D(x)$ is a function of x alone. Comparing these two possibilities, we conclude that $f = x^3 + 2xy^2 + 3y$ is a potential for the given field $\mathbf{F}$.

4. Let $\mathbf{F}$ be the vector field given below and let $a > 0$ be a constant.

$$\mathbf{F}(x, y) = \frac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$$

Let C be the circle of radius a centered at $(0, 0)$ and parametrized by $\mathbf{r}(t)$ as follows:

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} - a\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Compute the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution: Since $x(t) = a\cos t$ and $y(t) = -a\sin t$, it follows that $x^2 + y^2 = a^2$. Thus, the value of $\mathbf{F}$ along this circle C reduces to $\mathbf{F} = \frac{1}{a}\langle y(t), -x(t) \rangle$, which evaluates to $\langle -\sin t, -\cos t \rangle$.

The value of $\frac{\mathbf{r}}{dt}$ is equal to $\langle -a\sin t, -a\cos t$. Therefore, the integral reduces to

$$\int_0^{2\pi} \langle -\sin t, -\cos t \rangle \cdot \langle -a\sin t, -a\cos t \rangle\, dt = \int_0^{2\pi} a\, dt = 2\pi a.$$