Solutions to Homework 4

1. For each function below, identify the function’s domain and range. Then sketch at least three non-empty level curves of each function. Finally, determine if the domain is open, closed, or neither.

(a) \( f(x, y) = \sqrt{y - x} \)

Solution: \( y - x \geq 0 \) is the domain. This consists of all points in the plane above or on the line \( y = x \). This is a closed set. The range of the function is \([0, \infty)\). Here is a sketch of the level curves \( f = 0 \), \( f = 1 \), and \( f = 2 \):

![Graph of level curves](image)

The curves are the lines \( y - x = 0 \), \( y - x = 1 \), and \( y - x = 4 \), respectively.

(b) \( f(x, y) = y/x^2 \)

Solution: \( x^2 \neq 0 \) is the domain. This consists of all points in the plane which do not lie on the line \( x = 0 \). This is an open set. The range of the function is \((-\infty, \infty)\). Here is a sketch of the level curves \( f = -2 \), \( f = -1 \), \( f = 0 \), \( f = 1 \) and \( f = 2 \):

![Graph of level curves](image)
The figure above needs to be modified: all points on the $y$-axis, i.e. on the line $x = 0$, need to be excluded. This can be achieved by drawing an open circle at the origin in the figure above.

(c) $f(x, y) = \ln(x^2 + y^2)$

Solution: $x^2 + y^2 > 0$ is the domain. This consists of all points in the plane except $(0, 0)$. This is an open set. The range of the function is $(-\infty, \infty)$. Here is a sketch of the level curves $f = -1$, $f = 0$, and $f = 1$. 
2. By considering different paths of approach show that the function 
\[ f(x, y) = \frac{x^4}{x^4 + y^2} \] has no limit as \((x, y)\) approaches \((0, 0)\).

**Solution:** The limit as \(y = 0\) and \(x \to 0^+\) is equal to 1. The limit as \(x = 0\) and \(y \to 0^+\) is 0. Therefore, the limit of \(f(x, y)\) as \((x, y) \to (0, 0)\) does not exist.

3. Suppose that \(f(x_0, y_0) = 3\). What can you conclude about 
\[ \lim_{(x, y) \to (0, 0)} f(x, y) \]
if \(f\) is continuous at \((x_0, y_0)\)? What can you conclude if \(f\) is not continuous at \((x_0, y_0)\)? Explain your reasoning.

**Solution:** If \(f\) is continuous at \((x_0, y_0)\), then \(f(x_0, y_0) = 3\). If \(f\) is not continuous at this point, then either the limit of \(f(x, y)\) as \((x, y)\) approaches \((x_0, y_0)\) does not exist, or this limit does exist but its value is not equal to 3.

The reason is that the definition of \(f\) being continuous at this point is that the limit at this point exists and is equal to the value of the function at this point.

4. Determine \(f_x\), \(f_y\), and \(f_z\) for the following functions:

(a) \(f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}\)

**Solution:** 
\[ f_x = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \]
\[ f_y = -\frac{y}{(x^2 + y^2 + z^2)^{3/2}} \]
\[ f_z = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}} \]

(b) \(f(x, y, z) = yz \ln(xy)\)

**Solution:**
\[ f_x = \frac{(yz)/(xy)(y)}{y} = \frac{yz}{x} \]
\[ f_y = z \ln(xy) + yz(1/xy)(x) = z \ln(xy) + z \]
\[ f_z = y \ln(xy) \]

5. Determine the value of \(\frac{\partial x}{\partial z}\) at the point \((1, -1, -3)\) if the equation 
\[ xy + z^3 x - 2yz = 0 \]
defines \(x\) as a function of the two independent variables \(y\) and \(z\).

(Hint: implicit differentiation.)
Solution:

\[ \frac{\partial}{\partial z} \left( xy + z^3x - 2yz = 0 \right) \implies \frac{\partial x}{\partial z} y + (3z^2x + z^3 \frac{\partial x}{\partial z}) - 2y = 0, \]

where the parentheses are there to emphasize that the product rule has been applied to the term \( z^3x \) – both \( z^3 \) and \( x \) are functions of \( x \) in this problem.

Solving the above equation yields the following:

\[ \frac{\partial x}{\partial z} = \frac{2y - 3z^2x}{y + z^3}. \]

Finally, plugging in the point \((1, -1, -3)\), one obtains the answer

\( \frac{\partial x}{\partial z}(1, -1, -3) = \frac{2(-1) - 3(-3)^2(1)}{(-1 + (-3)^3) = 29/28}. \)

There is an error in the statement of the problem, however. This point does not lie on the surface! I should have given a point which makes the equation true. For, instance, \((1, 1, 1)\) would work.

There is another way to solve this problem by appealing to a formula derived in the text: if \( F(x, y, z) = 0 \) and \( x \) is implicitly a function of \( y \) and \( z \), then

\[ \frac{\partial x}{\partial z} = -\frac{F_z}{F_x}. \]

This formula is derived by the same process as the above: implicitly differentiate the equation \( F(x, y, z) = 0 \) and solve for \( \frac{\partial x}{\partial z} \). By the chain rule, we have that

\[ \frac{\partial}{\partial z} \left( F(x, y, z) = 0 = 0 \right) \implies F_x \frac{\partial x}{\partial z} + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial z} = 0. \]

Since \( \frac{\partial z}{\partial z} = 1 \) and \( \frac{\partial y}{\partial z} = 0 \) (since \( y \) and \( z \) are regarded as the independent variables in this setup), the equation above simplifies and upon solving for \( \frac{\partial x}{\partial z} \) one obtains the formula above.

6. The (two dimensional) Laplace equation is

\[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \]

Show that the function below is a solution to the Laplace equation:

\[ f(x, y) = \ln \sqrt{x^2 + y^2}. \]
**Solution:** Simplify first: \( \ln \sqrt{x^2 + y^2} = (1/2) \ln (x^2 + y^2) \). Then \( f_x = (1/2)(1/(x^2 + y^2))(2x) = x/(x^2 + y^2) \). By symmetry, \( f_y = y/(x^2 + y^2) \).

Differentiating again (using the quotient rule), one determines that

\[
\frac{\partial^2 f}{\partial x^2} = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.
\]

By symmetry, \( \frac{\partial^2 f}{\partial y^2} = \frac{(x^2 - y^2)}{(x^2 + y^2)^2} \). And so, \( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \).