LB 220 Homework 3 (due Monday, 01/28/13)

Directions. Please solve the problems below. Your solutions must begin with a clear statement (or re-statement in your own words) of the problem. You solutions should be clear, legible, and demonstrate at minimum partial progress towards a complete solution to the problem. Please refer to the syllabus for the policy on grading (communication, completeness, and correctness) and late homework (homework is due at the start of class, late homework is assessed a 20% penalty if submitted within the next 48 hours).

Collaboration. I encourage you to discuss the homework problems with your classmates. However, each student must write and submit his or her own homework solutions.

Calculators. You can use calculators to determine a numerical approximation to an answer to an application question, but you should use exact values until the very last step in the problem. Calculators are not, however, permitted on any quizzes or exams.

1. (a) Plot the point $P(-2, 2\sqrt{3}, 3)$ given in rectangular coordinates. Compute both the cylindrical coordinates and the spherical coordinates for $P$.

Solution: We are given that $x = -2$, $y = 2\sqrt{3}$, $z = 3$. Thus, $r = \sqrt{x^2 + y^2} = 4$. Since $\tan \theta = y/x = -\sqrt{3}$, $\theta = -\pi/3$ or $\theta = -\pi/3 + \pi$. Since $(x, y)$ is in the second quadrant, the correct value is $\theta = -\pi/3 + \pi = 2\pi/3$. Therefore, the cylindrical coordinates are $(4, 2\pi/3, 3)$.

Similarly, $\rho = \sqrt{r^2 + z^2} = 5$. And, $\tan \phi = z/r = 4/3$; so $\phi = \arctan(4/3)$. Therefore, the spherical coordinates are $(5, 2\pi/3, \arctan(4/3))$.

(b) Plot the point $Q(6, \pi/2, 3\pi/4)$ (using the order $(\rho, \theta, \phi)$) given in spherical coordinates. Then compute both the rectangular and cylindrical coordinates for $Q$.

Solution: We are given that $\rho = 6$, $\theta = \pi/2$, and $\phi = 3\pi/4$.

Thus, $r = \rho \sin \phi = 3\sqrt{2}$ and $z = \rho \cos \phi = -3\sqrt{2}$. So, the cylindrical coordinates are $(3\sqrt{2}, \pi/2, -3\sqrt{2})$.

Since $x = r \cos \theta = 0$ and $y = r \sin \theta = 3\sqrt{2}$, the rectangular coordinates are $(0, 3\sqrt{2}, -3\sqrt{2})$. 
2. Show that the trace of the function \( r(t) = (t \cos t, t \sin t, t) \) lies on the cone \( z^2 = x^2 + y^2 \) and use this fact to help you sketch the trace of this curve for \( 0 \leq t \leq 2\pi \).

Solution: Let \( x = t \cos t, y = t \sin t, \) and \( z = t \). Then
\[
x^2 + y^2 = t^2(\cos^2 t + \sin^2 t) = t^2 = z^2
\]
as claimed. We can discuss the sketch in class or during office hours as needed. The curve will spiral upwards along the cone, starting at the point \((0,0,0)\) and winding in a counter-clockwise direction. It is easiest if you first sketch the cone and then attempt to sketch the curve on the surface.

3. Sketch the trace of the function \( r(t) = (1 + \cos t, 2 + \sin t) \), compute \( r'(t) \), and sketch both the position vector \( r(t) \) (with tail at the origin) and the velocity vector \( r'(t) \) (with tail at the head of \( r'(t) \)) when \( t = \pi/6 \).

Solution: The trace of this function is a circle of radius 1 centered at the point \((1, 2)\) and oriented in a counter-clockwise direction. This can be seen by setting \( x = 1 + \cos t \) and \( y = 2 + \sin t \). It follows that \((x - 1)^2 + (y - 2)^2 = 1\), which is the equation of this circle.

The velocity is
\[
r'(t) = (-\sin t, \cos t).
\]
When \( t = \pi/6 \), the position vector is \((3/2, 2 + \sqrt{3}/2)\) and the velocity vector is \((-\sqrt{3}/2, 1/2)\).

4. Determine the two unit tangent vectors to the trace of the function \( r(t) = (\sin^2 t, \cos^2 t, \tan^2 t) \) at \( t = \pi/4 \). Which of these two tangent vectors indicates the direction of increasing values of \( t \)?

Solution: The unit tangent vector is computed by normalizing the velocity vector (meaning, make the velocity a vector a unit vector by dividing by its norm (a.k.a its length)):
\[
r'(t) = (2\sin t \cos t, -2 \cos t \sin t, 2 \tan t \sec^2 t).
\]
So, \( r'(\pi/4) = (1, -1, 4) \). Therefore, the unit tangent vector is
\[
T(\pi/4) = r'(\pi/4)/\sqrt{18}.
\]
Another unit tangent vector is the negative of this vector, but it is the first one which indicates the direction of increasing \( t \). This always the case; to see this, consider the definition of the velocity vector at \( t = t_0 \)
\[
r(t_0) = \lim_{t \to t_0} \frac{r(t_0 + \Delta t) - r(t_0)}{\Delta t}.
\]
If $\Delta t > 0$, then the difference quotient clearly points in the direction of increasing $t$. If $\Delta t < 0$, then negative denominator reverses the direction and so the difference quotient is also a vector in the direction of increasing $t$.

5. Evaluate the following integral:

$$\int_0^1 \left( 2i + \frac{4}{1 + t^2}j + \frac{2t}{1 + t^2}k \right).$$

Solution:

$$\left(2t i + 4 \arctan t j + \ln (1 + t^2) k\right) \bigg|_0^1 = \langle 2, \pi, \ln 2 \rangle.$$

6. Determine $r(t)$ if $r'(t) = \langle t, e^t, te^t \rangle$ and $r(0) = \langle 1, 1, 1 \rangle$.

Solution:

$$r(t) = \int \langle t, e^t, te^t \rangle \, dt = \left\langle \frac{1}{2} t^2, e^t, te^t - e^t \right\rangle + C$$

Since $r(0) = \langle 0, 1, -1 \rangle + C = \langle 1, 1, 1 \rangle$, it follows that $C = \langle 1, 0, 2 \rangle$. Therefore,

$$r(t) = \left\langle \frac{1}{2} t^2 + 1, e^t, te^t - e^t + 2 \right\rangle.$$