1 Vectors and the Scalar Product

1.1 Vector Algebra

• **vector** in \( \mathbb{R}^n \): an \( n \)-tuple of real numbers

\[
\mathbf{v} = \langle a_1, a_2, \ldots, a_n \rangle.
\]

For example, if \( n = 2 \) and \( a_1 = 1 \) and \( a_2 = -1 \), then \( \mathbf{w} = \langle 1, -1 \rangle \) is vector in \( \mathbb{R}^2 \). Vectors are represented by directed line segments. Every vector \( \mathbf{v} \) has a length \( \|\mathbf{v}\| \):

\[
\|\mathbf{v}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.
\]

• **vector addition** in \( \mathbb{R}^n \): if \( \mathbf{v} \) and \( \mathbf{w} \) are vectors in \( \mathbb{R}^n \) then the sum \( \mathbf{v} + \mathbf{w} \) is the vector whose components are the sum of the corresponding components of \( \mathbf{v} \) and \( \mathbf{w} \). For example, if \( n = 3 \) and \( \mathbf{x} = \langle 1, 2, 3 \rangle \) and \( \mathbf{y} = \langle -1, 0, 1 \rangle \), then

\[
\mathbf{x} + \mathbf{y} = \langle 0, 2, 4 \rangle \quad \text{and} \quad \|\mathbf{x} + \mathbf{y}\| = \sqrt{12} = 2\sqrt{3}.
\]

• **scalar multiplication** in \( \mathbb{R}^n \): if \( \mathbf{v} = \langle a_1, a_2, \ldots, a_n \rangle \) is a vector in \( \mathbb{R}^n \) and \( c \) is a real number (called a scalar), then \( c\mathbf{v} \) is the vector whose components are correspond to those of \( \mathbf{v} \) scaled by a factor of \( c \):

\[
c\mathbf{v} = \langle ca_1, ca_2, \ldots, ca_n \rangle.
\]

• **scalar product** of two vectors in \( \mathbb{R}^n \): if \( \mathbf{x} = \langle a_1, a_2, \ldots, a_n \rangle \) and \( \mathbf{y} = \langle b_1, b_2, \ldots, b_n \rangle \), then the scalar product (also called the dot product) of \( \mathbf{x} \) and \( \mathbf{y} \) is denoted by \( \mathbf{x} \cdot \mathbf{y} \) and is equal to

\[
\mathbf{x} \cdot \mathbf{y} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^{n} a_ib_i.
\]

Exercises: Let \( \mathbf{x} = \langle 1, 1, -1 \rangle \), \( \mathbf{y} = \langle 2, -3, 1 \rangle \) and \( \mathbf{z} = \langle 1, 0, -1 \rangle \).

1. \( \mathbf{x} + 2\mathbf{y} - \mathbf{z} = \)
   
   Answer: \( \langle 4, -5, 2 \rangle \)

2. \( \mathbf{x} \cdot \mathbf{y} = \)
   
   Answer: \(-2\)

3. \( (\mathbf{x} \cdot \mathbf{y})\mathbf{z} = \)
   
   Answer: \( \langle -2, 0, 2 \rangle \)

4. Determine scalars \( a \), \( b \), and \( c \) so that

\[
\langle 1, 0, 0 \rangle = a\mathbf{x} + b\mathbf{y} + c\mathbf{z}.
\]

Answer: \( a = 1, b = 1/3, c = -2/3 \)
1.2 Vector Geometry

- **directed line segment** in \( \mathbb{R}^n \): a line segment with an arrow at one end pointing away from the other end. The end with the arrow is called the head, the end without the arrow is called the tail. If the point \( A \) is the head and the point \( B \) is the tail, then the directed line segment is denoted by \( \overrightarrow{AB} \).

- **from vectors to directed line segments**: every vector \( v \) in \( \mathbb{R}^n \) can be represented by a directed line segment in \( \mathbb{R}^n \). For example, if \( v = (3, 4) \) is a vector in \( \mathbb{R}^2 \), let \( O(0, 0) \) be the origin and let \( P(3, 4) \). We say that \( \overrightarrow{OP} \) represents \( v \). When \( v \) is represented by a directed line segment with its tail at the origin, the vector is in standard position and is called the position vector of the point \( P \).

- **from directed line segments to vectors**: If \( A(a_1, a_2, \ldots, a_n) \) and \( B(b_1, b_2, \ldots, b_n) \) are points in \( \mathbb{R}^n \) and \( \overrightarrow{AB} \) is the directed line segment from \( A \) to \( B \), then to this is associated the vector
  \[
  v = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n).
  \]
  For example, if \( A(2, 1) \) and \( B(5, 5) \), then the directed line segment \( \overrightarrow{AB} \) represents the vector \( v = (3, 4) \). The convention is to regard \( \overrightarrow{AB} \) as being equal the vector \( v \), although formally it is but one of many representatives of \( v \).

- **correspondence between vectors and directed line segments**: each vector is associated to infinitely many directed line segments, namely if \( \overrightarrow{OP} \) is the directed line segment which represents \( v \) in standard position, then another directed line segment representing \( v \) can be obtained by translating \( \overrightarrow{OP} \) without changing its length or direction. On the other hand, each directed line segment is associated to one and only one vector, namely the one given by the formula above: subtract the coordinates of the tail from the coordinates of head and use this \( n \)-tuple of real numbers to define the vector.

**Exercises**:

1. Let \( O(0, 0) \), \( A(2, 0) \), and \( B(3, 4) \). Write the vectors associated to \( \overrightarrow{OA}, \overrightarrow{OB}, \) and \( \overrightarrow{AB} \) in component form.
   Answer: \( \overrightarrow{OA} = (2, 0), \overrightarrow{OB} = (3, 4), \overrightarrow{AB} = (1, 4) \)

2. Let \( v = (2, 2) \). If \( \overrightarrow{AB} \) represents \( v \) and \( B(-1, 3) \), what is \( A \)?
   Answer: \( A(-3, 1) \)

3. Verify that if \( x \) is represented by \( \overrightarrow{PQ} \) and if \( y \) is represented by \( \overrightarrow{QR} \), then \( x + y \) is represented by \( \overrightarrow{PR} \). (Hint: Try an example first: let \( P = (1, 0) \), \( Q = (2, 1) \), and \( R = (-1, 3) \).)
   Answer: If \( P(p_1, \ldots, p_n) \), \( Q(q_1, \ldots, q_n) \), and \( R(r_1, \ldots, r_n) \), then
   \[
   \overrightarrow{PQ} + \overrightarrow{QR} = (q_1 - p_1, \ldots, q_n - p_n) + (r_1 - q_1, \ldots, r_n - q_n) = (r_1 - p_1, \ldots, r_n - p_n) = \overrightarrow{PR}.
   \]
1.3 More vector geometry

Theorem (The law of cosines): Suppose that $A$, $B$ and $C$ are the vertices of a triangle. Denote the length of the sides opposite $A$, $B$, and $C$ by $a$, $b$, and $c$, respectively. Let $\theta$ denote the measure of the angle at $C$. Then

$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$ 

Theorem (Physicists’ definition of the scalar product): Let $x$ and $y$ be vectors in $\mathbb{R}^n$. Suppose that neither vector is non-zero. Let $\theta$ denote the measure of the angle between any two line segments representing $x$ and $y$ when both tails coincide. Then

$$x \cdot y = \|x\|\|y\| \cos \theta.$$ 

The two theorems above are essentially equivalent. Please refer to the textbook for a detailed discussion of how the second theorem is a consequence of the first.

The second theorem shows that the scalar product determines the angle between two vectors. This is because the scalar product also determines the length of a vector:

$$\|x\| = \sqrt{x \cdot x}.$$

- **unit vector**: if $x$ is a non-zero vector, then the unit vector in the direction of $x$ is $\frac{x}{\|x\|}$.
- **orthogonality**: if $x$ and $y$ are vectors in $\mathbb{R}^n$ and $x \cdot y = 0$ then the vectors are orthogonal. If the vectors are non-zero, then orthogonality is the same as the angle between the vectors having measure zero.
- **scalar projection**: if $x$ and $y$ are vectors in $\mathbb{R}^n$, then the scalar projection of $x$ onto $y$ is equal to

$$\|x\| \cos \theta = \frac{x \cdot y}{\|y\|}.$$ 

- **vector projection**: if $x$ and $y$ are vectors in $\mathbb{R}^n$, then the vector projection of $x$ onto $y$ is the scalar product of the scalar projection and the unit vector in the direction of $y$:

$$\left(\|x\| \cos \theta\right) \frac{y}{\|y\|} = \frac{x \cdot y}{\|y\|^2} y.$$ 

Exercises

1. Determine the unit vector in the direction of the vector represented by $\overrightarrow{AB}$ where $A = (1, 0, 2)$ and $B = (0, -1, -3)$.

Answer: $\frac{1}{3\sqrt{3}}(-1, -1, -5)$

2. Determine the two unit vectors that are orthogonal to $v = (1, 2)$. Which two unit vectors are parallel to $v$?

Answer: $\frac{1}{\sqrt{5}}(2, -1)$ and its negative.

3. Let $x$ and $y$ be non-zero vectors in $\mathbb{R}^n$. Let $z$ be the vector projection of $x$ onto $y$. Prove that $x - z$ is orthogonal to $y$. (Hint: draw a sketch which illustrates that this claim seems reasonable.)

Answer: Expand and simplify to conclude that

$$(x - z) \cdot y = (x - \frac{x \cdot y}{\|y\|^2} y) \cdot y = 0.$$
1.4 Applications of Vectors

**Tension:** Suppose that a 100 kg mass is suspended from a horizontal surface by two cables. The angles between the cables and the surface are 30° and 45°, respectively. Determine the magnitude of the tension in each wire.

Answer: The tension is about 90 N in the 45° wire and about 73 N in the other.

**Work:** A wagon is pulled by exerting a force of 45 N on a handle attached to a rod that is 30° above horizontal. Determine the amount of work done in pulling the wagon 100 m.

Answer: $2250\sqrt{3} \text{N} \cdot \text{m} \approx 3897 \text{J}$

**Bond angle (exercise #87, p. 791):** The molecule CH$_4$ has the shape of a regular tetrahedron. Determine the bond angles at the carbon atom. Here are some steps to follow:

1. Let $(0,0,0)$, $(k,k,0)$, $(k,0,k)$, and $(0,k,k)$ denote the locations of the hydrogen atoms, where $k$ is a positive real number. Sketch a diagram of this configuration. Where is the carbon atom?

   Answer: The carbon atom is at $(k/2,k/2,k/2)$.

2. Verify that the four points are in fact the vertices of a regular tetrahedron. (Hint: what are the lengths of the edges?)

   Answer: The length of each edge of the tetrahedron is $k\sqrt{2}$.

3. Determine the lengths of a directed line segment from the carbon atom to one of the hydrogen atoms.

   Answer: $\frac{\sqrt{3}}{2}k$

4. Determine the measure of the angle at the carbon atom between two directed line segments which are directed towards to different hydrogen atoms.

   Answer: $\arccos (1/3) \approx 109.5°$
2 Determinants and the Vector Product

2.1 Determinants

- **determinant of a $2 \times 2$ matrix**: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a matrix whose first row is $[a, b]$ and whose second row is $[c, d]$, then the determinant of $A$ is $ad - bc$. The determinant denote by $\det A$ or by $|A|$: 

  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

- **determinant of a $3 \times 3$ matrix**: if $A = \begin{bmatrix} a, b, c \\ d, e, f \\ g, h, i \end{bmatrix}$, then its determinant is 

  $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ef - fh) - b(di - fg) + c(dh - eg).$

- **determinants and linear combinations**: if $x = \langle a, b \rangle$ and $y = \langle c, d \rangle$, then $x$ is a scalar multiple of $y$ if and only if the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to zero. Similarly, if $u = \langle a, b, c \rangle$, $v = \langle d, e, f \rangle$, and $w = \langle g, h, i \rangle$, then $u$ is a linear combination of $v$ and $w$ if and only if the determinant of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is equal to zero.

- **determinants, area, and volume**: if the determinant of a $2 \times 2$ matrix is non-zero, then the determinant, in absolute value, is equal to the area of the parallelogram $\square OPQR$ where $O$ is the origin, $\overrightarrow{OP}$ represents $x = \langle a, b \rangle$, $\overrightarrow{OR}$ represents $y = \langle c, d \rangle$, and $\overrightarrow{OQ}$ represents $x + y$. A similar statement holds for the parallelepiped spanned by the row vectors of a $3 \times 3$ matrix.

**Exercises:**

1. Compute the determinant of the following three matrices:

   $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 2 \\ 4 & 5 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

   Answers: $\det A = 4$, $\det B = 0$, $\det C = 24$

2. Show that the quadrilateral $\square ABCD$, where $A(1, 2)$, $B(3, 4)$, $C(5, 9)$, and $D(3, 7)$, is a parallelogram. Then compute its area.

   Answer: $\overrightarrow{AB}$ and $\overrightarrow{AD}$ represent the same vector; $\overrightarrow{BC}$ and $\overrightarrow{DC}$ represent the same vector; the area equals 6.

3. Suppose $A$, $B$, and $C$ are points in $\mathbb{R}^2$. How can a determinant be used to test whether or not the points are collinear? (Hint: Think about $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and make a sketch of the problem.) Test your idea on the following two collections of points:

   (a) $A(1, 0)$, $B(2, 1)$, $C(3, 2)$ (collinear)
   (b) $A(1, 0)$, $B(2, 1)$, $C(3, 3)$ (not collinear)

   Answer: let $\overrightarrow{AB}$ be the first row of a matrix, $\overrightarrow{AC}$ the second row; if the determinant is zero, the points are collinear, otherwise non-collinear.
2.2 The Vector Product

- **vector product** of two vectors in $\mathbb{R}^3$: if $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, then the vector product $\mathbf{v} \times \mathbf{w}$ (also called the cross product) is equal to

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$  

where $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. In other words,

$$\mathbf{v} \times \mathbf{w} = ((v_2w_3 - w_2v_3), (v_3w_1 - w_3v_1), (v_1w_2 - w_1v_2)).$$

- **vector product of common vectors**:

  $$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

- **anti-commutative property**:

  $$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v};$$

  more generally, if two rows or columns of a matrix are swapped, then the determinant is changes by a factor of $(-1)$.

- **non-associative**: $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0$ but $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = -\mathbf{j}$.

Exercises:

1. Compute the vector product of $\mathbf{a} = (1, -1, 2)$ and $\mathbf{b} = (3, 2, 1)$.

   Answer: $-5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$

2. If $c \in \mathbb{R}$ is a scalar, is $(c\mathbf{a}) \times \mathbf{b}$ equal to $c(\mathbf{a} \times \mathbf{b})$?

   Answer: Yes.

3. Suppose that $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ and that $\mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}$. Using only the table of values for $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, etc., and the fact that $\mathbf{i} \times \mathbf{i} = \mathbf{0}$, etc. show that the formula for $\mathbf{v} \times \mathbf{w}$ is a consequence of the following distributive law for the vector product:

   $$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$  

   Answer: $\mathbf{v} \times \mathbf{w} = (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \times (w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}) = v_1 \mathbf{i} \times (w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}) + (\text{analogous terms for } v_2 \mathbf{j} \text{ and } v_3 \mathbf{k})$. The expansion of the one term is $v_1w_1 \mathbf{i} \times \mathbf{i} + v_1w_2 \mathbf{i} \times \mathbf{j} + v_1w_3 \mathbf{i} \times \mathbf{k}$. This term equals $v_1w_2 \mathbf{k} - v_1w_3 \mathbf{j}$ since $\mathbf{i} \times \mathbf{i} = \mathbf{0}$. The complete calculation is analogous.
2.3 Geometry and Applications of the Vector Product

**Theorem** (Physicists’ definition of the vector product): Let \( \mathbf{x} \) and \( \mathbf{y} \) be vectors in \( \mathbb{R}^3 \), and let \( \theta \) be the measure of the angle between these two vectors. The following three properties of the vector product hold:

1. \( \mathbf{x} \times \mathbf{y} \) is orthogonal to both \( \mathbf{x} \) and \( \mathbf{y} \),
2. \( \| \mathbf{x} \times \mathbf{y} \| = \| \mathbf{x} \| \| \mathbf{y} \| \sin \theta \),
3. the ordered triple \( [\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}] \) is a right-handed coordinate system.

In other words, the direction of \( \mathbf{x} \times \mathbf{y} \) is determined by the so-called right-hand rule. The magnitude of \( \mathbf{x} \times \mathbf{y} \) is equal to the magnitude of \( \mathbf{x} \) times the magnitude of \( \mathbf{y} \) times the sine of the angle between them.

- **torque**: suppose \( P \) and \( Q \) are two points in a system and that \( P \) is a point of rotation. Let \( \mathbf{r} \) be the position vector of \( Q \) relative to \( P \), i.e. \( \mathbf{r} \) is represented by \( \overrightarrow{PQ} \). If a force \( \mathbf{F} \) is applied at the point \( Q \), then the torque (with respect to \( P \)) is \( \mathbf{\tau} = \mathbf{r} \times \mathbf{F} \).

- **maximal torque**: maximum torque is achieved when a force is applied in a direction orthogonal to the displacement from the axis of rotation.

- **direction of torque**: the direction of the torque (up to sign) is aligned with the axis of rotation.

**Exercises:**

1. Suppose \( \mathbf{x} = \mathbf{i} + \mathbf{j} \) and \( \mathbf{y} = \mathbf{j} \) are two vectors in \( \mathbb{R}^3 \). Sketch these two vectors and use the physicists’ definition to compute the vector product \( \mathbf{x} \times \mathbf{y} \). Then check your answer by using a determinant to compute the vector product.
   
   Answer: \( \| \mathbf{x} \| = \sqrt{2}, \| \mathbf{y} \| = 1, \) and \( \sin \theta = 1/\sqrt{2}. \) So, \( \mathbf{x} \times \mathbf{y} = \mathbf{k}. \)

2. Suppose that the forward pedal of a bicycle is 30° above horizontal and that a force of 50 N is applied in a downward direction. Determine the magnitude of the torque about the center of the pedals assuming that the distance between the two pedals is 36 cm. What is the direction of the torque?
   
   Answer: \( \| \mathbf{\tau} \| = 4.5 \text{ N} \cdot \text{m}, \) directed towards the left side of the bicycle.

3. Let \( P(1, 1, 1), Q(1, 2, 3), \) and \( R(3, 2, 1) \) be points in \( \mathbb{R}^3 \). The following formula from trigonometry (easily derived using the definition of the sine) computes the area of triangle \( \triangle PQR \):

   \[
   \text{area}(\triangle PQR) = \frac{1}{2} \| \overrightarrow{PQ} \| \| \overrightarrow{PR} \| \sin \theta, \]

   where \( \theta \) is the measure of the angle at \( P \). Explain how this combined with the physicists’ definition of the vector product can be used to compute the area of the triangle. Then compute the area of the triangle.

   Answer: \( \overrightarrow{PQ} \times \overrightarrow{PR} = (-2, 4, -2); \) the area of the triangle equals \( \frac{1}{2} \| \overrightarrow{PQ} \times \overrightarrow{PR} \| = \sqrt{6}. \)
2.4 Geometry and Determinants

• **triple scalar product**: the product \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) (parentheses required!) is called the triple scalar product of \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \); in terms of the components of these vectors it is equal to a determinant:

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.
\]

The absolute value of this determinant represents the volume of the parallelepiped spanned by these three vectors.

• **algebraic properties of the determinant from its geometric properties**: if two rows of a \( 3 \times 3 \) matrix are interchanged then the absolute value of the determinant does not change. What happens if two columns are interchanged? (Hint: Picture the effect of interchanging two of the coordinate axes.)

• **matrices as transformations**: if \( \mathbf{A} \) is a \( 3 \times 3 \) matrix whose rows are \( \mathbf{A}_1 = \langle a_{11}, a_{12}, a_{13} \rangle \), \( \mathbf{A}_2 = \langle a_{21}, a_{22}, a_{23} \rangle \), and \( \mathbf{A}_3 = \langle a_{31}, a_{32}, a_{33} \rangle \) and if \( \mathbf{v} \) is a vector in \( \mathbb{R}^3 \), then the vector-valued function

\[
\mathbf{f}(\mathbf{v}) = \langle \mathbf{A}_1 \cdot \mathbf{v}, \mathbf{A}_2 \cdot \mathbf{v}, \mathbf{A}_3 \cdot \mathbf{v} \rangle
\]

takes vectors in \( \mathbb{R}^3 \) as input and returns vectors in \( \mathbb{R}^3 \) as output—in other words, \( \mathbf{f} \) is a transformation of \( \mathbb{R}^3 \). The absolute value of the determinant of \( \mathbf{A} \) measures the distortion of volume under the transformation; the sign of the determinant detects whether a right-handed system is transformed to a right-handed system or to a left-handed system. The result of applying \( \mathbf{f} \) to \( \mathbf{v} \) is denoted by \( \mathbf{A}\mathbf{v} \).

Exercises:

1. Sketch the parallelepiped having edges \( \overrightarrow{PQ}, \overrightarrow{PR}, \) and \( \overrightarrow{PS} \), where \( P(1,0,0), Q(1,1,0), R(0,1,0), \) and \( S(0,1,1) \). Then compute its volume.

   Answer: volume equals 1

2. How can a determinant be used to determine whether or not four points \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \) and \( \mathbf{D} \) in \( \mathbb{R}^3 \) are coplanar? (Hint: Refer back to how a \( 2 \times 2 \) determinant could be used to test whether three points in \( \mathbb{R}^2 \) are collinear.) Test your idea on the following two collections of points:

   (a) \( \mathbf{A}(1,0,0), \mathbf{B}(2,1,1), \mathbf{C}(3,2,1), \mathbf{D}(4,3,2) \) (coplanar)

   (b) \( \mathbf{A}(1,0,0), (2,1,1), \mathbf{C}(3,2,1) \) \( \mathbf{D}(4,2,3) \) (not coplanar)

   Answer: compute the determinant of the matrix having rows \( \overrightarrow{\mathbf{AB}}, \overrightarrow{\mathbf{AC}}, \) and \( \overrightarrow{\mathbf{AD}} \); if the determinant is zero, they are coplanar; if non-zero, then non-coplanar.

3. Let \( \mathbf{A} \) be the matrix below. Sketch the vectors \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \) and then sketch the vectors \( \mathbf{A}\mathbf{i}, \mathbf{A}\mathbf{j}, \) and \( \mathbf{A}\mathbf{k} \). Compute the determinant of \( \mathbf{A} \) and explain why the determinant measures the distortion of volume.

\[
\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{so the row vectors are} \quad \mathbf{A}_1 = \langle 2, 1, -1 \rangle, \quad \mathbf{A}_2 = \langle 0, 1, 1 \rangle, \quad \mathbf{A}_3 = \langle 0, 0, 1 \rangle.
\]

Answer: The parallelepiped spaned by the unit coordinate vectors is a cube of volume 1; the parallelepiped spaned by the images of these vectors under the transformation \( \mathbf{A} \) has volume equal to the determinant of the matrix having rows \( \mathbf{A}\mathbf{i}, \mathbf{A}\mathbf{j}, \) and \( \mathbf{A}\mathbf{k} \); the determinant of this matrix is the same as the determinant of \( \mathbf{A} \). (A matrix and its transpose have equal determinant.)
3 Lines and Planes in Space

3.1 Lines in Space

- **vector equation of a line**: Let \( \mathbf{r}_0 \) be the position vector of a point \( P_0 \), i.e. \( \mathbf{r}_0 \) is represented by \( \overrightarrow{OP}_0 \) where \( O \) is the origin. Let \( \mathbf{v} \) be a given non-zero vector. Then a vector equation of the line through \( P_0 \) and parallel to \( \mathbf{v} \) is given by

\[
\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.
\]

This means that the position vector \( \mathbf{r} \) of a point \( P \) on the line is equal to \( \mathbf{r}_0 + t\mathbf{v} \) for some (unique) real number \( t \). In particular, if \( t = 0 \), then we see that the point \( P_0 \) having position vector \( \mathbf{r}_0 \) lies on the line.

Here is an example. Suppose that the point \( P_0 = (1, 0, -1) \) lies on a line in \( \mathbb{R}^3 \) and that the vector \( \mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \) is parallel to this line. Then a vector equation for this line is given by

\[
\mathbf{r} = \mathbf{i} - \mathbf{k} + (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})t
\]

Simplifying and writing \( \mathbf{r} = \langle x, y, z \rangle \), we see that

\[
x = 1 + 3t, \quad y = 0 + 4t, \quad z = -1 + 5t,
\]

which are the more familiar parametric equations of a line.

- **from parametric equations to a vector equation**: Suppose \( x = 1 + t, \quad y = 2 - t, \quad \text{and} \quad z = 3 + 5t \), where \( -\infty < t < \infty \). Then by varying the parameter \( t \) the corresponding points \( (x, y, z) \) trace a line in \( \mathbb{R}^3 \). A vector equation is recovered by selecting any point on the line and then using the direction numbers (the coefficients of \( t \)) to form a vector. Let \( \mathbf{v} = \langle 1, -1, 5 \rangle \). Choose a point, for instance, by setting \( t = 2 \): \( x = 3, \quad y = 0, \quad z = 13 \). Let \( P_0 = (3, 0, 13) \) and write its position vector \( \mathbf{r}_0 = \langle 3, 0, 13 \rangle \).

Finally, write the vector equation:

\[
\mathbf{r} = \langle 3, 0, 13 \rangle + t\langle 1, -1, 5 \rangle.
\]

The above can be written as a single vector: \( \mathbf{r} = \langle 3 + t, -t, 13 + 5t \rangle \). The components are just another set of parametric equations. (Parametric equations are not unique!)

**Exercises:**

1. Determine a vector equation and set of parametric equations for the line passing through the points \( A(1,2,3) \) and \( B(1,0,-1) \).
   Answer: \( \mathbf{r} = \langle 1,2,3 \rangle + t\langle 0,-2,-4 \rangle; \quad x = 1, \quad y = 2 - 2t, \quad z = 3 - 4t \).

2. Determine a vector equation and a set of parametric equations for the line passing through \( P(1,-1,1) \) and which is parallel to the line having parametric equations \( x = 1 + t, \quad y = 2 + 3t, \quad z = -1 + 4t \). Does this set of parametric equations describe the same line?
   Answer: \( \mathbf{r} = \langle 1,-1,1 \rangle + t\langle 1,3,4 \rangle, \quad x = 1 + t, \quad y = -1 + 3t, \quad z = 1 + 4t \)

3. Determine a vector equation and a set of parametric equations for the line which is the intersection of the planes \( x + y + z = 1 \) and \( x + 2y + 3z = 6 \). (Hint: Let \( x = t \), then eliminate \( z \) to find an equation for \( y \) in terms of \( t \), then do the same to find \( z \) in terms of \( t \).)
   Answer: \( x = t, \quad y = -3 - 2t, \quad z = 4 + t, \quad \mathbf{r} = \langle 0,-3,4 \rangle + t\langle 1,-2,1 \rangle \)
3.2 Planes in Space

- **vector equation of a plane in space**: Let \( \mathbf{r}_0 \) be the position vector of a point \( P_0 \) and let \( \mathbf{n} \) be a non-zero vector. A vector equation of the plane passing through \( P_0 \) and normal to \( \mathbf{n} \) is

\[
(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0.
\]

Here is an example. Let \( P_0 = (1, 1, 1) \) and let \( \mathbf{n} = \langle 2, 3, 4 \rangle \). Then the position vector \( \mathbf{r} \) of a point in the plane through \( P_0 \) and normal to \( \mathbf{n} \) satisfies an equation of the above form. Let \( \mathbf{r} = \langle x, y, z \rangle \), so that \( \mathbf{r} - \mathbf{r}_0 = \langle x - 1, y - 1, z - 1 \rangle \). The equation above expands to the following:

\[
2(x - 1) + 3(y - 1) + 4(z - 1) = 0,
\]

which is an equation for the plane in rectangular coordinates.

- **finding a normal vector to a plane**: if \( a, b, \) and \( c \) are not all zero, then \( \mathbf{n} = \langle a, b, c \rangle \) is a normal vector to any plane having equation \( ax + by + cz = d \). Example: a normal vector to the plane given by \( x + 2y + 3z = 4 \) is \( \langle 1, 2, 3 \rangle \).

- **parallel planes**: Two planes are parallel if and only if they have parallel normal vectors.

Exercises:

1. Determine a vector equation and an equation in rectangular coordinates for the plane passing through the points \( A(1, 0, 0), B(0, 2, 0) \) and \( C(0, 0, 3) \). Hint: Use the vector product to determine a normal vector.
   
   Answer: \( (\mathbf{r} - \langle 1, 0, 0 \rangle) \cdot \langle 6, 3, 2 \rangle = 0, \) \( 6(x - 1) + 3y + 2z = 0 \)

2. Determine an equation for the plane which is orthogonal to the line \( x = 1 + t, y = 2 + t, z = 3 - t \) and which passes through the point \( (1, 2, 3) \).
   
   Answer: \( (x - 1) + (y - 2) - (z - 3) = 0 \)
3.3 Geometry Problems Involving Lines and Planes

- **distance from a point to a line**: The distance from a point \(Q\) to the line through \(P\) and parallel to \(v\) can be computed using the vector product: the distance \(d\) from \(Q\) to the line is equal to \(\|\overrightarrow{PQ}\| \sin \theta\), where \(\theta\) is the angle between \(\overrightarrow{PQ}\) and \(v\). Draw a sketch and then work out a formula for \(d\) which involves the vector product instead of the sine of \(\theta\).

  Answer: \(d = \|\mathbf{v} \times \overrightarrow{PQ}\|/\|\mathbf{v}\|\)

- **distance from a point to a plane**: The distance from a point \(Q\) to the plane through \(P\) and normal to \(n\) can be computed using the scalar product: the distance \(d\) from \(Q\) to the plane is equal to \(\|\overrightarrow{PQ}\| |\cos \theta|\). (Absolute value of cosine of \(\theta\) is needed since the normal vector may not point to the same halfspace as the one containing \(P\). Draw a sketch and then work out a formula for \(d\) which involving the vector product instead of the cosine of \(\theta\).

  Answer: \(d = \|\mathbf{n} \cdot \overrightarrow{PQ}\|/\|\mathbf{n}\|\)

Exercises:

1. Determine the distance from the point \(P(1, 2, 3)\) to the line given by \(x = 1 + t\), \(y = -1 + 2t\), and \(z = 5t\). How can you test if this line passes through \(P\) without first computing the distance?

   Answers: \(d = \sqrt{99}/\sqrt{29}\), if \(1 = x = 1 + t\), then \(t = 0\), which implies that \(-1 + 2(0) = -1\), so \((1, 2, 3)\) does not satisfy the parametric equations.

2. Determine the distance from the point \(P(1, 2, 3)\) to the plane \(x - y + 5z = 8\).

   Answer: \(6/\sqrt{27}\)
4 Surfaces in $\mathbb{R}^3$

4.1 Cylinders

- **non-singular surface** in $\mathbb{R}^3$: a collection of points in $\mathbb{R}^3$ which locally are like $\mathbb{R}^2$; this means that $S \subset \mathbb{R}^3$ is a surface if for each point $P \in S$, a ball of sufficiently small radius and centered at $P$ intersects $S$ in a set which looks like a disk; this is a correct definition, but one which is difficult to apply. A double cone is an example of a singular surface: at the point where the two cones meet, locally the surface is like two disks glued at their centers.

- **cylinder in $\mathbb{R}^3$**: let $C$ be a planar curve, i.e. $C$ is contained in some plane; let $L$ be a line not in the plane of $C$; the cylinder generated by $C$ and $L$ is the collection of points which belong to some line parallel to $L$ and which passes through $C$. A cylinder is a non-singular surface. For example, let $C$ be the parabola $y = x^2$ in the $xy$-plane. Let $L$ be the $z$-axis. Then the cylinder generated by this pair is the parabolic cylinder $y = x^2$. IMPORTANT: When is it understood that an equation defines a surface in $\mathbb{R}^3$, then all coordinate variable $x$, $y$, and $z$ are allowed to vary so long as they satisfy the equation. In this example, $z$ is a free variable. As a set, the parabolic cylinder consists of all points $(x, y, z)$ such that $y = x^2$ (and $z$ is free to have any value).

- **recognizing a cylinder**: an equation in $x$, $y$, and $z$ for which one of these variables is missing represents a cylinder; for example $y^2 + z^2 = 1$ represents a circular cylinder with the $x$-axis as its radial axis.

Exercises:

1. Sketch the cylinder $y = x^2$.
   
   Answer: parabolic cylinder (use Wolfram Alpha)

2. Sketch the cylinder $y^2 + z^2 = 1$.
   
   Answer: cylinder with $x$-axis as its central axis

3. Sketch the cylinder generated by the curve $x^2 + y^2 = 1$ and the line $x = 0$, $y = t$, $z = t$.
   
   Answer: “slanted” cylinder: the lines of the cylinder meet the $xy$-plane at a $45^\circ$ angle while the horizontal cross sections remain circles
4.2 Traces, Conic Sections, and Quadric Surfaces

- **trace**: if $S$ is a surface in $\mathbb{R}^3$, then its trace with the $xy$-plane is the intersection of $S$ and the $xy$-plane. For example, the cylinder $x^2 + y^2 = 1$ has $xy$-trace equal to the circle $x^2 + y^2 = 1$. The $xz$-trace is a pair of lines: the lines $x = 1$ and $x = -1$ in the $xz$-plane. What is the trace in the $yz$-plane?

- **traces in family of planes**: the equation $z = x^2 + y^2$ describes a surface in $\mathbb{R}^3$. The trace in the plane $z = 1$ is the unit circle $x^2 + y^2 = 1$. The trace in the plane $z = c$, where $c > 0$ is $x^2 + y^2 = c$, a circle of radius $\sqrt{c}$. The trace in the $xy$-plane is the point $(0, 0)$. The trace in the plane $z = -1$ is the empty set.

- **conic sections**: these are the curves which are the intersections of a plane with a double cone; the generic conic sections are represented by the following curves in the $xy$-plane: an ellipse (equation $Ax^2 + By^2 = C$, where $A, B, C > 0$), a hyperbola (equation $Ax^2 - By^2 = C$, where $A, B, C > 0$), and a parabola (equation $Ax^2 + By = C$, where $A, B \neq 0$).

- **traces and conic sections**: the surface given by $x^2 - y^2 + z^2 = 1$ has the following traces: in the $xy$-plane, the trace is the hyperbola $x^2 - y^2 = 1$, in the $yz$-plane the trace is the hyperbola $z^2 - y^2 = 1$, in the $xz$-plane the trace is the circle $x^2 + z^2 = 1$. This is an example of a hyperboloid of one sheet. It is the surface of revolution generated by rotating the curve $z = \sqrt{y^2 + 1}$ in the $yz$-plane about the $y$-axis.

- **quadric surface**: a surface given by a degree two polynomial in $x$, $y$, and $z$ is called a quadric surface; the generic quadric surfaces are the following:

1. paraboloids such as $z = x^2 + y^2$
2. ellipsoids such as $x^2 + 2y^2 + 3z^2 = 1$
3. hyperbolic paraboloids such as $z = x^2 - y^2$
4. hyperboloids of one sheet such as $x^2 + y^2 - z^2 = 1$
5. hyperboloids of two sheets such as $z^2 - x^2 - y^2 = 1$
6. elliptic cones such as $z^2 = 2x^2 + 3y^2$

**Exercises**:

1. Sketch the traces of the surface $z^2 = x^2 + y^2$ in the coordinate planes and in the planes of the form $z = c$, where $c \in \mathbb{R}$. If the trace is a conic section, identify its type.

   Answers: Traces in the planes $z = c$ are circles of radius $|c|$ if $c \neq 0$; trace in the $xy$-plane ($z = 0$) is the point $(0, 0)$; trace in the $xz$-plane is a pair of lines ($z = \pm x$); trace in the $yz$-plane is also a pair of lines $z = \pm y$. The surface is a double cone.

2. Sketch the traces of the surface $z^2 = 1 + x^2 + y^2$ in the coordinate planes and in the planes of the form $z = c$, where $c \in \mathbb{R}$. If the trace is a conic section, identify its type.

   Answers: Traces in the planes $z = c$ are empty if $|c| < 1$ (so the trace in the $xy$-plane ($z = 0$) is empty), are a single point if $|c| = 1$, and are circles of radius $\sqrt{c^2 - 1}$ if $|c| > 1$; trace in the $xz$-plane is the hyperbola $z^2 - x^2 = 1$; trace in the $yz$-plane is the hyperbola $z^2 - y^2 = 1$. The surface is a two-sheeted hyperboloid.

3. Choose one or more of the quadratic surfaces on the list above, sketch the traces in the coordinate planes and in planes parallel to the coordinate planes. (This is more of a study project than an exercise.)

   Answers: check your answers using Wolfram Alpha
5 Cylindrical and Spherical Coordinates

5.1 From rectangular coordinates to cylindrical or spherical coordinates

- **polar coordinates**: if the point $P(x, y)$ is given in the $xy$-plane in terms of its rectangular coordinates $x$ and $y$, then $(r, \theta)$ are polar coordinates for $P$ if $x^2 + y^2 = r^2$ and the ray from the origin to the point $(\cos \theta, \sin \theta)$ on the unit circle contains $P$ if $r \geq 0$ or the ray in the opposite direction contains $P$ if $r \leq 0$.

  Polar coordinates are not unique: for example, the point $P(1, 1)$ has polar coordinates $(\sqrt{2}, \pi/4)$ (the ray has vector equation $r = t(1 + j)$, $t \geq 0$), $(-\sqrt{2}, -\pi/4)$ (the ray has vector equation $r = -t(1 + j)$, $t \geq 0$), and $(\sqrt{2}, 9\pi/4)$ (because $\theta = 2\pi + \pi/4$).

  Since the polar coordinate of a point $P$ uses the letter $r$ and the position vector represented by $\overrightarrow{OP}$ uses the letter $r$, it is important to use arrows over vectors (or use boldface type).

- **cylindrical coordinates**: if the point $P(x, y, z)$ is given in $\mathbb{R}^3$ in terms of its rectangular coordinates $x$, $y$, and $z$, then $(r, \theta, z)$ are cylindrical coordinates for $P$ if $(r, \theta)$ are polar coordinates for $(x, y)$.

  Equations of some surfaces are much simpler in terms of cylindrical coordinates. For example the cylinder $x^2 + y^2 = 1$ has equation $r = 1$. Some are not so simple; for example the cylinder $x^2 + z^2 = 1$ has equation $(r \cos \theta)^2 + z^2 = 1$.

- **spherical coordinates**: the point $P(x, y, z)$ in $\mathbb{R}$ given in rectangular coordinates has spherical coordinates $(\rho, \theta, \phi)$ if $\rho \geq 0$ is the distance from $P$ to the origin, $\phi$ is the angle between the vectors $\overrightarrow{OP}$ and $k$ and $\theta$ is the polar angle of $(x, y)$. Spherical coordinates also allow for $\rho < 0$, and in such a case, the angles need to be suitably interpreted as in the case of polar coordinates. Our convention will be to use $0 \leq \rho$, $0 \leq \phi \leq \pi$, and $0 \leq \theta < 2\pi$ or $-\pi < \theta \leq \pi$ as is convenient.

**Exercises**: Determine the cylindrical an spherical coordinates of each given point. Sketch the point and use trigonometry to determine the correct values. The answer are not unique.

1. $P(1, 1, 1)$
   Answers: cylindrical $(\sqrt{2}, \pi/4, 1)$, spherical $(\sqrt{3}, \pi/4, \arccos(1/\sqrt{3}))$

2. $Q(1, 1, 0)$
   Answers: cylindrical $(\sqrt{2}, \pi/4, 0)$, spherical $(\sqrt{2}, \pi/4, \pi/2)$

3. $R(1, 0, 1)$
   Answers: cylindrical $(1, 0, 1)$, spherical $(\sqrt{2}, 0, \pi/4)$

4. $S(0, 1, 1)$
   Answers: cylindrical $(1, \pi/2, 1)$, spherical $(\sqrt{2}, \pi/2, \pi/4)$
5.2 From cylindrical and spherical to rectangular

- **polar to rectangular**: the transformations \( x = r \cos \theta \) and \( y = r \sin \theta \) convert polar coordinates into rectangular coordinates.

- **cylindrical to rectangular**: using the above transformations also transforms \((r, \theta, z)\) to \((x, y, z)\)

- **spherical to rectangular**: the following transformations transform spherical coordinates \((\rho, \theta, \phi)\) into rectangular coordinates \((x, y, z)\):
  \[
  x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.
  \]

- **from spherical to cylindrical**: the spherical coordinates are easy to translate into cylindrical coordinates using the following equations:
  \[
  z = \rho \cos \phi \quad \text{and} \quad r = \rho \sin \phi.
  \]

  The above combined with the transformations from cylindrical to rectangular makes the equations for spherical to rectangular easy to apply– just be patient and use two steps.

**Exercises:** Determine the rectangular coordinates of the points given.

1. \(P\) has cylindrical coordinates \((2, \pi/3, -1)\)

2. \(Q\) has cylindrical coordinates \((5, 3\pi/4, 2)\)

3. \(R\) has spherical coordinates \((2, \pi/3, \pi/3)\)

4. \(S\) has spherical coordinates \((5, 3\pi/4, \pi/6)\)
6 Vector-valued Functions

- **vector-valued function**: \( \mathbf{r}(t) = (x(t), y(t), z(t)) \) is a function of a independent variable \( t \). The dependent variable is the vector \( \mathbf{r}(t) \), which is interpreted as the position vector of a point \( P \) (depending on \( t \)). So, the coordinates of \( P \) are \((x(t), y(t), z(t))\). Vector-valued functions are essentially the same as parametric equations.

- **general example**: The graph of the function \( y = f(x) \) is the trace of the vector-value function \( \mathbf{r}(t) = (t, f(t)) \). Note that \( \mathbf{r} \) is a function which maps real numbers to vectors in \( \mathbb{R}^2 \), not \( \mathbb{R}^3 \).

- **example in \( \mathbb{R}^3 \)**: The trace of the vector-valued function \( \mathbf{r} = (\cos t, \sin t, t) \) is a helix.

- **limits of vector-valued functions**: defined in the usual way: the limit of \( \mathbf{r}(t) \) is equal to \( \mathbf{L} \) as \( t \) approaches \( a \) if the values of \( \mathbf{r}(t) \) become arbitrarily close to \( \mathbf{L} \) provided \( t \) is sufficiently close to \( a \). The only change is that the limit is a vector. Closeness is measured by distance in \( \mathbb{R}^3 \).

- **continuity of vector-valued functions**: defined in the usual way: \( \mathbf{r}(t) \) is continuous at \( t = a \) if \( \mathbf{r}(a) \) is defined and is equal to the limit of \( \mathbf{r}(t) \) as \( t \) approaches \( a \). The trace of a discontinuous vector-valued function is either disconnected or has wildly oscillating behavior.

**Exercises**: Sketch the trace of each vector-valued function. Also, identify the domain of the function.

1. \( \mathbf{r}(t) = (\cos t, \sin t, \sqrt{t}) \)

2. \( \mathbf{r}(t) = (2 \cos t, 3 \sin t, 5) \)

3. \( \mathbf{r}(t) = (t - \sin t, 1 - \cos t) \)
7 Differentiation and Integration of Vector-valued Functions and Velocity & Acceleration

• Suppose the \( \mathbf{r}(t) \) is a vector-valued function. Let \( \Delta t \) denote a small change in the value of \( t \), for instance if \( t = 1 \) and \( \Delta t = 0.1 \), then \( t + \Delta t = 1.1 \) represents the result of a small increase in the value of \( t \). Likewise, if \( \Delta t = -0.1, t + \Delta t = 0.9 \) represents the result of a small decrease in \( t \). In general \( t + \Delta t \) represents a (small or not so small) change in \( t \).

• The vector \( \mathbf{r}(t + \Delta t) - \mathbf{r}(t) \) represents the change in the values of \( \mathbf{r}(t) \). If this is divided by \( \Delta t \), assuming \( \Delta t \neq 0 \), then this represents a relative change. The following limit is called the derivative or velocity of \( \mathbf{bfr}(t) \):

\[
\lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}
\]

The above is denoted by \( \mathbf{r}'(t) \) or \( \mathbf{v}(t) \).

• The derivative is computed by applying familiar differentiation rules to each component. For example, if

\[
\mathbf{r} = 3 \sin t \mathbf{i} + \cos (2t) \mathbf{j}
\]

then

\[
\mathbf{r}'(t) = 3 \cos t \mathbf{i} - 2 \sin (2t) \mathbf{j}.
\]

• The integral of \( \mathbf{r}(t) \) is computed by integrating each component. The constant of integration is a vector in the case of an indefinite integral. The value of an integral is a vector in the case of a definite integral.

• Example:

\[
\int (2 \mathbf{i} + 4t \mathbf{j} + 2e^{-2t} \mathbf{k}) \, dt = 2t \mathbf{i} + 2t^2 \mathbf{j} - e^{-2t} \mathbf{k} + \mathbf{C},
\]

where \( \mathbf{C} = C_1 \mathbf{i} + C_2 \mathbf{j} + C_3 \mathbf{k} \) is the (vector) constant of integration.

• Example:

\[
\int_0^1 (2 \mathbf{i} + 4t \mathbf{j} + 2e^{-2t} \mathbf{k}) \, dt = 2t \mathbf{i} + 2t^2 \mathbf{j} - e^{-2t} \mathbf{k} \bigg|_0^1 = 2 \mathbf{i} + 2 \mathbf{j} + (1 - e^{-2}) \mathbf{k}.
\]

• The velocity of \( \mathbf{r}(t) \) is the derivative: \( \mathbf{v}(t) = \mathbf{r}'(t) \).

• The acceleration of \( \mathbf{r}(t) \) is the second derivative, i.e. the derivative of the velocity: \( \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \).

Exercises:

1. Sketch the cycloid

\[
\mathbf{r}(t) = (t - \sin t, 1 - \cos t)
\]

and its velocity and acceleration vectors at \( t = 0, \pi/4, \pi/2, 3\pi/4, \) and \( \pi \).

2. Solve the differential equation

\[
\frac{d^2 \mathbf{r}}{dt^2} = -g \mathbf{j}
\]

for the position \( \mathbf{r}(t) \) of a projectile in 2-dimensions, where the initial velocity is

\[
\frac{d\mathbf{r}}{dt}(0) = v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j}
\]

and the initial position is \( \mathbf{r}(0) = h_0 \mathbf{j} \). Here, \( v_0, h_0, \theta, \) and \( g \) are constants representing the initial speed, the initial height, the initial angle of the trajectory, and the acceleration of gravity, respectively.
8  Tangent Vectors and Normal Vectors

• The principal unit tangent vector $T(t)$ of $r(t)$ is the unit vector $r'(t)/\|r'(t)\|$.  

• The principal unit normal vector $N(t)$ of $r(t)$ is unit vector $T'(t)/\|T(t)\|$. This vector is normal to the trace of $r(t)$ since $T(t) \cdot T'(t) = \|T(t)\|^2 = 1$ and so differentiating both sides and using the product rule implies that 

$$T'(t) \cdot T(t) + T(t) \cdot T'(t) = 2T(t) \cdot T(t) = 0,$$

i.e. $T(t) \cdot T'(t) = 0$ so that these vectors are orthogonal.

• Repeating the above process produces a third vector $B(t)$ called the principal binormal vector of $r(t)$.

Exercises:

1. Compute the principal unit tangent, normal, and binormal vectors of 

$$r(t) = 3 \cos t \, \mathbf{i} + 3 \sin t \, \mathbf{j} + t \, \mathbf{k}.$$
9 Functions of Several Variables

- A function having more than one independent variable is called a function of several variables. For example, the area of a rectangle is a function of the length and width of the rectangle:
  \[ \text{area}(\text{length}, \text{width}) = \text{length} \times \text{width}. \]

  If \( A \) represents the area, \( l \) the length, and \( w \) the width, then the function can be expressed as
  \[ A(l, w) = lw. \]

  What is the domain of this function?

- If \( x \) and \( y \) are independent variables and \( f \) is a function of \( x \) and \( y \), this is expressed by writing \( f(x, y) \). For example, the distance from the point \( P(2, 3) \) to a arbitrary point \( Q(x, y) \) is given by the function
  \[ d(x, y) = \sqrt{(x - 2)^2 + (y - 3)^2}. \]

  What is the domain of this function?

- The graph of \( f(x, y) \) is the surface \( z = f(x, y) \) in \( \mathbb{R}^3 \). But the function can studied geometrically by looking at its level curves. A curve of the form \( f(x, y) = c \), where \( c \) is a constant, is called a level curve.

  Sketch the level curve \( d(x, y) = 2 \) for the function \( d(x, y) \) above. What do the level curves \( d(x, y) = c \) look like? What do they represent?

- More generally, a function of three variables, such as
  \[ g(x, y, z) = x + 2y + 3z \]

  can be studied geometrically by looking at its level surfaces, \( g(x, y, z) = c \), where \( c \) is a constant. This is much simpler than studying the graph \( w = g(x, y, z) \) which is a subset of \( \mathbb{R}^4 \).

  Sketch the level surfaces of the function \( g \) above. What kind of surfaces are these? If \( g \) represents the temperature at a point \( P(x, y, z) \), then what does the surface \( g(x, y, z) = c \) represent if \( c = 25^\circ C \)?

**Exercises:**

1. If \( xy + yz + zx = 1 \), is \( z \) a function of \( x \) and \( y \)? What is the definition of the term “function” in mathematics? For instance, if \( y = \sqrt{x} \), in what sense is \( y \) a function of \( x \)? If \( y^2 = x \), is \( y \) a function of \( x \)?

2. If \( f(x, y) = x^2 + y \), then what is \( f(2, -1) \)? What is the domain of \( f \)? Describe and sketch the level curves \( f(x, y) = c \).

3. If \( f(x, y) = 1/\sqrt{x^2 - y^2} \), then what is \( f(2, -1) \)? What is the domain of \( f \)? Describe and sketch the level curves \( f(x, y) = c \).

4. If \( g(x, y, z) = x^2 + y^2 - z \), then what is \( g(1, 2, -1) \)? What is the domain of \( g \)? Describe and sketch the level surfaces \( g(x, y, z) = c \).

5. Write a function \( V \) which computes the volume of a rectangular box which has a height \( h \) and has a square base of length \( l \).

6. Suppose that a rectangular box has a height of 2 m and a square base of length 1 m. Compute the change in the volume when
   (a) the height is increased to 2.1 m
   (b) the length of the sides of the square base are increased to 1.1 m
   (c) the length of the sides of the square base are increased to 1.05 m
   (d) the height and the lengths are increased by 10% from their initial values.
   (e) What is a reasonable definition of the derivative of \( V \)?
10 Limits of Functions of Several Variables

- Throughout, the point \( P(x_1, \ldots, x_n) \) in \( \mathbb{R}^n \) will be identified with its position vector \( \mathbf{x} = (x_1, \ldots, x_n) \).
  Let \( r > 0 \) be a positive number and let \( \mathbf{x} \in \mathbb{R}^n \). The open ball of radius \( r \) centered at \( \mathbf{x} \) is
  \[
  B(\mathbf{x}, r) = \{ \mathbf{y} \in \mathbb{R}^n \mid \| \mathbf{x} - \mathbf{y} \| < r \}.
  \]
  In words, \( B(\mathbf{x}, r) \) is the set of all points in \( \mathbb{R}^n \) having distance less than \( r \) to the point \( \mathbf{x} \).
  A set \( U \subset \mathbb{R}^n \) is said to be open if for each \( \mathbf{x} \in U \) there is a positive number \( r > 0 \) such that \( B(\mathbf{x}, r) \subset U \). The value of \( r \) typically depends on the point \( \mathbf{x} \).

- Suppose that \( U \subset \mathbb{R}^n \) is an open set and that \( f : U \to \mathbb{R} \) is a function. Let \( \mathbf{x} \in U \). If for every \( \mathbf{y} \in U - \{ \mathbf{x} \} \), the value \( f(\mathbf{y}) \) is arbitrarily close to a number \( L \) provided \( \mathbf{y} \) is sufficiently close to \( \mathbf{x} \), then we say that the limit of \( f(\mathbf{y}) \) as \( \mathbf{y} \) approaches \( \mathbf{x} \) is equal to \( L \).

- Suppose that \( f(x, y) = x^2 + y^2 \); the domain of this function is \( \mathbb{R}^2 \). Is \( \mathbb{R}^2 \) open?
  The limit of \( f(x, y) \) as \( (x, y) \) approach \((0, 0)\) is equal to 0. Formally, the reason is that the difference between \( f(x, y) \) and 0 can be made arbitrarily small provided \( (x, y) \) is sufficiently close to \((0, 0)\). In fact, if the distance between \( (x, y) \) and \((0, 0)\) is \( r = \sqrt{x^2 + y^2} \), then the difference between \( f(x, y) \) and 0 is \( r^2 \). Since \( r^2 \) can be made smaller than any positive number provided \( r \) is chosen to be sufficiently small, we say the limit of \( f(x, y) \) as \( (x, y) \) approaches \((0, 0)\) is 0. Here’s how to write this in symbols:
  \[
  \lim_{(x, y) \to (0, 0)} f(x, y) = 0.
  \]

- Suppose that \( g(x, y) = \frac{xy}{x^2 + y^2} \). Then the limit as \( (x, y) \) approaches \((0, 0)\) does not exist. One way to see this is to first let \((x, y)\) approach \((0, 0)\) along the positive x-axis, i.e. through points of the form \((x, 0)\), where \( x > 0 \). The expression for \( f(x, 0) \) is equal to 0 for all such points. But if \((x, y)\) approaches \((0, 0)\) along the ray \( y = x, x > 0 \), then \( f(x, x) = \frac{1}{2} \). So, there is no single limiting value. This is analogous to left-handed and right-handed limits.

- A function \( f(y) \) is continuous at \( \mathbf{x} \) if \( f \) is defined on some open set \( U \) containing \( \mathbf{x} \) and if the limit of \( f(y) \) as \( y \) approaches \( \mathbf{x} \) exists and is equal to the value \( f(\mathbf{x}) \).
  For example, the function \( f(x, y) = \lfloor x + y \rfloor \) is not continuous at \((0, 0)\); here \( \lfloor z \rfloor \) is the greatest integer less than or equal to \( z \). The value \( f(0, 0) = \lfloor 0 + 0 \rfloor = 0 \). But if \((x, y)\) is close to \((0, 0)\) but such that \( x < 0 \) and \( y < 0 \), i.e. in the 3rd quadrant, then \( f(x, y) = -1 \). So, the limit of \( f(x, y) \) as \((x, y)\) approaches \((0, 0)\) does not exist. And so \( f(x, y) \) is not continuous at \((0, 0)\).

1. What is the domain of \( f(x, y) = \frac{1}{x^2 + y^2} \)? Is the domain an open set? At which points in this open set is \( f \) continuous?
2. Is the function \( f(x, y) = \lfloor x + y \rfloor \) continuous at the point \((1/2, 1/2)\)? What about at the point \((1/4, 1/4)\)?
3. Show that the following limit does not exist by considering the limiting values of \( f(x, y) \) as \((x, y)\) approaches \((0, 0)\) along different rays based at \((0, 0)\):
   \[
   \lim_{(x, y) \to (0, 0)} \frac{xy^2}{x^3 + y^3}
   \]
4. Does the following limit exist?
   \[
   \lim_{(x, y) \to (0, 0)} \frac{xy}{x^3 + y^3}
   \]
11 Partial Derivatives

- The partial derivative with respect to $x$ of $f(x, y)$ at the point $(a, b)$ is, by definition, the following limit:
  \[ \lim_{\Delta x \to 0} \frac{f(a + \Delta x, b) - f(a, b)}{\Delta x}. \]
  The partial derivative is denoted by $f_x(a, b)$ or by $\frac{\partial f}{\partial x}(a, b)$. The partial derivative with respect to $y$ is defined analogously.

- The value of $f_x(a, b)$ is the slope of the line tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ and lying in the plane $y = b$. In other words, the computation of the partial derivative with respect to $x$ is computed by holding $y$ equal to a constant value and computing the derivative with respect to $x$ in the usual manner.

- The partial derivative is itself a function of several variables, and so you can take the derivative of the derivative. There are many such higher derivatives. For instance, $f_{xy}(a, b)$ represents the result of first computing $f_x$ and then computing its partial derivative with respect to $y$ and finally evaluating this function at the point $(a, b)$. The functions $f_{xx}$, $f_{xy}$, $f_{yx}$, and $f_{yy}$ are called the second order partial derivatives.

  How many second order partial derivatives does a function of three variables have? How many third order partial derivatives?

- Many important scientific observations can be expressed as a partial differential equation, i.e., an equation which involves partial derivatives. Here are a few examples:

  1. Laplace equation: $f_{xx} + f_{yy} = 0$
  2. wave equation: $f_{tt} = c^2 f_{xx}$
  3. heat equation: $f_t = c^2 f_{xx}$

  1. Compute the partial derivatives of each function:

    (a) $f(x, y) = x^2 - 2y^2 + 4$
    (b) $f(x, y) = 4x^3y^{-2}$

  2. Compute the second order partial derivatives of each function:

    (a) $f(x, y) = e^{x/y}$
    (b) $\ln(x - y)$

  3. Show that the following function is a solution to the wave equation:

    $f(x, t) = \cos(ax + bct)$, where $a$ and $b$ are constants.

  4. Show that the following equation is a solution to the heat equation:

    $f(x, t) = e^{-t}\sin(x/c)$. 
12 Tangent Planes and the Chain Rule

- **Differentiability**: If a function \( f(x, y) \) can be suitably approximated by a tangent plane at a point \((a, b)\), then the function is said to be differentiable at \((a, b)\). The approximation is suitable if the relative error tends to zero as \((x, y)\) tends to \((a, b)\).

- **Tangent Plane**: The vectors \( v = i + f_x(a, b)k \) and \( w = j + f_y(a, b)k \) are tangent to the surface \( z = f(x, y) \) at the point \((a, b)\). Therefore, \( v \times w \) is normal to this surface. This calculation leads to the following equation of the tangent plane to the surface \( z = f(x, y) \) at the point \((a, b)\):

\[
z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b).
\]

- **Differentiability, the tangent plane, and the chain rule**: Let \( \Delta x = (x-a) \), \( \Delta y = (y-b) \), and \( \Delta z = f(x, y) - f(a, b) \). Then the tangent line approximation is the statement that

\[
\Delta z \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y
\]

Suppose that both \( x \) and \( y \) are functions of a parameter \( t \).

For example, \( f(x, y) = x^2 + y^2 \) might represent the temperature at the point \((x, y)\); if \( x = \cos t \) and \( y = \sin t \), then \( f(x(t), y(t)) \) represents the temperature at time \( t \) of the point \((x, y)\) moving along the unit circle in a counter-clockwise direction.

It is natural to ask what is the derivative of \( f \) with respect to \( t \). Dividing by \( \Delta t \) in the above approximation and letting \( \Delta t \) tend to zero yields the following equation for the so-called chain rule of \( f(x(t), y(t)) \):

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.
\]

- If the point \((x, y)\) depends on two parameters, say \( s \) and \( t \), then a function \( f(x, y) \) may be viewed as a function of the variables \( s \) and \( t \): \( f(x(s, t), y(s, t)) \).

For example, if \( f(x, y) = x^2 + y^2 \) represents temperature and \( x = \cos st \), \( y = \sin st \) represents a parameterization of the unit circle with parameter \( t \) representing time and the parameter \( s \) representing an angular speed (frequency), then it is natural to ask how the temperature changes when the frequency is changed while holding time constant and, vice-versa, how the temperature changes when time varies while frequency is held constant. Arguments similar to the above lead to the following chain rules for \( f(x(s, t), y(s, t)) \):

\[
\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
\]

**Exercises:**

1. Suppose that \( f(x, y) = x^2 + y^2 \). Find equations for two lines: one is tangent to the surface \( z = f(x, y) \) at the point \((1, 2, 5)\) and and parallel to the \( xz \)-plane, the other is also tangent at the same point but is parallel to the \( yz \)-plane. Then find a vector that is normal to \( z = f(x, y) \) at \((1, 2, 5)\).

2. The plane \( z = 13 + 4(x-2) + 6(y-3) \) is tangent to the surface \( z = f(x, y) = x^2 + y^2 \) at the point \((2, 3, 13)\). What is the tangent plane approximation of the value of \( f(2.1, 2.9) \)?

3. Use the chain rule to compute \( f'(t) \), where \( f(x, y) = \sin x + y^2 \), \( x = 2t \), and \( y = 3t \). Check your answer by first determining an expression for \( f(t) \) and then computing \( f'(t) \).

4. Use the chain rule to compute the partial derivative of \( f(x, y, z) = x^2 + y^2 \) with respect to \( \theta \), where \( x = \cos \theta \cos \phi \), \( y = \sin \theta \cos \phi \) and \( z = \cos \phi \).

5. Compute the partial derivative of \( f \) with respect to \( \phi \) using \( f(x, y, z) \) as in the previous question.
13 Directional Derivatives and the Gradient

- Suppose that \( \mathbf{u} \) is a unit vector in \( \mathbb{R}^n \) and that \( f : \mathbb{R}^n \to \mathbb{R} \) is a function of several variables. Let \( \mathbf{x} \in \mathbb{R}^n \). It is natural to compute a derivative which measures how the value of \( f(\mathbf{x}) \) changes when \( \mathbf{x} \) is moved by a small amount \( t \) in the direction of \( \mathbf{u} \). Such a derivative is called the directional derivative \( D_\mathbf{u}f(\mathbf{x}) \) and is computed by the following limit:

\[
D_\mathbf{u}f(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t}
\]

- The gradient vector \( \nabla f(\mathbf{x}) \) is the vector in \( \mathbb{R}^n \) whose components are the partial derivatives of \( f \) at \( \mathbf{x} \). More precisely, for a function of two variables,

\[
\nabla f(a, b) = \frac{\partial f}{\partial x}(a, b) \mathbf{i} + \frac{\partial f}{\partial y}(a, b) \mathbf{j}
\]

- The following formula holds:

\[
D_\mathbf{u}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u},
\]

i.e. the dot product of the gradient vector at a point with a unit vector \( \mathbf{u} \) returns the directional derivative at this point in the direction of \( \mathbf{u} \).

- The gradient vector is the direction in which \( f(\mathbf{x}) \) increases most quickly, i.e. among all unit vectors \( \mathbf{u} \), the one which produces the largest directional derivative is the unit vector obtained by taking the gradient vector and normalizing it.

- Suppose that \( f(\mathbf{x}) = c \) is a level set (e.g. curve or surface if \( \mathbf{x} \in \mathbb{R}^2 \) or \( \mathbb{R}^3 \), respectively) of \( f \). Then the gradient vector \( \nabla f(\mathbf{x}) \) is normal to this level set.

This follows from the following observation: suppose that \( \mathbf{x}_0 \) is a point in the level set \( f(\mathbf{x}) = c \) and suppose that \( \mathbf{x}(t) \) is a parameterization of a curve such that \( \mathbf{x}(0) = \mathbf{x}_0 \) and such that this curve lies in the level set for all \( t \) between \(-1\) and \( 1 \). Then \( \frac{df}{dt}(\mathbf{x}(0)) = 0 \) since \( f \) is constant for \( t \) between \(-1\) and \( 1 \). On the other hand, the chain rule implies that \( \frac{df}{dt}(\mathbf{x}(0)) = \nabla f(\mathbf{x}_0) \cdot \frac{d\mathbf{x}}{dt}(0) \). Therefore, \( \nabla f(\mathbf{x}_0) \) is orthogonal to the tangent vector \( \frac{d\mathbf{x}}{dt}(0) \).

Since \( \nabla f(\mathbf{x}_0) \) is orthogonal to the tangent vector of every such curve in the level set, the gradient vector is normal to the level set.

Here’s another way to interpret the above. Suppose you are reading a hiking map which shows contours of constant elevation. The direction of steepest incline (the direction of the gradient of the elevation function) is orthogonal to the contour lines. This should agree with your intuition from reading such a map.

Exercises:

1. Let \( f(x, y) = x^2 + y^2 \). Compute the directional derivative \( D_\mathbf{u}f(1, 1) \) when \( \mathbf{u} = \mathbf{i} \), when \( \mathbf{u} = \mathbf{i} - \mathbf{j} \) and when \( \mathbf{u} = \mathbf{i} + \mathbf{j} \). Do your calculations agree with your intuition if you interpret \( f(x, y) \) as the temperature at the point \((x, y)\)? Which direction should be the direction in which the directional derivative is largest?

2. Let \( g(x, y) = 4x^2 + 9y^2 \) Compute the gradient of \( g \) at the point \((1, 1)\). Then sketch the level curve \( g = 13 \) and compute the unit vector in the direction of maximal increase. Add this unit vector to your sketch, placing the tail of the vector at the point \((1, 1)\).

3. (Important Example!) The problem of finding the equation of a tangent plane to a surface can be both simplified and generalized by using the gradient vector.

(a) Determine an equation of the plane tangent to the ellipsoid \( x^2 + 2y^2 + 3z^2 = 6 \) at the point \((1, 1, 1)\) by using the fact that \( x^2 + 2y^2 + 3z^2 = 6 \) is a level surface of the function \( f(x, y, z) = x^2 + 2y^2 + 3z^2 \) and therefore \( \nabla f(1, 1, 1) \) is normal to this surface at this point.

(b) Prove that the equation below for the tangent plane to the surface \( z = f(x, y) \) at the point \((a, b)\) by using the fact that \( f(x, y) - z = 0 \) is a level surface of the function \( g(x, y, z) = f(x, y) - z \).

\[
f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0.
\]
14 Extrema of functions of two variables

- **critical points**: if \( f_x(a, b) = 0 \) and \( f_y(a, b) = 0 \), then \((a, b)\) is said to be a critical point of \( f \). For example, if \( f(x, y) = x^2 + xy + y^2 \), then \( f_x(x, y) = 2x + y \) and \( f_y(x, y) = 3y \). If both quantities are zero, then \( 2x + y = 0 \) and \( 3y = 0 \). The second equation holds true if \( x = 0 \) or if \( y = 0 \). If \( x = 0 \), the first equation implies that \( y = 0 \); if \( y = 0 \), then the first equation also implies that \( x = 0 \). So, the only critical point of \( f \) is \((0, 0)\).

- **the Hessian**: The \( 2 \times 2 \) matrix having first row \([f_{xx}, f_{xy}]\) and second row \([f_{yx}, f_{yy}]\) is called the hessian of \( f \). Let \( H(a, b) \) be the determinant of the hessian evaluated at the point \((a, b)\). For example, using the function above, \( f_{xx} = 2, f_{xy} = 3y, f_{yx} = 3y, \) and \( f_{yy} = 6y. \) So, \( H(0, 0) \) is the determinant of the matrix having first row \([2, 0]\) and second row \([0, 0]\). Therefore, \( H(0, 0) = 0. \)

- **local extrema**: A function \( f : \mathbb{R}^n \to \mathbb{R} \) has a local maximum at the point \( x \) if there is an open ball \( U \) centered at \( x \) such that \( f(y) \leq f(x) \) for every \( y \in U \). In other words, \( f(x) \) is the largest value of \( f \) in a neighborhood of \( x \). A local minimum is a point at which \( f \) is smallest compared with all points in some neighborhood.

- **Classification of Local Extrema**: If \((a, b)\) is a critical point of \( f(x, y) \) and \( H(a, b) > 0 \), then \( f \) has a local extremum at \((a, b)\). If \( f_{xx}(a, b) > 0 \), then this is a local minimum; if \( f_{xx}(a, b) < 0 \), then this is a local maximum.

- **saddle points**: A function \( f : \mathbb{R}^n \to \mathbb{R} \) has a saddle point at \( x \) if for every neighborhood of \( x \) there are two points in this neighborhood, one having larger \( f \)-value than \( f(x) \) and the other having smaller \( f \)-value than \( f(x) \).

- **Classification of Saddle Points**: If \((a, b)\) is a critical point of \( f(x, y) \) and \( H(a, b) < 0 \), then \( f \) has a saddle point at \((a, b)\).

- **when \( H(a, b) = 0 \)**: if \((a, b)\) is a critical point of \( f(x, y) \) and \( H(a, b) = 0 \), then \( f \) may have a local extremum or a saddle point; however, this must be determined by using some other test—this test of looking at the sign of the hessian is inconclusive if \( H(a, b) = 0 \). For example, the function above has \( H(0, 0) = 0 \). Nonetheless, this function has a saddle point at \((0, 0)\). This follows from the observation that \( f(x, x) > 0 \) for \( x \neq 0 \) and that \( f(x^2, -2x) < 0 \) for \( x \neq 0 \).

**Exercises:**

1. Determine the critical points of each function:
   
   (a) \( f(x, y) = x^2 + xy + y^2 + y \)
   
   (b) \( f(x, y) = e^x \cos y \)
   
   (c) \( f(x, y) = e^y(y^2 - x^2) \)

2. Determine the critical points of the function and classify each critical point as being either a local minimum, a local maximum, or a saddle point:

\[
g(x, y) = 2x^3 + 2y^3 + 9x^2 - 24y + 12x.
\]

3. You might wonder why the case when \( H(a, b) > 0 \) and \( f_{xx}(a, b) = 0 \) is not addressed in the classification of local extrema. Prove this situation cannot occur.
15 Applications of Critical Points and Local Extrema

How to find global extrema:

Given \( f(x, y) \) defined on a closed region \( R \) with boundary curve \( g(x, y) = 0 \), you can find the global extrema, i.e. the absolute maximum and minimum values of \( f \) over \( R \) by the method outlined below.

To have an example to keep in mind, consider the following example:

\[
f(x, y) = y^2 + x^2 + 2x + 1
\]

\[
R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}
\]

So, \( g(x, y) = x^2 + y^2 - 4 = 0 \) describes the boundary of \( R \).

**Step one: critical points in the interior of \( R \).** Identify the critical points in the interior of \( R \) by solving \( f_x = 0 \) and \( f_y = 0 \) (or look for points where the derivative fails to exist).

In our example, \( f_x = 2x + 2 \) and \( f_y = 2y \). Thus, \((-1, 0)\) is the only critical point and this point lies in the interior of \( R \).

**Step two: restrict \( f \) to the boundary of \( R \).** Using the fact that on the boundary \( g(x, y) = 0 \), we can (in principle, under certain hypotheses), eliminate one of the variables in the expression for \( f(x, y) \). This will make more sense using the example:

If \( g(x, y) = x^2 + y^2 - 4 = 0 \), then \( f(x, y) = y^2 + x^2 + 2x + 1 \) is really just a function of the variable \( x \). Using \( g(x, y) = 0 \), we find that \( y^2 = 4 - x^2 \). Substituting this into \( f(x, y) \), we see that \( f(x) = (4 - x^2) + x^2 + 2x + 1 \) or, after simplification, \( f(x) = 2x + 5 \). This function is valid only if \( x \) corresponds to a point on the boundary of \( R \): this means that \(-2 \leq x \leq 2 \) since \( R \) is a closed disk of radius 2.

Therefore, to determine the global extrema of \( f \) restricted to the boundary of \( R \), we need to find the global extrema of \( f(x) = 2x + 5 \) on the interval \([-2, 2]\). As one learns in a first semester calculus course, the extreme values occur at critical points or at end points. \( f'(x) = 2 \), and so there are no critical points in \((-2, 2)\) (but there very well could be in more complicated examples). The values at the end points are \( f(-2) = 1 \) and \( f(2) = 9 \).

**Step three: compare to values at the critical points interior to \( R \).** Finally, we compute the value of \( f(x, y) \) at the critical points in the interior of \( R \) and compare these values with the extreme values of \( f \) restricted to the boundary.

In the example, our only critical point in the interior of \( R \) was \((-1, 0)\). Computing, we have that \( f(-1, 0) = 0 \). Comparing this with the extreme values (max 9, min 1) of \( f \) on the boundary, we conclude that the maximum value of \( f \) on \( R \) is 9 and that the minimum value of \( f \) on \( R \) is 0.

**Exercises:**

1. Determine the absolute maximum and minimum values of \( f(x, y) = 2x^3 + y^4 \) on the region \( R = \{(x, y) \in \mathbb{R} \mid x^2 + y^2 \leq 1\} \).

2. Determine the absolute maximum and minimum values of \( f(x, y) = x^2 + y^2 + x^2 y + 4 \) on the region \( R = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1, |y| \leq 1\} \). (Hint: There are four boundary curves, not just one; you will need to consider each of the four cases separately.)

3. Find the points on the cone \( z^2 = x^2 + y^2 \) that are closest to the point \((4, 2, 0)\). (Hint: Rather than minimize the distance, minimize the square of the distance.)
16 Lagrange Multipliers

- 
- 
- 
- 

Exercises:

1. 
2. 
3.
17 Iterated Integrals

Exercises:

1.

2.

3.
18 Double Integrals

Exercises:

1.

2.

3.
19 Double Integrals in Polar Coordinates

- 
- 
- 

Exercises:

1.

2.

3.
Application: Center of Mass and Moments of Inertia

- 
- 
- 

Exercises:

1. 
2. 
3.
21 Surface Area for Surfaces of the Form $z = f(x, y)$

- 
- 
- 

Exercises:

1. 
2. 
3. 
22 Triple Integrals

Exercises:
1.
2.
3.
23 Triple Integrals in Cylindrical or Spherical Coordinates

Exercises:

1.

2.

3.
24 Change of Variables and the Jacobian Determinant

Exercises:
1.
2.
3.