A MISCELLANY OF MATHEMATICAL FORMULAE

Triple Integrals in Cylindrical & Spherical Coordinates
\[
\begin{align*}
\rho^2 &= x^2 + y^2 + z^2 \\
r^2 &= x^2 + y^2 \\
z &= \rho \cos \phi \\
r &= \rho \sin \phi \\
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= \rho \cos \phi \\
r &= \rho \sin \phi \\
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= \rho \cos \phi \\
\end{align*}
\]
\[dV = dx \, dy \, dz = r \, dz \, dr \, d\theta = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta\]

Change of Variables for Double Integrals
\[
\iint_R f(x, y) \, dx \, dy = \iint_S f(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv
\]

Curl, and Divergence
Let \( F = M \, i + N \, j + P \, k \) and \( \nabla = \frac{\partial}{\partial x} \, i + \frac{\partial}{\partial y} \, j + \frac{\partial}{\partial z} \, k \).
\[
\begin{align*}
\text{div} \, F &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \nabla \cdot F \\
\text{curl} \, F &= (\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}) \, i + (\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}) \, j + (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) \, k = \nabla \times F.
\end{align*}
\]

Vector Fields and Differential Forms
Let \( F = M \, i + N \, j + P \, k \). The form \( \omega = M \, dx + N \, dy + P \, dz \) is the associated differential form.

We say that \( \omega \) is closed if \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \, \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \) and \( \frac{\partial P}{\partial z} = \frac{\partial M}{\partial x} \). Thus \( \omega \) is closed if and only if \( \text{curl} \, F = 0 \).

We say that \( \omega \) is exact if there exists a potential \( f \) for \( F \), i.e. \( \nabla f = F \), i.e. \( F \) is a gradient field, i.e. \( F \) is conservative.

**Theorem:** Let \( F \) be a smooth vector field on an open connected set \( R \), and let \( \omega \) be the associated differential form.

- \( \omega \) is exact \( \Rightarrow \) \( \omega \) is closed, i.e. conservative vector fields are curl-free.
- If \( R \) is simply connected, then \( \omega \) closed \( \Rightarrow \) \( \omega \) is exact.
- The analogous statements holds if \( F(x, y) \) is a vector field in \( \mathbb{R}^2 \).

Line Integrals
Let \( F = M \, i + N \, j + P \, k \).
\[
\text{Work} = \int_C F \cdot dr = \int_C M \, dx + N \, dy + P \, dz
\]
If \( C \) is closed curve in \( \mathbb{R}^2 \), then the flux of \( F \) through \( C \) is \( \oint_C F \cdot N \, ds \), where \( C \) is oriented in a counterclockwise sense and \( N \) is an outward pointing unit normal vector.
Surface Integrals

Let $\mathbf{r}(u,v)$ be a vector-valued function which parametrizes the surface $S$. Then $dS = \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv$.

If $S$ is the graph of $f(x,y)$, then $dS = \sqrt{1 + (f_x)^2 + (f_y)^2} \, dx \, dy$.

If $S$ is a closed surface in $\mathbb{R}^3$, then the flux of $\mathbf{F}$ through $S$ is $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$, where $\mathbf{N}$ is an outward pointing unit normal vector.

Green’s Theorem

Suppose that $R$ is a region in $\mathbb{R}^2$ with piecewise smooth boundary $C$, oriented so that the region lies to the left of each boundary curve. Suppose that $\mathbf{F} = M \mathbf{i} + N \mathbf{j}$ is a $C^1$ vector field on an open region containing $R$. Then

$$\oint_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA.$$  

An equivalent formulation is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl} \mathbf{F} \cdot \mathbf{k} \, dA.$$  

Another equivalent formulation, where $\mathbf{N}$ is the outward unit normal vector, is

$$\oint_C \mathbf{F} \cdot \mathbf{N} \, ds = \iint_R \text{div} \mathbf{F} \, dA.$$  

Stokes’s Theorem

Suppose that $S$ is a smooth oriented surface in $\mathbb{R}^3$ with continuously varying unit normal vector $\mathbf{N}$ and with piecewise smooth boundary $C$ oriented using the right-hand rule with $\mathbf{N}$. Suppose that $\mathbf{F} = M \mathbf{i} + N \mathbf{j} + P \mathbf{k}$ is a $C^1$ vector field on an open region containing $S$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl} \mathbf{F} \cdot \mathbf{N}) \, dS.$$  

The Divergence Theorem

Suppose that $Q$ is a solid region in $\mathbb{R}^3$ bounded by a closed surface $S$ oriented by its outward unit normal vector $\mathbf{N}$. Suppose that $\mathbf{F} = M \mathbf{i} + N \mathbf{j} + P \mathbf{k}$ is a $C^1$ vector field on an open region containing $Q$. Then

$$\iiint_Q \mathbf{F} \cdot dV = \iiint_S \text{div} \mathbf{F} \, dV.$$  

2