Solutions to Homework 7

Section 12.2 # 8: Compute the integral of $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

Solution: Sketch the triangle. Holding x constant and varying y, one sees that $0 \leq y \leq 1 - x$, where the quantity $1 - x$ is found by writing an equation for the line joining $(1, 0)$ to $(0, 1)$. Therefore, the integral is equal to

$$\int_0^1 \int_0^{1-x} (x^2 + y^2)\, dy\, dx.$$

The integral is easily computed. Answer: 1/6.

Section 12.2 # 18: Sketch the region of integration and write an equivalent double integral with the order of integration reversed:

$$\int_0^1 \int_{1-x}^{1-x^2} dy\, dx.$$

Solution: The region is the set of points which lie above the line $y = 1 - x$ and below the parabola $y = 1 - x^2$ and whose x-coordinates lie in between 0 and 1. Varying x while holding y constant, one sees that $1 - y \leq x \leq \sqrt{1 - y}$ and that the y-coordinates of points in the region lie between 0 and 1. Answer:

$$\int_0^1 \int_{1-y}^{\sqrt{1-y}} dx\, dy.$$

Section 12.2 # 28: Sketch the region, reverse the order of integration, and evaluate the integral:

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4 - y}\, dy\, dx.$$

Solution: The region is the set of points which lie above the line $y = 0$ and below the parabola $y = 4 - x^2$ and whose x-coordinates lie between 0 and 2. Varying x and holding y constant, one sees that $0 \leq x \leq \sqrt{4 - y}$ and $0 \leq y \leq 4$. The computation of the integral is fairly straight forward.

(Once you integrate with respect to x and evaluate, you will see a cancellation of terms of the form $(4 - y)$. Answer: $(e^8 - 1)/4$.

Section 12.2 # 50: Approximate the double integral below by using the rectangular partition formed by the vertical lines $x = 1, 3/2, 2, 5/2, 3$ and the horizontal lines $y = 2, 5/2, 3, 7/2, 4$ and with (x_k, y_k) the center of each rectangle.

$$\iint_R (x + 2y)\, dx\, dy$$

where R is the unit disc centered at $(2, 3)$. Only use subrectangles which are contained in R.

Solution: It is easy to see that the area of each rectangle is $1/4$ as every rectangle in the partition is a square with an edge of length $1/2$. There are sixteen squares in the partition. However, only the center four lie inside the region R. The center of each of these four squares is easy to identify. The points are, proceeding from left to right with the bottom two squares and then left to right with the upper two squares, as follows:

$$(1.75, 2.75), (2.25, 2.75), (1.75, 3.25), (2.25, 3.25).$$

The approximation is computed by evaluating $(x + 2y)$ at each of the sixteen points, adding these values, and multiplying the total by $1/4$. Answer: 8.

Section 12.3 # 16: Which is larger: the average value of xy over the square $[0, 1] \times [0, 1]$ or the average value of xy over the portion of the unit disc centered at the origin which lies in the first quadrant?

Solution: The first average is equal to $\iint_R xy\, dA$, where R is the square $[0, 1] \times [0, 1]$, divided by the area of this square (which equals one). Answer: $1/4$.

The second average is equal to $\iint_R xy\, dA$, where R is the quarter of the unit disc which lies in the first quadrant, divided by the area of this quarter disc (which equals $\pi/4$). Answer: $1/2\pi$.

So, the first average is greater.

(To compute the integral, the easiest approach would be to use polar coordinates; however, as this appears in section 12.3, the author expected you to use rectangular coordinates. This is an unpleasant integral. Try it again now that you know how to integrate in polar coordinates.)

Additional Problem: What is the average height of a point which lies in the solid pyramid with square base having corners $(1, 0, 0)$, $(0, 1, 0)$ $(-1, 0, 0)$, and $(0, -1, 0)$ and having the point $(0, 0, 1)$ as its summit?

Solution: Let $h(x, y, z)$ be the height of a point (x, y, z) which belongs to the pyramid. It is clear that we can compute the average height of a point in one quarter of the pyramid by symmetry. It is also clear that $h(x, y, z) = z$. The average value of this function over the quarter pyramid is equal to

$$\frac{1}{\text{volume}(\Omega)} \iiint_\Omega z\, dV,$$

where Ω represents the solid quarter pyramid. This is a triple integral, and it computation is as follows:

$$\frac{1}{\text{volume}(\Omega)} \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx.$$

The answer is $1/4$.

Many of you (including me when we discussed this problem during class!) computed the solution to a different problem:

"What is the average height of a point on the boundary of the pyramid (excluding the base)?" The answer to this problem is found by computing the average z-coordinate of a point on the top part of the pyramid. Again, we can restrict our attention to one quarter of the pyramid. The boundary of the pyramid consists of the base and four triangular regions. The triangular region in the first octant is a portion of the plane $x + y + z = 1$. So, $z = 1 - x - y$ is the z-coordinate. Now find its average value over the triangular region R which lies below the plane:

$$\frac{1}{\text{area}(R)} \iint_R (1 - x - y)\, dA,$$

where R is the triangle in the xy-plane with having vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Answer: $1/12$.

Here is another way to compute the answer to the question that was originally asked. View the pyramid as composed of horizontal cross sections, each cross section being a square. The square at height z has area $A(z) = 2(1-z)^2$. (You'll need to work this out to follow the discussion.) So, for instance, the volume of the pyramid is $\int_0^1 A(z)\,dz = 2/3$, which agrees with formulas from geometry (the volume of a pyramid is one third of the height times the area of the base). The average height of a point in the solid pyramid is computed by $\frac{1}{1-0}\int_0^1 A(z)z\,dz$, since height of the cross section is z. This integral is easy to compute and one obtains the answer 1/4 as claimed above.