Section 11.7, Exercise # 4: Determine the local extrema and saddle points of \( f(x, y) = 2xy - 5x^2 - 2y^2 + 4x - 4 \).

Strategy: Compute the partial derivatives, solve for when both partial derivatives are equal to zero, then apply the test using the Hessian.

\[
f_x = 2y - 10x + 4 = 0 \\
f_y = 2x - 4y = 0
\]

Double the first equation and add it to the second to obtain

\[
4y - 20x + 8 + 2x - 4y = 0 \implies -18x + 8 = 0 \implies x = 4/9
\]

Therefore, using the second equation,

\[
8/9 - 4y = 0 \implies y = 2/9
\]

So, the only critical point is \((4/9, 2/9)\).

\[
f_{xx} = -10, \quad f_{xy} = 2 = f_{yx}, \quad f_{yy} = -4. \quad \text{Therefore, the Hessian,} \\
H = f_{xx}f_{yy} - f_{xy}f_{yx}, \quad \text{is equal to} \quad 40 - 4 = 36. \quad \text{So, } H(4/9, 2/9) = 36. \\
\text{Therefore } f \text{ has a local extremum at this point. Since } f_{xx}(4/9, 2/9) < 10, \quad f \text{ has a local maximum.}
\]

Section 11.7, Exercise # 18: Determine the local extrema and saddle points of \( f(x, y) = x^3 + 3xy + y^3 \).

The strategy is the same as in the exercise above.

\[
f_x = 3x^2 + 3y = 0 \\
f_y = 3x + 3y^2 = 0
\]

The simplest way to proceed is to solve for one of the variables and substitute into the other equation. Solving for \( y \) in the first equation, \( y = -x^2 \), and then substituting into the second yields \( 3x + 3x^4 = 0 \). To solve this equation, factor:

\[
3x(1 + x^3) = 3x(x + 1)(x^2 - x + 1) = 0
\]
The quadratic $x^2 - x + 1$ is always positive (as can be seen from the quadratic formula, or more efficiently by just looking at the discriminant $b^2 - 4ac$, which is negative, and so there are no real roots).

So, the only solutions are $x = 0$ and $x = -1$. If $x = 0$, use the first equation to deduce that $y = 0$ also. If $x = -1$, use the first equation again to deduce that $y = -1$. So, the only critical points are $(0, 0)$ and $(-1, -1)$.

$f_{xx} = 6x, f_{xy} = 3 = f_{yx}, f_{yy} = 6y$. So, the hessian is $H = 36xy - 9$. $H(0, 0) < 0$, so there is a saddle point at $(0, 0)$. But $H(-1, -1) > 0$, so there is a local maximum (since $f_{xx}(-1, -1) < 0$) at $(-1, -1)$.

Section 11.7, Exercise # 32: Find the absolute maxima and minima of the function $D(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0, y = 4$, and $y = x$.

Strategy: First check to see if there are any critical points in the interior of the triangular plate. Then analyze the values of $D$ when restricted to the sides of the triangle. (There will be three separate cases for the second part.)

$D_x = 2x - y = 0$

$D_y = -x + 2y = 0$

Double the second and add to the first to obtain $2x - y - 2x + 4y = 0$, which implies that $y = 0$. Using either equation yields that $x = 0$ also. So, $(0, 0)$ is the only critical point of $D$ and this point does not lie in the interior of the triangular plate (it lies on the boundary). Note: I am using words like “closed”, “interior”, and “boundary” because these have precise mathematical meaning. Please review the terminology on p. 587 of our textbook.

First, restrict $D$ to the line $x = 0$. To stay on the plate, we have that $0 \leq y \leq 4$. On this interval, $D(0, y) = y^2 + 1$, and this function clearly is maximal when $y = 4$ and minimal when $y = 0$. So, $D(0, 4) = 17$ is possibly the maximum value of $D$ on the triangular plate. Similarly, $D(0, 0) = 1$ is possibly the minimum value. To be certain, we need to analyze the other two cases.

Restricted to the line $y = 4$, we have that $0 \leq x \leq 4$, and
\(D(x, 4) = x^2 - 4x + 17\). It is less clear what the extreme values of this function are on the interval \([0, 1]\). Find the critical points:

\[D'(x, 2) = 2x - 4 = 0 \implies x = 2.\]

And test the end points:

\[x = 0 : D(0, 4) = 17,\] and \(x = 4 : D(4, 4) = 17\), both of which were already computed. Updating the possibilities, we see that 17 is the largest value of \(D\) found so far, whereas 1 is the smallest value found so far.

Restricted to the line \(y = x\), we have, after choosing to eliminate \(y\), that

\[0 \leq x \leq 4\] and \(D(x, x) = x^2 - x^2 + x^2 + 1 = x^2 + 1\). This is clearly maximal when \(x = 4\) and minimal when \(x = 0\). \(D(4, 4) = 17\) and \(D(0, 0) = 1\) (as we had seen previously).

This completes the analysis. The maximum value of \(D\) is 17 and the minimum value of \(D\) is 1.

Section 11.7, Exercise # 34: Determine the extreme value of

\[T(x, y) = x^2 + xy + y^2 - 6x\]
on the rectangular plate \(0 \leq x \leq 5, -3 \leq y \leq 3\).

The strategy is the same as in the previous problem.

\[T_x = 2x + y - 6 = 0\]

\[T_y = x + 2y = 0\]

Subtract twice the second from the first: \(2x + y - 6 - 2x - 4y = 0\), which implies that \(-3y - 6 = 0\), and so \(y = -2\). This implies that \(x = 4\). So, \((4, -2)\) is the only critical point. This point lies inside the rectangle. So, we compute \(T(4, -2) = -12\).

Next check the boundaries. If \(x = 0\), then \(-3 \leq y \leq 3\), and \(T(0, y) = y^2\). It is easy to see that on \([-3, 3]\), the maximum value is 9 and the minimum value is 0.

If \(x = 5\), then \(-3 \leq y \leq 3\), and \(T(5, y) = 25 + 5y + y^2 - 30 = y^2 + 5y - 5\).

Check for critical points: \(T'(5, y) = 2y + 5\), so there is one at \(y = -5/2\).

\(T(5, -5/2) = -5 - 25/4 = -11.25\). At the endpoints: \(T(5, -3) = -11\) and \(T(5, 3) = 19\).

If \(y = -3\), then \(0 \leq x \leq 5\), and \(T(x, -3) = x^2 - 9x + 9\). Critical points:
\[ T'(x, -3) = 2x - 9, \text{ so } x = 9/2. \ T(9/2, -3) = 9 - 81/4 = -11.25. \]"}Endpoints: \( T(0, -3) = 9, T(5, -3) = 11. \)

If \( y = 3 \), then \( 0 \leq x \leq 5 \), and \( T(x, 3) = x^2 - 3x + 9 \). Critical points:
\[ T'(x, 3) = 2x - 3 = 0 \text{ when } x = 3/2. \ T(3/2, 3) = 9 - 9/4 = 6.75. \]Endpoints: \( T(0, 3) = 9 \) and \( T(5, 3) = 19. \)

So, the maximum value of \( T \) is 19 and the minimum value of \( T \) is \(-12.\)

Section 11.7, Exercise \# 55: This problem was solved in class on February 10. Here’s a recap:

\[ w = \sum_{k=1}^{n} (mx_k + b - y_k)^2 \] is a function of \( m \) and \( b \). We are to find \( m \) and \( b \) which minimize \( w \). This will occur at a critical point. So, compute \( w_m \) and \( w_b \):

\[
\begin{align*}
w_m &= \sum_{k=1}^{n} 2(mx_k + b - y_k)x_k = 2[m\sum x_k^2 + b\sum x_k - \sum x_k y_k] = 0 \\
w_b &= \sum_{k=1}^{n} 2(mx_k + b - y_k) = 2[m\sum x_k + bn - \sum y_k] = 0
\end{align*}
\]

Cancel the two’s and solve the second equation for \( b \):

\[ b = (1/n)(\sum y_k - m\sum x_k). \] This is one of the equations.

Then substitute into the first equation:

\[ m\sum x_k^2 + (1/n)(\sum y_k - m\sum x_k)\sum x_k - \sum x_k y_k = 0 \]

and solve for \( m \):

\[ m = [(1/n)(\sum x_k)(\sum y_k) - \sum x_k y_k]/[(1/n)(\sum x_k)^2 - \sum x_k^2]. \]

Finally, multiply the numerator and the denominator by \( n \) to obtain the formula in the textbook.

Section 11.7, Exercises \# 56 and 58: Use the formula from Exercise 55 to find the least square line:

56: \((-2, 0), (0, 2), (2, 3)\).

\[ \sum x_k = 0, \ \sum y_k = 5, \ \sum x_k y_k = 6, \ \sum x_k^2 = 8, \text{ and } n = 3. \]
Using $m = [(\sum x_k)(\sum y_k) - n \sum x_k y_k]/[(\sum x_k)^2 - n \sum x_k^2]$, and
$b = (1/n)[\sum y_k - m \sum x_k]$, we have that

$m = 3/4$ and $b = 5/3$.

58: (0, 0), (1, 2), (2, 3)

$\sum x_k = 3$, $\sum y_k = 5$, $\sum x_k y_k = 8$, $\sum x_k^2 = 5$, and $n = 3$.

$m = 3/2$ and $b = 1/6$. 