Solutions to Homework 10

Section 12.8 # 6: Use the transformation $u = x - y$ and $v = 2x + y$ to evaluate the integral below, where $R$ is the region in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

$$\iint_{R} (2x^2 - xy - y^2) \, dx \, dy$$

Solution: Solving for $x$ and $y$ in terms of $u$ and $v$, one obtains $x = (1/3)(u + v)$ and $y = (1/3)(v - 2u)$. The lines which bound the quadrangle (parallelogram) above are $v = 4$ (corresponding to $2x + y = 4$), $v = 7$ (corresponding to $2x + y = 7$), $u = 2$ (corresponding to $x - y = 2$) and $u = -1$ (corresponding to $x - y = -1$). The Jacobian determinant $\partial(x,y)/\partial(u,v)$ is equal to $1/3$. The integrand $(2x^2 - xy - y^2)$ $= (2x + y)(x - y)$ is equal to $uv$. Therefore, the integral above is equal to the integral below by the change of variables formula.

$$\int_{-1}^{2} \int_{4}^{7} uv |1/3| \, du \, dv.$$ 

The above is easy to compute. Answer: $33/4$. A sketch of the region $R$ reveals that the original integral, while not difficult, would be tedious to compute. The region $R$ is neither $x$-simple nor $y$-simple. Furthermore, after computing the anti-derivative of the integrand with respect to either $x$ or $y$ and evaluating at the appropriate bounds, one would then need to expand the resulting third degree equations before arriving at an expression which can be integrated term by term.

Section 12.8 # 8: Use the transformation $u = 2x - 3y$ and $v = -x + y$ to evaluate the integral below, where $R$ is the parallelogram bounded by the lines $x = -3$, $x = 0$, $y = x$, and $y = x + 1$.

$$\iint_{R} 2(x - y) \, dx \, dy$$

Solution: Solving for $x$ and $y$ in terms of $u$ and $v$ results in $x = -(u + 3v)$ and $y = -(u + 2v)$. The Jacobian determinant $\partial(x,y)/\partial(u,v)$ is equal to $-1$. The transformation carries $R$ to a region in the $uv$-plane bounded by
the lines \( u + 3v = 3 \) (corresponding to \( x = -3 \)), \( u + 3v = 0 \) (corresponding to \( x = 0 \)), \( v = 0 \) (corresponding to \( y - x = 0 \)), and \( v = 1 \) (corresponding to \( y - x = 1 \)). This transformed region is again a parallelogram, and this region is \( u \)-simple: \( u \) is bounded below by the line \( u + 3v = 0 \) and above by the line \( u + 3v = 3 \). The integrand \( 2(x - y) \) is equal to \(-2v\). Therefore, by the change of variables formula, the integral above is equal to the following integral:
\[
\int_0^1 \int_{-3v}^{3-3v} (-2v) | -1 | du dv
\]
This integral is surprisingly easy to compute: the anti-derivative with respect to \( u \) when evaluated at the inner limits of integration is equal to
\[
-2v[(3 - 3v) - (-3v)] = -6v,
\]
and so the second integral \( \int_0^1 -6v dv \) is equal to \(-3\). The original integral is not difficult, but it is tedious. The region \( R \) is \( y \)-simple, but the algebra used in the computation is more complicated than the above.

(A more geometric solution is the following: sketch the region \( R \) and observe that the integrand \( 2(x - y) \) is measuring the negative of twice the length of a vertical line segment from from a point in \( R \) to the line \( x - y = 0 \), which happens to be one side of the parallelogram \( R \). One can then deduce that the value of the integral is equal to \(-2\int_0^1 t dt \) times the width of the parallelogram (which is equal to 3); answer: \(-3\).

Section 12.8 # 10: Find the Jacobian of the transformation \( x = u \) and \( y = uv \) and sketch the region \( G \) in the \( uv \)-plane defined by the inequalities \( 1 \leq u \leq 2 \) and \( 1 \leq uv \leq 2 \). Then use this transformation to write an integral over \( G \) that is equivalent to the one below. Finally, compute the value of these integrals.
\[
\int_1^2 \int_1^2 \frac{y}{x} dy dx
\]
Solution: Since \( x \) and \( y \) are given as functions of \( u \) and \( v \), the Jacobian determinant \( \partial(x, y)/\partial(u, v) \) can be computed directly. Answer: \( u \). The region \( G \) is the region in the \( uv \)-plane bounded by the vertical lines \( u = 1 \) and \( u = 2 \) and by the hyperbolas \( uv = 1 \) (below) and \( uv = 2 \) (above). Since \( y/x = uv/u = v \). And since \( u > 0 \) in this region, \( |u| = u \). Therefore,
by the change of variables formula, the integral above is equal to
\[ \int_1^2 \int_{1/u}^{2/u} uv \, dv \, du. \]

The integral is straight-forward to compute. Answer: \( \frac{3}{2} \ln 2 \). (The original integral is even easier to compute; so, in practice, it seems likely that the inverse of this transformation would be more useful: it transforms hyperbolas of the form \( uv = c \) to straight lines.)

Section 10.1 # 4: Determine an equation in \( x \) and \( y \) (independent of \( t \)) for the curve
\[ r(t) = (t^2 + 1) \mathbf{i} + (2t - 1) \mathbf{j}. \]
Then compute the velocity and the acceleration when \( t = 1/2 \) of a particle whose position is given by \( r(t) \).

Solution: Let \( x = t^2 + 1 \) and \( y = 2t - 1 \). Solving for \( t \) in the second equation yields \( t = (1/2)(y + 1) \). Substituting into the first gives \( x = (1/4)(y + 1)^2 + 1 \), or \( 4(x - 1) = (y + 1)^2 \), which is the equation of a parabola having vertex \((1, -1)\) and which has \( x = 1 \) as an axis of symmetry. The velocity is \( \mathbf{v} = < 2t, 2 > \), and the acceleration is \( \mathbf{a} = < 2, 0 > \). Evaluating these at \( t = 1/2 \), shows that the velocity vector is \( < 1, 2 > \) and the acceleration vector is \( < 2, 0 > \).

Section 10.1 # 6: Determine the velocity and acceleration vectors of
\[ r(t) = (4 \cos \frac{t}{2}) \mathbf{i} + (4 \sin \frac{t}{2}) \mathbf{j} \]
when \( t = \pi \) and when \( t = 3\pi/2 \). Sketch the curve parametrized by \( r(t) \) and sketch the velocity and accelerations vectors on this curve so that the tail of the vectors coincides with the point on the curve at the corresponding values of \( t \).

Solution: Let \( x = 4 \cos (t/2) \) and \( y = 4 \sin (t/2) \). Then \( (x/4)^2 + (y/4)^2 = 1 \). So, \( r(t) \) traces the circle \( x^2 + y^2 = 16 \). The curve is traced in a counter-clockwise direction. The velocity is given by \( \mathbf{v}(t) = < -2 \sin (t/2), 2 \cos (t/2) > \) and the acceleration vector is given by \( \mathbf{a}(t) = < -\cos (t/2), -\sin (t/2) > \).
Therefore, when \( t = \pi \), the position vector is equal to \(< 0, 4 >\), i.e. the particle is at the point \((0, 4)\) on the circle, the velocity vector is \(< -2, 0 >\), i.e. motion is in the direction of \(-\mathbf{j}\) with speed 2, and the acceleration vector is \(< 0, -1 >\), i.e. acceleration is directed towards the center of the circle (as might be expected if the particle is moving at a constant speed—but not if the particle is moving at a variable speed).

When \( t = \frac{3\pi}{2} \), the position vector is \(< \frac{4}{\sqrt{2}}, \frac{4}{\sqrt{2}} >\), the velocity vector is \(< -\frac{2}{\sqrt{2}}, -2\sqrt{2} >\), and the acceleration vector is \(< \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} >\).

A sketch of the above will show two points on a circle together with velocity vectors tangent to the circle and in the direction of a particle moving in a counter-clockwise direction, and both points have acceleration vectors directed towards the center of the circle.

That the acceleration vectors are directed towards the center is not a coincidence, since it is easy to check that the particle moves at a constant speed: \( ||\mathbf{v}(t)|| = 2 \). Moreover, as \( t \) varies from 0 to \( 4\pi \), a particle completes one revolution. Since \( 2 \times 4\pi = 8\pi \) is the distance traveled over one revolution, this ought to be the circumference of the circle. Is it?

Section 10.1 # 8: As in the exercise #6 above, except use \( t = -1, 0, \) and 1 for the vector-valued function

\[
\mathbf{r}(t) = t \mathbf{i} + (t^2 + 1) \mathbf{j}.
\]

Solution: This problem is very similar to the above. Answers:

When \( t = -1 \), the particle has position \(< -1, 1 >\), velocity \(< 1, -2 >\), and acceleration \(< 0, 2 >\).

When \( t = 0 \), the particle has position \(< 0, 1 >\), velocity \(< 1, 0 >\) and acceleration \(< 2, 0 >\).

When \( t = 1 \), the particle has position \(< 1, 2 >\), velocity \(< 1, 2 >\) and acceleration \(< 2, 0 >\).

In fact, the particle has constant acceleration \( \mathbf{a}(t) =< 2, 0 >\). The speed, naturally, is not constant. Indeed, a sketch shows that the velocity and acceleration vectors are not orthogonal, rather the angle between these
vectors changes as \( t \) changes. However, it is always the case that the velocity vector is tangent to the curve.

Section 10.1 # 16: Find the angle between the velocity and acceleration vectors of the vector-valued function below when \( t = 0 \):

\[
\mathbf{r}(t) = \frac{\sqrt{2}}{2} t \mathbf{i} + (\frac{\sqrt{2}}{2} t - 16 t^2) \mathbf{j}.
\]

Solution: The velocity at \( t = 0 \) is equal to \( < \sqrt{2}/2, \sqrt{2}/2 > \). The acceleration at \( t = 0 \) is equal to \( < 0, -32 > \). The angle between these two vectors is clearly equal to 135° or \( 3\pi/4 \). In case you should encounter two vectors \( \mathbf{a} \) and \( \mathbf{b} \) such that the angle between them is not geometrically apparent, remember to use what I like to call the “physicist’s definition of the dot product”:

\[
\|\mathbf{a}\||\|\mathbf{b}\| \cos \theta = \mathbf{a} \cdot \mathbf{b}.
\]

Section 10.1 # 20: Determine parametric equations for the line tangent to the curve traced by the vector-valued function below when \( t = 4\pi \).

\[
\mathbf{r}(t) = 2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + 5t \mathbf{k}
\]

Solution: The strategy is to use the fact that the velocity vector at \( t = 4\pi \) is tangent to the curve at the point determined by \( \mathbf{r}(4\pi) \). Since the velocity is equal to \( \mathbf{v}(t) = < 2 \cos t, -2 \sin t, 5 > \), the vector \( \mathbf{v}(4\pi) = < 2, 0, 5 > \) is tangent to the curve at the point whose position vector is \( \mathbf{r}(4\pi) = < 0, 2, 20\pi > \). Therefore, a vector-valued function which traces the tangent line is the following:

\[
\mathbf{g}(t) = \mathbf{r}(4\pi) + t \mathbf{v}(4\pi) = < 0, 2, 20\pi > + t < 2, 0, 5 >, \quad -\infty < t < \infty.
\]

This can be expressed equivalently in terms of three parametric equations:

\[
x = 0 + 2t, \quad y = 2 + 0t, \quad z = 20\pi + 5t, \quad \text{where } t \in (-\infty, \infty).
\]

Section 10.1 # 22: As in exercise # 20 above, except \( t = \pi/2 \) and

\[
\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \sin 2t \mathbf{k}.
\]
Solution: At $t = \pi/2$ the position vector is $<0,1,0>$ and (as is easily checked) the velocity vector is $<-1,0,-2>$. Therefore, the tangent line is parametrized by $x = -t$, $y = 1$, and $z = -2t$, where $t \in (-\infty, \infty)$.

Section 10.3 # 2: Determine a vector-valued function whose value is the unit tangent vector to the curve below. Also, determine the length of this curve as $t$ ranges from 0 to $\pi$.

$$r(t) = 6 \sin 2t \hat{i} + 6 \cos 2t \hat{j} + 5t \hat{k}$$

Solution: To determine the unit tangent vector, compute the velocity vector and divide by its length. The velocity is $v(t) = <12 \cos 2t, -12 \sin 2t, 5>$ and $\|v(t)\| = \sqrt{144 \cos 2t + 144 \sin 2t + 25}$ $= \sqrt{144 + 25} = 13$. Therefore, the unit tangent vector $T(t)$ is given by

$$T(t) = \frac{12}{13} \cos 2t \hat{i} - \frac{12}{13} \sin 2t \hat{j} + \frac{5}{12} \hat{k}.$$ 

The length of the curve is determined by integrating the length of the velocity vector over the given range (in this case $t$ ranges from 0 to $\pi$):

$$\text{length} = \int_{0}^{\pi} \|v(t)\| \, dt = \int_{0}^{\pi} 13 \, dt = 13\pi.$$ 

Section 10.3 # 4: As in exercise # 2 above, except $t$ ranges from 0 to 3 and

$$r(t) = (2 + t) \hat{i} - (t + 1) \hat{j} + t \hat{k}.$$ 

Solution: The velocity vector is $v(t) = <1, -1, 1>$, and its length is $\sqrt{3}$. The length over the range $t \in [0, 3]$ is therefore equal to $3\sqrt{3}$. The unit tangent vector is $T(t) = \frac{1}{\sqrt{3}} \hat{i} - \frac{1}{\sqrt{3}} \hat{j}$.

Section 10.3 # 8: As in the previous two exercises, except $t$ ranges from $\sqrt{2}$ to 2 and

$$r(t) = (t \sin t + \cos t) \hat{i} + (t \cos t - \sin t) \hat{j}.$$
Solution: The velocity vector is \( \mathbf{v}(t) = <t \cos t, t \sin t> \); you will want to write this out (don’t forget to use the product rule). The length of the velocity vector is \( \sqrt{(t \cos t)^2 + (t \sin t)^2} = |t| \). Since \( t > 0 \) over the given range, the length is the integral from \( \sqrt{2} \) to 2 of \( t \) with respect to \( t \).

Answer: 1. The unit tangent vector is \( \mathbf{T}(t) = <\cos t, \sin t> \) if \( t > 0 \) The unit tangent vector is the negative of this vector if \( t < 0 \) because \( t/|t| = -1 \) if \( t < 0 \). And the unit tangent vector does not exist if \( t = 0 \) since the curve is not smooth at the point where \( t = 0 \). Have you tried plotting this curve on Wolfram Alpha? I have suggested a different range of values:

\[
\text{ParametricPlot[}\{t \cdot \text{Sin}[t] + \text{Cos}[t], t \cdot \text{Cos}[t] - \text{Sin}[t]\}, \{t, 0, 16\}]\]

Section 10.3 # 15: Determine the arc length of the curve traced by the vector-valued function below between the points \((0, 0, 1)\) and \((\sqrt{2}, \sqrt{2}, 0)\).

\[
\mathbf{r}(t) = \sqrt{2} t \mathbf{i} + \sqrt{2} t \mathbf{j} + (1 - t^2) \mathbf{k}
\]

Solution: The velocity vector is \( \mathbf{v}(t) = <\sqrt{2}, \sqrt{2}, -2t> \). The length of the velocity vector is \( ||\mathbf{v}(t)|| = \sqrt{2^2 + 2^2 + (-2t)^2} = 2 \sqrt{1 + t^2} \). Since \( \mathbf{r}(0) = <0, 0, 1> \) and \( \mathbf{r}(1) = <\sqrt{2}, \sqrt{2}, 0> \), the length is computed by \( \int_0^1 2 \sqrt{1 + t^2} \, dt \). To compute this integral, let \( t = \tan \theta \) so that \( dt = \sec^2 \theta \, d\theta \). When \( t = 0, \theta = 0 \), and when \( t = 1, \theta = \pi/4 \). Thus, after some simplification, one determines that the length is equal to

\[
\int_0^{\pi/4} \sec^3 \theta \, d\theta.
\]

This is a very challenging integral. One method to compute this is to use integration by parts. See example 6 in section 5.2 for details. Answer: \( \sqrt{2} + \ln (1 + \sqrt{2}) \).

Additional Problem: Write down a parametrization of the straight line segment which joins the point \((1, 2, 3)\) to the point \((a, b, c)\) using a parameter \( t \) which ranges from \( 0 \leq t \leq 1 \). (Hint: If you think in terms of vectors, this is conceptually easier.) Then write down a second parametrization of this line segment using a parameter \( s \) which ranges from \( 0 \leq s \leq L \), where \( L \) is the length of the line segment. (The numbers \( a, b, \) and \( c \) are constants.)
Solution: The pair of points determines a vector \( v \) directed from the point \((1, 2, 3)\) to the point \((a, b, c)\).

\[
v = (a - 1) \mathbf{i} + (b - 2) \mathbf{j} + (c - 3) \mathbf{k}
\]

A vector-valued function which traces the line segment is

\[
r(t) = (i + 2j + 3k) + t v, \quad 0 \leq t \leq 1.
\]

Thus, the line segment is parametrized by the following equations:

\[
x = 1 + (a - 1)t, \quad y = 2 + (b - 2)t, \quad z = 3 + (c - 3)t,
\]

where \( 0 \leq t \leq 1 \).

To parametrize the line with respect to arclength, first compute the distance from \((1, 2, 3)\) to \((a, b, c)\) and call this value \( L \):

\[
L = \sqrt{(a - 1)^2 + (b - 2)^2 + (c - 3)^2}.
\]

The goal is to write a parametrization \( r(s) = x(s) \mathbf{i} + y(s) \mathbf{j} + z(s) \mathbf{k} \) such that \( r(0) = \langle 1, 2, 3 \rangle \) and \( r(L) = \langle a, b, c \rangle \) and, more generally, \( r(s) \) is the point on the line segment which lies at distance \( s \) from the point \((1, 2, 3)\). Since the curve is a line segment, the equations should still be linear. The following equations work:

\[
x(s) = x_0 + (L-s)x_1 = 1+(L-s)a, \quad y(s) = 2+(L-s)b, \quad z(s) = 3+(L-s)c,
\]

where \( 0 \leq s \leq L \).

In general, if \( r(t) \) is a smooth parametrization (which means that in addition to being a continuously differentiable parametrization that \( r'(t) = v(t) \) is never the zero vector), then it is possible (in theory) to reparametrize the same curve using the arclength parameter \( s \), where

\[
s(t) = \int_a^t \|v(u)\| \, du,
\]

and where \( r(t) \) is defined for \( a \leq t \leq b \). It is clear from the above that \( s(b) = L \), the length of the curve. Moreover, \( s(\tau) \) is the length of the curve between \( t = a \) and \( t = \tau \). Using the fundamental theorem of calculus, \( s'(t) = \|v(t)\| \), which is assumed to be non-zero. So \( s'(t) > 0 \) for all \( t \in [a, b] \). Therefore, \( s(t) \) is an increasing function, which means that there
is an inverse function, which we might write as $t(s)$. This allows for a new parametrization $r(t(s))$, where $0 \leq s \leq L$. This is called the archlength parametrization of the curve. It is in many respects the most natural parametrization of the curve. However, this parametrization can be difficult to write down in practice.

Here’s an interesting problem: what is the arc length parametrization of the circle of radius $a > 0$ having center at the origin?

If you would like to explore this topic further, the remainder of chapter 10 is suitable for self-study over the summer time.