Solutions to Homework 4

1. The following formulas are easy to remember due to their similarity:

\[
\int \sin^2 x \, dx = \frac{1}{2} (x - \sin x \cos x) + C
\]

\[
\int \cos^2 x \, dx = \frac{1}{2} (x + \sin x \cos x) + C.
\]

Show that the above formulas are true. (Hint: You may want to use more than one trigonometric identity.)

**Solution:** Use the half-angle identity: \( \sin^2 x = \frac{1}{2} (1 - \cos 2x) \). The integral of the right-hand side is equal to \( \frac{1}{2} (x - \frac{1}{2} \sin 2x) \). Now apply the double-angle identity: \( \sin 2x = 2 \sin x \cos x \) to the previous expression. The result is the first formula in this problem. To obtain the second formula, do the same procedure except begin with the half-angle identity: \( \cos^2 x = \frac{1}{2} (1 + \cos 2x) \).

2. In this problem you will investigate Wallis’ product representation for \( \pi/2 \).

   (a) Use integration by parts to show that if \( m \geq 2 \) is an integer, then

   \[
   \int_0^{\pi/2} \sin^m x \, dx = (m - 1) \int_0^{\pi/2} (\sin^{m-2} x)(\cos^2 x) \, dx
   \]

   (Hint: Let \( dv = \sin x \, dx \).)

   **Solution:** Let \( u = \sin^{m-1} x \) and let \( dv = \sin x \, dx \). Then \( du = (m - 1)\sin^{m-2} x \cos x \, dx \) (don’t forget the chain rule!) and \( v = -\cos x \). Integration by parts yields

   \[
   \int_0^{\pi/2} \sin^m x \, dx = [-\sin^{m-1} x \cos x]_0^{\pi/2} + (m-1) \int_0^{\pi/2} (\sin^{m-2} x)(\cos^2 x) \, dx
   \]

   The term after the equal sign evaluates to zero since \( m - 1 > 0 \) and \( \sin (0) = 0 \) and \( \cos (\pi/2) = 0 \). The remaining terms comprise the desired formula.
(b) Use part (a) and the identity $\cos^2 x + \sin^2 x = 1$ to deduce that

$$\int_0^{\pi/2} \sin^m x \, dx = \frac{m-1}{m} \int_0^{\pi/2} \sin^{m-2} x \, dx.$$ 

**Solution:** Replace $\cos^2 x$ by $1 - \sin^2 x$ in the formula of part (a). Expand the integrand and write this as a sum of two integrals:

$$\int_0^{\pi/2} \sin^m x \, dx =
\left( m - 1 \right) \int_0^{\pi/2} (\sin^{m-2} x) \, dx - \left( m - 1 \right) \int_0^{\pi/2} (\sin^m x) \, dx.$$ 

Group like terms and solve: add $\left( m - 1 \right) \int_0^{\pi/2} \sin^m x \, dx$ to both sides and then divide both sides by $m$. The result is the desired formula.

(c) Let $J_m = \int_0^{\pi/2} \sin^m x \, dx$. Thus, the formula in part (c) may be rewritten as follows:

**if** $m \geq 2$, **then** $J_m = \frac{m-1}{m} J_{m-2}.$

Use part (b) deduce that

$$J_1 = 1, \quad J_3 = \frac{2}{3} \cdot 1, \quad \ldots, \quad J_{2k+1} = \frac{2k - 2}{2k + 1} \cdot \frac{2}{3} \cdot 1,$$

and that

$$J_2 = \frac{1}{2} \cdot \frac{\pi}{2}, \quad J_4 = \frac{3}{4} \cdot \frac{\pi}{2}, \quad \ldots, \quad J_{2k} = \frac{2k - 3}{2k} \cdot \frac{3}{4} \cdot \frac{\pi}{2}$$

(Hint: First compute $J_1$ and $J_2$ directly, and then use the formula in part (b).)

(Remark: The above shows that the values of $J_m$ are determined according to whether $m$ is even or odd. You should think of these two cases as $m = 2k$ and $m = 2k + 1$, where $k$ is a positive integer.)

**Solution:** The definition of $J_m$ makes sense for any integer $m$. If $m = 0$, then $J_0 = \int_0^{\pi/2} \sin^0 x \, dx = \int_0^{\pi/2} \, dx = \pi/2$. If $m = 1$, then $J_1 = \int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2} = 1$. 
Now, having shown that $J_0 = \frac{\pi}{2}$ and $J_1 = 1$, we can use the formula in (b) to determine $J_2$ and $J_3$:

$$J_2 = \frac{2 - 1}{2} J_0 = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$J_3 = \frac{3 - 1}{3} J_1 = \frac{2}{3} \cdot 1$$

If we write a few more terms, we can see that this pattern continues and yields the formulas as claimed.

**Remark:** A more formal argument uses the principle of mathematical induction. Here is how such an argument would go. We have shown that the formulas hold true for $J_0$, $J_1$, $J_2$, and $J_3$. Suppose that the formulas hold true for $J_{2k-2}$ and $J_{2k-1}$. Then by applying the formula from (b), we must agree that $J_{2k} = \frac{2k-1}{2k} J_{2k-2}$ and that $J_{2k+1} = \frac{2k+1-1}{2k+1} J_{2k-1}$. Now substitute the formulas that we have supposed to hold true for $J_{2k-2}$ and $J_{2k-1}$. The result is the formulas for $J_{2k}$ and $J_{2k+1}$. Thus, we have shown that if the formulas are known to hold for a previous stage, then they must hold true for the next stage. Since the formulas do hold true for the initial stages ($J_0$, $J_1$, $J_2$, and $J_3$), the principle of mathematical induction implies that these formulas hold true for every integer $m$ greater than or equal to zero.

(d) Explain why the following is true: if $c$ is a positive number that is less than one, then for any positive integer $m$,$$c^{m+1} < c^m < c^{m-1}.$$**Solution:** Positive real numbers have the following property: if $a < b$ and $0 < c$, then $ac < ab$.

Now suppose that $c < 1$. Multiply both sides by $c$. The result is that $c^2 < c$. Together with the statement that $c < 1$, we have that $c^2 < c < 1$. Now multiply this inequality by $c$: $c^3 < c^2 < c$.

If I multiply both sides by $c$ again and again ($(m - 2)$-times to be precise), then the result is the desired inequality.

**Remark:** As in the previous step, this argument could be formalized by appealing to the principle of mathematical induction. Can you do it? (You will start using the principle of mathematical induction regularly in class and on exams and...
homework once you begin to study 300-level mathematics courses.

(e) If $0 < x < \pi/2$, then $0 < \sin x < 1$. Use this observation and part (d) to deduce that $J_{2k+1} < J_{2k} < J_{2k-1}$.

**Solution:** If $f(x) < g(x)$ for every $x \in (a, b)$, then $\int_a^b f(x) \, dx < \int_a^b g(x) \, dx$. (This is geometrically clear for non-negative functions, but it also holds true for arbitrary functions as can be shown using Riemann sums; regardless, it is a property of integrals that you are (or are now) aware of.) Since $\sin x < 1$ for $x \in (0, \pi/2)$, we have, by part (d), that $\sin^{2k+1} x < \sin^{2k} x < \sin^{2k-1} x$ for $x \in (0, \pi/2)$. Integrate these three quantities. By the aforementioned property of integrals, the integrals of these quantities are ordered the same. Therefore $J_{2k+1} < J_{2k} < J_{2k-1}$.

(f) Use part (e) to show that
$$1 < \frac{J_{2k}}{J_{2k+1}} < \frac{J_{2k-1}}{J_{2k+1}}$$

**Solution:** Divide the previous inequality by the positive (this is important!) quantity $J_{2k+1}$.

(g) Use part (c) and part (f) to deduce that for each positive integer $k$,
$$1 < \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots (2k-1)(2k+1)}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2k)(2k)} \cdot \frac{\pi}{2} < \frac{2k+1}{2k}$$

**Solution:** Substitute the formulas from part (c) and simplify.

(h) Finally, use part (f) and the squeeze theorem to deduce that
$$\lim_{k \to \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2k)(2k)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots (2k-1)(2k+1)} = \frac{\pi}{2}$$

**Solution:** Multiply the previous inequality by $2/\pi$. The left-most quantity is $\frac{2}{\pi}$. The right-most quantity is $\frac{2}{\pi} \cdot \frac{2k+1}{k}$; this limits to $\frac{2}{\pi}$ as $k$ tends to infinity. Therefore, by the squeeze theorem, the middle term also limits to $\frac{2}{\pi}$. Finally, take the reciprocal.