1. Use a cross section method (e.g. disk, washer, or shell method – your choice) to demonstrate that the volume of a right circular cone of radius $r$ and height $h$ is equal to $\frac{1}{3}\pi r^2 h$.

**Solution:** Consider the figure below.

![Diagram](image)

The region shown is bounded by the $x$ and $y$ axes and the line having slope $-r/h$ and $y$-intercept $r$, i.e. the line having equation $y = -(r/h)x + r$. If we rotate this region about the $x$-axis we obtain a solid of revolution which is a solid right circular cone of radius $r$ and height $h$. The volume of this solid is computed using the disk method:

$$
\int_0^h \pi \left[ -\frac{r}{h}x + r \right]^2 \, dx
$$

$$
= \pi \int_0^h \left[ \frac{r^2}{h^2}x^2 - 2\frac{r^2}{h}x + r^2 \right] \, dx
$$

Since $r$ and $h$ are constants, the above is easily computed. It simplifies to $(1/3)\pi r^2 h$.

2. Suppose that $S$ is a planar region enclosed by a square having an edge of length $l$, that $C$ is a planar region enclosed by a circle having radius $r$, and that $H$ is a planar region enclosed by a regular hexagon having an edge of length $l$. In each case, compute the ratio of the area of the region to the area of the region obtained by scaling it by a factor of $c$, where $c > 0$. For example, if $c = 1/2$, then to scale $S$ by a factor of $c$ means that its edges now have length $l/2$. Sketch an
example in each case (square, circle, regular hexagon) of such a region and scaled copy of each region. Your sketch should be accurate (measure the distances) and you should state which value of c you chose for your sketches.

Suppose that R is a planar figure enclosing a finite area. What do you conjecture is the ratio of the area of R to the area of the figure obtained by scaling R by a factor of c, where c > 0?

**Solution:** If S is a square whose edges have length l, then when we scale the figure by a factor of c, the scaled figure has edges of length cl. Thus, the ratio of the area of the region to the area of the scaled region is $l^2 : (cl)^2$ which simplifies to $1 : c^2$. Thus, area is scaled by a factor of $c^2$.

For C, the scaled figure has radius cr. So, the ratio of the original area to the scaled area is $\pi r^2 : \pi (cr)^2$ which simplifies again to $1 : c^2$.

For H, the area can be computed by dividing the hexagon into six equilateral triangles each having sides of length l. The area of one such triangle is $(1/2) \times l \times \sqrt{3}l = \frac{\sqrt{3}}{2}l^2$ (as can be seen by dropping an altitude and using the resulting 30-60-90 triangle whose side opposite the 30 degree angle is $l/2$. Thus, the area of hexagon is six times this or $3\sqrt{3}l^2$. The area of the scaled hexagon is $3\sqrt{3}(cl)^2$. The ratio of the original to the scaled simplifies to $1 : c^2$.

It is reasonable to conjecture that that the above observations hold true for any planar figure which encloses a region of finite area: the ratio of the area of the original region to the area of the region scaled by a factor of c is equal to $1 : c^2$. In other words, scaling by a factor of c results in area being scaled by a factor of $c^2$.

**Notes:** The difficulty in proving such a conjecture is that we would need to precisely define what qualifies as a “planar figure which encloses a finite area” and we would need to define precisely what is meant by “scaling a figure by a factor of c”. These are surprisingly challenging difficulties. For instance if X is a subset of the plane, what is meant by its area? Using calculus we have defined the area of subsets of the plane which are enclosed by sufficiently well-behaved curves. But since the notion of area is fundamental to geometry we might hope to assign some numerical measure of area to any region in the plane. For further reading, include some relatively modern mathematics (second link) check out:
3. Let \( R \) be the triangular region in the plane enclosed by the lines \( y = 0 \), \( y = x - 1 \) and \( y = 3 - x \). Let \( S \) be the solid of revolution generated by rotating the region \( R \) about the \( y \)-axis. Compute the volume of \( S \) in two different ways: once using the washer method and once using the shell method. Which of these two methods seems more efficient? Why? Finally, compute the volume in a third way by applying the Theorem of Pappus (Theorem 7.1 in Section 7.6); you do not need to show any work for the computation of the centroid—just state what it is—the answer is geometrically obvious.

**Solution:** The region \( R \) is shown below.

\[
\text{http://en.wikipedia.org/wiki/Area}
\]

and

\[
\text{http://en.wikipedia.org/wiki/Jordan_measure}
\]

If we consider cross sections parallel to the axis of rotation (vertical cross sections in this case since the axis of rotation is the \( y \)-axis), then the cross sections of \( S \) are cylindrical shells each having area \( 2\pi rh \), where the radius \( r \) and height \( h \) are determined by the figure. The thickness of the cross sections is \( dx \); so all formulas should be in terms of \( x \). The radius of a is equal to the \( x \)-coordinate. The height of a cross section, however, is sometimes equal to \( y = x - 1 \) and sometimes equal to \( y = 3 - x \). Two integrals are required:

\[
\text{volume}(S) = \int_1^2 2\pi x(x-1) \, dx + \int_2^3 2\pi x(3-x) \, dx.
\]
The above integrals are easy to compute. Answer: $4\pi$.

If we consider cross sections perpendicular to the axis of rotation (horizontal cross sections in this case), then the cross sections of $S$ are washers of thickness $dy$. The larger radius is equal to $x = 3 - y$ (solving for $x$ in $y = 3 - x$) and the smaller radius is equal to $x = y + 1$ (solving for $x$ in $y = x - 1$). Since the area of a washer of large radius $R$ and small radius $r$ is equal to $\pi R^2 - \pi r^2$, the volume of $S$ is equal to

$$\pi \int_0^1 ((3 - y)^2 - (y + 1)^2) \, dy = 4\pi.$$

Since the washer method requires only one integral, this method seems more efficient for this particular problem. Another way to solve the problem is to apply Pappus’s Theorem. The centroid of the region $R$ lies on the line $x = 2$ since the triangular region is clearly symmetric about this line. Therefore, the distance from the $y$-axis to the centroid is equal to 2. The area of the triangular region is $(1/2)(2)(1)$ (the base and height are easy to determine). By the Theorem of Pappus, the volume of the solid of revolution is $(2\pi)(\text{distance to centroid})(\text{area of cross section})$, so this equals $(2\pi)(2)(1) = 4\pi$.

4. Let $a$ and $c$ be positive constants. Let $L$ be the locus of points in the plane whose coordinates $(x, y)$ satisfy $y = c\sqrt{x}$ and $0 \leq x \leq a$. Determine the surface area of the surface of revolution generated by revolving the curve $L$ about the $x$-axis.

**Solution:** The surface area is computed by adding up the surface areas of cross sections of the surface. If these cross sections are perpendicular to the $x$-axis, then each such cross section has surface area $2\pi y \sqrt{1 + (y')^2} \, dx$. These are summed (integrated) to compute the surface area:

$$\text{area} = \int_a^b 2\pi y \sqrt{1 + (y')^2} \, dy.$$

$$1 + (y')^2 = 1 + \left(\frac{c}{2\sqrt{x}}\right)^2 = 1 + \frac{c^2}{4x}$$
Thus, we are to compute the following integral:

$$2\pi \int_0^a (c\sqrt{x})\sqrt{1 + \frac{c^2}{4x}} \, dx$$

Combine the square roots to obtain the following:

$$2\pi c \int_0^a \sqrt{x(1 + \frac{c^2}{4x})} \, dx = 2\pi c \int_0^a \sqrt{x + \frac{c^2}{4}} \, dx$$

If we let $u = x + \frac{c^2}{4}$, then the integral is easy to compute. The answer is somewhat awkward:

$$\frac{4\pi}{3} c \left[ \left(a + \frac{c^2}{4}\right)^{3/2} - \frac{c^3}{8} \right].$$

**Notes:** The origin of the formula for surface area is that the surface area of a slice (cut parallel to the base of the cone) of a right circular cone is equal to $\pi(r_1 + r_2)s$, where $r_1$ is the radius of the smaller circle of the slice, $r_2$ is the radius of the larger circle of the slice, and $s$ is the slant height (the length of the slanted part of the slice). If the slice is very thin then $r_1$ and $r_2$ are approximately the same value, say $r$, and the slant height is a small part of the arclength, written $ds$, of the side of the cone; thus it is that the surface area of small slice of a cone is equal to $2\pi r \, ds$.

5. Suppose that a lamina, i.e. a thin plate, is described as the region $R$ in the plane enclosed by a circle of radius $r$ centered at the origin. Suppose further that the density of the lamina is given as a function of the $x$-coordinate of the points in $R$ by the function $\delta(x) = 1 + |x|$. Compute the mass of the lamina. Is it necessary in this case to first compute the mass if your objective was to determine the center of mass of the lamina? Explain.

**Solution:** The center of mass of the lamina clearly lies somewhere on the $y$-axis because of the symmetry of the figure and because the density is symmetric about the $y$-axis (since $\delta(x)$ is an even function). So, if we were only trying to compute $\bar{x}$, there would be no need to compute the mass; we could simply state that for the reasons above (basically, symmetry) it is geometrically clear that $\bar{x} = 0$. 

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To compute the mass we integrate the density over the region. To do this, the region is sliced into thin vertical strips. Each such strip lies above some $x$-coordinate; the mass of such a strip is equal to its area times the density at its $x$-coordinate. The total mass is found by integrating (summing) these values:

$$
\text{mass} = \int_{-r}^{r} \delta(x) (\text{height of the slice at } x) \, dx
$$

Since the equation of the circle is $x^2 + y^2 = r^2$, the height is equal to twice $y = \sqrt{r^2 - x^2}$. Thus,

$$
\text{mass} = \int_{-r}^{r} (1 + |x|)(2\sqrt{r^2 - x^2}) \, dx
$$

To deal with the absolute value, we use symmetry: the integral is equal to twice the integral from 0 to $r$ since the integrand is an even function; since for value of $x$ between 0 and $r$ the $x$ values are nonnegative, we can drop the absolute value sign:

$$
\text{mass} = 4 \int_{0}^{r} (1 + x) \sqrt{r^2 - x^2} \, dx
$$

The first integral can be computed geometrically:

$$
\int_{0}^{r} \sqrt{r^2 - x^2} \, dx = \frac{1}{4} \pi r^2
$$

since it represents the area of one fourth of a circle. The second integral can be computed using the substitution $u = r^2 - x^2$ so that $du = -2x \, dx$ (since $r^2$ is a constant).

Check that the final answer to this problem is equal to the following:

$$
\text{mass} = \pi r^2 + \frac{4}{3} r^3.
$$