Directions. Please work on the problems below. Your solutions must begin with a clear statement (or re-statement in your own words) of the problem. You solutions should be clear, legible, and demonstrate at minimum partial progress towards a complete solution to the problem. Please refer to the syllabus for the policy on grading (communication, completeness, and correctness) and late homework (late homework is not accepted; homework is collected during class).

Collaboration. I encourage you to discuss the homework problems with your classmates. However, each student must write and submit his or her own homework solutions.

Grading: Only the first problem and each of parts (a), (b), and (c) of the second problem will count towards your grade for this homework assignment. (So, if you make a strong attempt at each of these, you earn full points for “completeness”.) The remaining parts of the second problem are worth extra credit (to a maximum of 2 full points) towards your homework total.

1. The following formulas are easy to remember due to their similarity:

$$\int \sin^2 x \, dx = \frac{1}{2}(x - \sin x \cos x) + C$$

$$\int \cos^2 x \, dx = \frac{1}{2}(x + \sin x \cos x) + C.$$ 

Show that the above formulas are true. (Hint: You may want to use more than one trigonometric identity.)

2. In this problem you will investigate Wallis’ product representation for $\pi/2$.

(a) Use integration by parts to show that if $m \geq 2$ is an integer, then

$$\int_0^{\pi/2} \sin^m x \, dx = (m - 1) \int_0^{\pi/2} (\sin^{m-2} x)(\cos^2 x) \, dx$$

(Hint: Let $dv = \sin x \, dx.$)
(b) Use part (a) and the identity \( \cos^2 x + \sin^2 x = 1 \) to deduce that
\[
\int_0^{\pi/2} \sin^m x \, dx = \frac{m-1}{m} \int_0^{\pi/2} \sin^{m-2} x \, dx.
\]

(c) Let \( J_m = \int_0^{\pi/2} \sin^m x \, dx \). Thus, the formula in part (c) may be rewritten as follows:

if \( m \geq 2 \), then \( J_m = \frac{m-1}{m} J_{m-2} \).

Use part (b) deduce that
\[
J_1 = 1, \quad J_3 = \frac{2}{3} \cdot 1, \quad \ldots, \quad J_{2k+1} = \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \ldots \frac{2}{3} \cdot 1,
\]
and that
\[
J_2 = \frac{1}{2} \cdot \frac{\pi}{2}, \quad J_4 = \frac{3}{4} \cdot \frac{\pi}{2}, \quad \ldots, \quad J_{2k} = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \ldots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}.
\]

(Hint: First compute \( J_1 \) and \( J_2 \) directly, and then use the formula in part (b).)

(Remark: The above shows that the values of \( J_m \) are determined according to whether \( m \) is even or odd. You should think of these two cases as \( m = 2k \) and \( m = 2k + 1 \), where \( k \) is a positive integer.)

(d) Explain why the following is true: if \( c \) is a positive number that is less than one, then for any positive integer \( m \), \( c^{m+1} < c^m < c^{m-1} \).

(e) If \( 0 < x < \pi/2 \), then \( 0 < \sin x < 1 \). Use this observation and part (d) to deduce that \( J_{2k+1} < J_{2k} < J_{2k-1} \).

(f) Use part (e) to show that
\[
1 < \frac{J_{2k}}{J_{2k+1}} < \frac{J_{2k-1}}{J_{2k+1}}.
\]

(g) Use part (c) and part (f) to deduce that for each positive integer \( k \),
\[
1 < \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k-1)(2k+1)}{2 \cdot 2 \cdot 4 \cdot 6 \cdots (2k)(2k)} \cdot \frac{\pi}{2} < \frac{2k + 1}{2k}.
\]
(h) Finally, use part (f) and the squeeze theorem to deduce that

$$\lim_{k \to \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2k)(2k)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots (2k - 1)(2k + 1)} = \frac{\pi}{2}$$

**Reminder:** You should be working through the recommended textbook exercises. A list of these problems can be found on our course web page: