Math 421 / Homework 9.4

 # 3 Suppose that A is open in $\mathbf{R}^n$ and $\mathbf{f}\colon A \to \mathbf{R}^m$. Prove that $\mathbf{f}$ is continuous on A if and only if $\mathbf{f}^{-1}(V)$ is open in $\mathbf{R}^n$ for every open subset V of $\mathbf{R}^m$.

 # 4 Suppose that A is closed in $\mathbf{R}^n$ and $\mathbf{f}\colon A \to \mathbf{R}^m$. Prove that $\mathbf{f}$ is continuous on A if and only if $\mathbf{f}^{-1}(E)$ is closed in $\mathbf{R}^n$ for every closed subset E of $\mathbf{R}^m$.

 # 6 Prove that

$$f(x,y) = \begin{cases} e^{-1/|x-y|} & x \neq y \\ 0 & x = y \end{cases}$$

is continuous on $\mathbf{R}^2$.

 # 7 Let H be a nonempty, closed, bounded subset of $\mathbf{R}^n$.

(a) Suppose that $\mathbf{f}\colon H \to \mathbf{R}^m$ is continuous. Prove that

$$\|\mathbf{f}\|_H := \sup_{\mathbf{x}\in H} \|\mathbf{f}(\mathbf{x})\|$$

is finite and there exists an $\mathbf{x}_0 \in H$ such that $\|\mathbf{f}(\mathbf{x}_0)\| = \|\mathbf{f}\|_H$.

(b) A sequence of functions $\mathbf{f}_k\colon H \to \mathbf{R}^m$ is said to **converge uniformly on** H to a function $\mathbf{f}\colon H \to \mathbf{R}^m$ if for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that

$$k \geq N \quad \text{and} \quad \mathbf{x} \in H \quad \text{imply} \quad \|\mathbf{f}_k(\mathbf{x}) - \mathbf{f}(\mathbf{x})\| < \epsilon.$$

Show that $\|\mathbf{f}_k - \mathbf{f}\|_H \to 0$ as $k \to \infty$ if and only if $\mathbf{f}_k$ converges uniformly to $\mathbf{f}$ on H.

(c) Prove that a sequence of functions $\mathbf{f}_k$ converges uniformly on H if and only if for every $\epsilon > 0$ there exists an $n \in N$ such that

$$k,\ j \geq N \quad \text{implies} \quad \|\mathbf{f}_k - \mathbf{f}_j\|_H < \epsilon.$$