Math 421 / Homework 5.4

 # 1 Evaluate the following improper integrals.

(a)
$$\int_1^\infty \frac{1+x}{x^3}\,dx$$

(c)
$$\int_0^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin x}}\,dx$$

(d)
$$\int_0^1 \ln x\,dx$$

 # 2 For each of the following, find all values of $p \in \mathbf{R}$ for which f is improperly integrable on I.

(a) $f(x) = 1/x^p, \;\; I = (1,\infty)$
(b) $f(x) = 1/x^p, \;\; I = (0,1)$
(d) $f(x) = 1/(1+x^p), \;\; I = (0,\infty)$

 # 7 (a) Suppose that f is improperly integrable on $(0,\infty)$. Prove that if $L = \lim_{x\to\infty} f(x)$ exists, then $L = 0$.

(b) Let
$$f(x) = \begin{cases} 1 & n \le x < n + 2^{-n},\ n \in \mathbf{N} \\ 0 & \text{otherwise.} \end{cases}$$
Prove that f is improperly integrable on $(0,\infty)$ but $\lim_{x\to\infty} f(x)$ does not exist.

 # 8 Prove that if f is absolutely integrable on $[1,\infty)$, then
$$\lim_{n\to\infty} \int_1^\infty f(x^n)\,dx = 0.$$

 # 10 (a) Prove that
$$\int_0^{\pi/2} e^{-a\sin x}\,dx \le \frac{2}{a}$$
for all $a > 0$.

(b) What happens if $\cos x$ replaces $\sin x$?