Math 421 / Homework 5.3

 # 1 Let f be continuous, find $F'(x)$ for each of the following functions.

(a)
$$F(x) = \int_{x^2}^{1} f(t)\,dt$$

(b)
$$F(x) = \int_{x^2}^{x^3} f(t)\,dt$$

(c)
$$F(x) = \int_{0}^{x\cos x} tf(t)\,dt$$

(d)
$$F(x) = \int_{0}^{x} f(t-x)\,dt$$

 # 2 Suppose that f is nonnegative and continuous on $[1,2]$ and that $\int_1^2 x^k f(x)\,dx = 5 + k^2$ for $k = 0,1,2$. Prove that each of the following statements is correct.

(a)
$$\int_1^4 f(\sqrt{x})\,dx = 12$$

(b)
$$\int_{\sqrt{2}/2}^{1} f(\frac{1}{x^2})\,dx \le \frac{5}{2}$$

(c)
$$\int_0^1 x^2 f(x+1)\,dx = 2$$

 # 6 If f is continuous on $[a,b]$ and there exist numbers $\alpha \neq \beta$ such that
$$\alpha \int_a^c f(x)\,dx + \beta \int_c^b f(x)\,dx = 0$$
holds for all $c \in (a,b)$, prove that $f(x) = 0$ for all $x \in [a,b]$.

 # 9 Suppose that $f\colon [a,b] \to \mathbf{R}$ is continuously differentiable and 1-1 on $[a,b]$. Prove that
$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a).$$