Math 421 / Homework 11.6

 # 1 For each of the following functions, prove that $\mathbf{f}^{-1}$ exists and is differentiable in some nonempty, open set containing (a,b), and compute $D(\mathbf{f}^{-1})(a,b)$.

(a) $\mathbf{f}(u,v) = (3u - v, 2u + 5v)$ at any $(a,b) \in \mathbf{R}^2$.

(c) $\mathbf{f}(u,v) = (uv, u^2 + v^2)$ at $(a,b) = (2,5)$.

 # 5 Given nonzero numbers $x_0, y_0, u_0, v_0, s_0, t_0$ which satisfy the simultaneous equations

$$(*) \qquad \begin{aligned} u^2 + sx + ty &= 0 \\ v^2 + tx + sy &= 0 \\ 2s^2x + 2t^2y - 1 &= 0 \\ s^2x - t^2y &= 0, \end{aligned}$$

prove that there exist functions $u(x,y), v(x,y), s(x,y), t(x,y)$, and an open ball B containing (x_0, y_0), such that u, v, s, t are continuously differentiable and satisfy $(*)$ on B, and such that $u(x_0,y_0) = u_0, v(x_0,y_0) = v_0, s(x_0,y_0) = s_0$, and $t(x_0,y_0) = t_0$.

 # 6 Let $E = \{(x,y) : 0 < y < x\}$ and set $\mathbf{f}(x,y) = (x+y,\ xy)$ for $(x,y) \in E$.

(a) Prove that $\mathbf{f}$ is 1-1 from E onto $\{(s,t) : s > 2\sqrt{t},\ t > 0\}$ and find a formula for $\mathbf{f}^{-1}(s,t)$.

(b) Use the Inverse Function Theorem to compute $D(\mathbf{f}^{-1})(\mathbf{f}(x,y))$ for $(x,y) \in E$.

(c) Use the formula you obtain from part (a) to compute $D(\mathbf{f}^{-1})(s,t)$ directly. Check to see that this agrees with the one you found in part (b).

 # 9 Let $F\colon \mathbf{R}^3 \to \mathbf{R}$ be continuously differentiable in some open set containing (a,b,c) with $F(a,b,c) = 0$ and $\nabla F(a,b,c) \neq 0$.

(a) Prove that the graph of the relation $F(x,y,z) = 0$; that is, the set $\mathcal{G} = \{(x,y,z) : F(x,y,z) = 0\}$, has a tangent plane at (a,b,c).

(b) Prove that a normal of the tangent plane to $\mathcal{G}$ at (a,b,c) is given by $\nabla F(a,b,c)$.

 # 11 Let $\mathcal{H}$ be the hyperboloid of one sheet, given by $x^2 + y^2 - z^2 = 1$.

(a) Use Exercise # 9 above to prove that at every point $(a,b,c) \in \mathcal{H}$, $\mathcal{H}$ has a tangent plane whose normal is given by $(-a,-b,c)$.

(b) Find an equation of each plane tangent to $\mathcal{H}$ which is perpendicular to the xy-plane.

(c) Find an equation of each plane tangent to $\mathcal{H}$ which is parallel to the plane $x + y - z = 1$.