§3.4

7\. Sol. Let $S = \{(a+b,\ a-b+2c,\ b,\ c)^T \mid a, b, c \in \mathbb{R}\}$

Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

then $S = \{a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 \mid a, b, c \in \mathbb{R}\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

that is, $\{\vec{v}_1\ \vec{v}_2\ \vec{v}_3\}$ is a spanning set of S ⋯ (*)

Moreover, $(\vec{v}_1\ \vec{v}_2\ \vec{v}_3) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\overset{R_2 - R_1}{\approx} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{\approx} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{R_1 - R_2}{\overset{R_3 \leftrightarrow R_4}{\approx}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$

$$\overset{R_4 + 2R_2 - 3R_3}{\approx} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

that is, $\vec{v}_1$, $\vec{v}_2$ & $\vec{v}_3$ are linearly independent. ⋯ (**)

by (*) & (**), we know that

$\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}\right\}$ is a basis of S

and $\dim S = 3$

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15 Sol: $P_3 = \{ax^2+bx+c \mid a, b, c \in \mathbb{R}\} = \text{span}\{1, x, x^2\}$

(a) $S = \{p(x) \in P_3 \mid p(0) = 0\}$

$= \{ax^2+bx+c \mid a, b \in \mathbb{R},\ c=0\}$

$= \{ax^2+bx \mid a, b \in \mathbb{R}\}$

$= \text{span}\{x, x^2\}$

Note that x & x^2 are linearly independent.

Therefore, $\{x, x^2\}$ is a basis of S #

(b) $T = \{p(x) \in P_3 \mid p(1) = 0\}$

$= \{ax^2+bx+c \mid a+b+c=0,\ a, b, c \in \mathbb{R}\}$

$= \{ax^2+bx-a-b \mid a, b \in \mathbb{R}\}$

$= \{a(x^2-1)+b(x-1) \mid a, b \in \mathbb{R}\}$

$= \text{span}\{x^2-1, x-1\}$

Note $ax^2+bx-a-b=0 \iff \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\iff \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\iff a=b=0$

Therefore, x^2-1 & $x-1$ are linearly independent

As a result, $\{x^2-1, x-1\}$ is a basis of T #

(c) $S \cap T = \{ p(x) \in P_3 \mid p(0) = p(1) = 0 \}$

$= \{ ax^2 + bx + c \mid c = 0, \ a+b+c = 0, \ a, b \in \mathbb{R} \}$

$= \{ ax^2 - ax \mid a \in \mathbb{R} \}$

$= \text{span}\{ x^2 - x \} = \text{span}\{ x(x-1) \}$

Thus $\{ x(x-1) \}$ is a basis of $S \cap T$ #

§ 3.5

5\. Sol: (a) The transition matrix from $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ to $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is

$$U = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}$$

$$(U \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{R_3 - R_1}{\overset{R_2 - R_1}{\approx}} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\underset{R_3 - R_2}{\overset{R_1 - R_2}{\approx}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

$$\underset{R_2 - R_3}{\overset{R_1 - R_3}{\approx}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) = (I_3 \mid U^{-1})$$

Thus the transition matrix from $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ to $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is

$$U^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

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(b) (i) $\begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

10. Sol. $(1, 1+x, 1+x+x^2) = (1, x, x^2)\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Thus the transition matrix from $\{1, 1+x, 1+x+x^2\}$ to $\{1, x, x^2\}$ is

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(U \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{R_2 - R_3}{\overset{R_1 - R_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = (I_3 \mid U^{-1})$$

As a result, the transition matrix from $\{1, x, x^2\}$ to $\{1, 1+x, 1+x+x^2\}$ is

$$U^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

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