§5.4

8 (a) sol: $\|1\| = \left(\int_0^1 1^2 dx\right)^{\frac{1}{2}} = 1$

$\|x\| = \left(\int_0^1 x^2 dx\right)^{\frac{1}{2}} = \sqrt{\frac{1}{3}}$

$\langle 1, x\rangle = \int_0^1 x\, dx = \frac{1}{2}$

Thus $\cos\theta = \frac{\langle 1, x\rangle}{\|1\|\,\|x\|} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$. #

(b) sol: $p = \frac{\langle 1, x\rangle}{\|x\|^2} x = \frac{3}{2}x$

$1-p = 1-\frac{3}{2}x$

$\langle 1-p, p\rangle = \int_0^1 (1-\frac{3}{2}x)x\, dx$

$= \left(\frac{x^2}{2} - \frac{1}{2}x^3\right)\Big|_0^1 = 0$

$\Rightarrow 1-p \perp p$ #

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9. Proof. $\langle \cos(mx), \sin(nx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\, dx$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{\sin[(m+n)x] + \sin[(n-m)x]\right\} dx$

$= 0$

$\Rightarrow \cos(mx) \perp \sin(nx)$

$\|\cos(mx)\| = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(mx)\, dx\right)^{\frac{1}{2}} = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(2mx)+1}{2} dx\right)^{\frac{1}{2}}$

$= (0+1)^{\frac{1}{2}} = 1$

$\|\sin(nx)\| = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(nx)\, dx\right)^{\frac{1}{2}} = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2} dx\right)^{\frac{1}{2}} = 1$

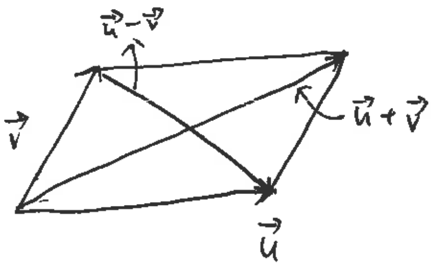
$\|\cos(mx) - \sin(nx)\| = \sqrt{\|\cos(mx)\|^2 + \|\sin(nx)\|^2} = \sqrt{1+1} = \sqrt{2}$ #

26 Proof: $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2$

$$= (\|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\langle \vec{u}, \vec{v}\rangle) + (\|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\langle \vec{u}, \vec{v}\rangle)$$

$$= 2(\|\vec{u}\|^2 + \|\vec{v}\|^2)$$

In $\mathbb{R}^2$, the equality shows that for a parallelogram, the sum of the squares of side lengths equals the sum of the squares of the lengths of the two diagonals #



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33. Proof: (a) $\langle A\vec{x}, \vec{y}\rangle = (A\vec{x})^T\vec{y} = \vec{x}^T A^T \vec{y}$

$$= \vec{x}^T (A^T\vec{y})$$

$$= \langle \vec{x}, A^T\vec{y}\rangle$$

(b) $\langle A^TA\vec{x}, \vec{x}\rangle \overset{(a)}{=} \langle A\vec{x}, (A^T)^T\vec{x}\rangle$

$$= \langle A\vec{x}, A\vec{x}\rangle$$

$$= \|A\vec{x}\|^2$$ #

§5.5

2 Sol: (a) $\|\vec{u}_1\| = [(\frac{1}{3\sqrt{2}})^2 + (\frac{1}{3\sqrt{2}})^2 + (-\frac{4}{3\sqrt{2}})^2]^{\frac{1}{2}} = (\frac{1}{18} + \frac{1}{18} + \frac{16}{18})^{\frac{1}{2}} = 1$

$\|\vec{u}_2\| = [(\frac{2}{3})^2 + (\frac{2}{3})^2 + (\frac{1}{3})^2]^{\frac{1}{2}} = (\frac{4}{9} + \frac{4}{9} + \frac{1}{9})^{\frac{1}{2}} = 1$

$\|\vec{u}_3\| = [(\frac{1}{\sqrt{2}})^2 + (-\frac{1}{\sqrt{2}})^2 + 0^2]^{\frac{1}{2}} = (\frac{1}{2} + \frac{1}{2})^{\frac{1}{2}} = 1$

$\langle \vec{u}_1, \vec{u}_2 \rangle = \frac{1}{3\sqrt{2}} \cdot \frac{2}{3} + \frac{1}{3\sqrt{2}} \cdot \frac{2}{3} - \frac{4}{3\sqrt{2}} \cdot \frac{1}{3} = \frac{1}{9\sqrt{2}}(2+2-4) = 0$

$\langle \vec{u}_1, \vec{u}_3 \rangle = \frac{1}{3\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{2}} (-\frac{1}{\sqrt{2}}) - \frac{4}{3\sqrt{2}} \cdot 0 = 0$

$\langle \vec{u}_2, \vec{u}_3 \rangle = \frac{2}{3} \cdot \frac{1}{\sqrt{2}} + \frac{2}{3} \cdot (-\frac{1}{\sqrt{2}}) + \frac{1}{3} \cdot 0 = 0$

Thus, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an <u>ON</u> basis for $\mathbb{R}^3$. #

(b) $\vec{x} = \sum_{i=1}^{3} \langle \vec{x}, \vec{u}_i \rangle \vec{u}_i$

$= (\frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} - \frac{4}{3\sqrt{2}}) \vec{u}_1 + (\frac{2}{3} + \frac{2}{3} + \frac{1}{3}) \vec{u}_2 + (\frac{1}{\sqrt{2}} + (-\frac{1}{\sqrt{2}}) + 0) \vec{u}_3$

$= -\frac{2}{3\sqrt{2}} \vec{u}_1 + \frac{5}{3} \vec{u}_2 + 0 \vec{u}_3 = -\frac{\sqrt{2}}{3} \vec{u}_1 + \frac{5}{3} \vec{u}_2$

$\|\vec{x}\| = \sqrt{(-\frac{\sqrt{2}}{3})^2 + (\frac{5}{3})^2 + 0^2} = \sqrt{\frac{2}{9} + \frac{25}{9}} = \sqrt{3}$

#

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5. Sol: $\{\vec{u}_1, \vec{u}_2\}$ is an <u>ON</u> basis of $\mathbb{R}^2 \Rightarrow \vec{u} = \langle \vec{u}, \vec{u}_1 \rangle \vec{u}_1 + \langle \vec{u}, \vec{u}_2 \rangle \vec{u}_2$

$= (\vec{u}^T \vec{u}_1) \vec{u}_1 + (\vec{u}^T \vec{u}_2) \vec{u}_2$

By Parseval's equality $\|\vec{u}\|^2 = (\vec{u}^T \vec{u}_1)^2 + (\vec{u}^T \vec{u}_2)^2$

$\vec{u}$ is a unit vector $\Rightarrow \|\vec{u}\| = 1$

$\vec{u}^T \vec{u}_1 = \frac{1}{2}$

$\Rightarrow |\vec{u}^T \vec{u}_2| = \sqrt{1 - \frac{1}{4}} = \sqrt{3}/2$

#

7. Sol: $\left.\begin{array}{l}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} \text{ is an } \underline{ON} \text{ basis} \\ \vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3\end{array}\right\}$

$$\Rightarrow \begin{cases} c_1 = \langle \vec{x}, \vec{u}_1 \rangle \\ c_2 = \langle \vec{x}, \vec{u}_2 \rangle \\ c_3 = \langle \vec{x}, \vec{u}_3 \rangle \\ \text{Parseval's equality: } \|\vec{x}\|^2 = c_1^2 + c_2^2 + c_3^2 \end{cases}$$

Note
$$\left.\begin{array}{l} \langle \vec{u}_1, \vec{x} \rangle = 4. \Rightarrow c_1 = \langle \vec{x}, \vec{u}_1 \rangle = \langle \vec{u}_1, \vec{x} \rangle = 4 \\ \vec{x} \perp \vec{u}_2 \Rightarrow c_2 = \langle \vec{x}, \vec{u}_2 \rangle = 0 \\ \|\vec{x}\| = 5 \Rightarrow c_1^2 + c_2^2 + c_3^2 = \|\vec{x}\|^2 = 25 \end{array}\right\}$$

$$\Rightarrow c_3 = \pm 3$$

Thus,
$$\begin{cases} c_1 = 4 \\ c_2 = 0 \\ c_3 = \pm 3 \end{cases}$$