§5.2

3 (a) Proof.

$$S^{\perp} = \{\vec{v} \in \mathbb{R}^3 \mid \vec{v}\cdot\vec{u} = 0 \quad \forall\, \vec{u} \in S\}$$

$$= \{\vec{v} \in \mathbb{R}^3 \mid \vec{v}\cdot\vec{x} = \vec{v}\cdot\vec{y} = 0\}$$

$$= \{\vec{v} \in \mathbb{R}^3 \mid \begin{pmatrix} \vec{x}^T \\ \vec{y}^T \end{pmatrix}\vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

$$= \{\vec{v} \in \mathbb{R}^3 \mid A\vec{v} = \vec{0}\}$$

$$= N(A)$$

#

(b) Sol. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

$$S = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$$

$$S^{\perp} = N(A) = \{\vec{v} \in \mathbb{R}^3 \mid A\vec{v} = \vec{0}\}$$

$$= \{\vec{v} \in \mathbb{R}^3 \mid \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}\vec{v} = \vec{0}\}$$

$$= \{\vec{v} \in \mathbb{R}^3 \mid \vec{v} = (-5t, t, 3t)^T,\ t \in \mathbb{R}\}$$

$$= \operatorname{Span}\left\{ \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \right\}$$

#

8. Proof: $\vec{y} \in S^{\perp} \Leftrightarrow \vec{y} \cdot \vec{x} = 0 \quad \forall \vec{x} \in S$

$$\Leftrightarrow \vec{y} \cdot \left(\sum_{i=1}^{k} a_i \vec{x}_i\right) = 0 \quad \forall a_i \in \mathbb{R},\ i = 1, 2, \cdots k$$

$$\Leftrightarrow \sum_{i=1}^{k} a_i (\vec{y} \cdot \vec{x}_i) = 0 \quad \forall a_i \in \mathbb{R} \quad i = 1, 2, \cdots k$$

$$\Leftrightarrow \vec{y} \cdot \vec{x}_i = 0 \quad \forall i = 1, 2, \cdots k$$

$$\Leftrightarrow \vec{y} \perp \vec{x}_i \quad \forall i = 1, 2, \cdots k.$$

#

9. Sol:

$$\left.\begin{aligned} & r = \operatorname{rank}(A) = \dim R(A) = \dim R(A^T) \\ & \dim N(A) + \dim R(A) = n \\ & \dim N(A^T) + \dim R(A^T) = m \end{aligned}\right\}$$

$$\Rightarrow \dim N(A) = n - r, \quad \dim N(A^T) = m - r.$$

#

11. Proof: Note $\mathbb{R}^n = N(A) \oplus R(A^T)$

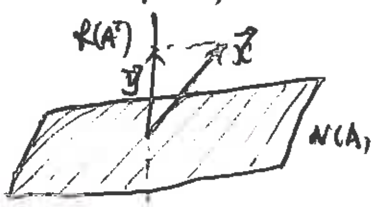
for $\vec{x} \in \mathbb{R}^n$, $\exists!\ \vec{z} \in N(A)$ & $\vec{y} \in R(A^T) \ni$

$$\vec{x} = \vec{y} + \vec{z}$$

If $\vec{y} = 0$, then $\vec{x} = \vec{z} \in N(A)$, that is $A\vec{x} = \vec{0}$

If $\vec{y} \neq 0$ then $\vec{x}^T\vec{y} = (\vec{y}^T + \vec{z}^T)\vec{y} = \|\vec{y}\|^2 \neq 0$

Therefore, either $A\vec{x} = \vec{0}$ or $\exists\ \vec{y} \in R(A^T) \ni \vec{x}^T\vec{y} \neq 0$



#

§ 5.3

3 (b) $A^TA = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 6 \\ 0 & 14 & 14 \\ 6 & 14 & 26 \end{pmatrix}$

$A^T\vec{b} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 26 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 3 & 0 & 6 & 6 \\ 0 & 14 & 14 & 14 \\ 6 & 14 & 26 & 26 \end{array}\right) \underset{R_3/2}{\overset{R_1/3 \;\; R_2/14}{\approx}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 3 & 7 & 13 & 13 \end{array}\right)$$

$$\overset{R_3-3R_1}{\approx} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 7 & 7 & 7 \end{array}\right)$$

$$\overset{R_3-7R_2}{\approx} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

That is
$$\begin{cases} x_1 + 2x_3 = 2 \\ x_2 + x_3 = 1 \end{cases}$$

let $x_3 = t$. then $\begin{cases} x_1 = 2-2t \\ x_2 = 1-t \end{cases}$

Thus the set of least square sol's

is $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

#

5 (a) Sol: General line equation

$$y = ax + b$$

Thus the over-determined linear system

$$\begin{cases} -a+b = 0 \\ b = 1 \\ a+b = 3 \\ 2a+b = 9 \end{cases}$$

$$\underbrace{\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix}}_{\vec{b}}$$

$$A^T A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 21 \\ 13 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 21 \\ 13 \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} 21 \\ 13 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 29 \\ 18 \end{pmatrix}$$

Thus the best least square fit is

$$y = \frac{1}{10}(29x + 18)$$

#

9. Proof: (a) $\vec{b} \in R(A) \Rightarrow \exists\, \vec{x} \in \mathbb{R}^n \ni A\vec{x} = \vec{b}$

$$\Rightarrow P\vec{b} = \left[A(A^TA)^{-1}A^T\right](A\vec{x})$$
$$= A\left[(A^TA)^{-1}A^TA\right]\vec{x}$$
$$= A I_n \vec{x}$$
$$= A\vec{x}$$
$$= \vec{b}$$

Explaination: $P\vec{b}$ is the projection of $\vec{b}$ onto $R(A)$, therefore, when $\vec{b} \in R(A)$

$P\vec{b} = \vec{b}$ #

(b) $\left.\begin{array}{l} \vec{b} \in R(A)^{\perp} \\ R(A)^{\perp} = N(A^T) \end{array}\right\} \Rightarrow \vec{b} \in N(A^T)$

$\Rightarrow A^T\vec{b} = \vec{0}$

$\Rightarrow P\vec{b} = A(A^TA)^{-1}A^T\vec{b} = \vec{0}$ #

(c)