A multiple level set method for three-dimensional inversion of magnetic data

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SUMMARY

We propose a multiple level-set method for inverting three-dimensional magnetic data induced by magnetization only. To alleviate inherent non-uniqueness of the inverse magnetic problem, we assume that the subsurface geological structure consists of several uniform magnetic mass distributions surrounded by homogeneous non-magnetic background such as host rock, where each magnetic mass distribution has a known constant susceptibility and is supported on an unknown sub-domain. This assumption enables us to reformulate the original inverse magnetic problem into a domain inverse problem for those unknown domains defining the supports of those magnetic mass distributions. Since each uniform mass distribution may take a variety of shapes, we use multiple level-set functions to parameterize these domains so that the domain inverse problem can be further reduced to an optimization problem for multiple level-set functions. To compute rapidly gradients of the nonlinear functional arising in the multiple-level-set formulation, we utilize the fact that the kernel function in the field-susceptibility relation decays rapidly off the diagonal so that matrix-vector multiplications for evaluating the gradients can be speeded up.
significantly. Numerical experiments with synthetic and field data sets are carried out to illustrate the effectiveness of the new method.

**Key words:** Magnetic inversion – level set method – gradient descent method.

1 INTRODUCTION

In exploration geophysics, one important problem is to interpret a great amount of three-dimensional (3-D) magnetic survey data measured above the surface of the earth in order to map subsurface geological structure. Robust and efficient inversion methods for magnetic data are called for when quantitative interpretation is required in practice. Mathematically, it is well-known that there exist infinitely many subsurface structures that reproduce a given set of magnetic data within the same level of accuracy so that the solution for the magnetic inverse problem is theoretically non-unique. To overcome this difficulty, a prior information is necessarily incorporated in the inversion process so that a geologically reasonable model can be selected from the family of mathematically acceptable models.

Existing methods for dealing with such non-uniqueness of the magnetic inverse problem can be categorized into three categories. The first is the parametric inversion that assumes a geometrically simple causative body represented by a small number of parameters consisting of the geometry and the magnetic properties. These parameters are determined by solving a nonlinear parameter estimation problem (Ballantyne 1980; Bhattacharyya 1980; Silva & Hohmann 1983). Since only a small subset of possible solutions is involved in the inversion, non-uniqueness is not generally an issue due to the restrictive nature of the parametrization.

The second approach constructs a 3-D distribution of magnetic susceptibility by dividing the earth into a large number of cells with fixed sizes and unknown constant susceptibilities, and those in-cell susceptibilities are obtained in the inversion (Cribb 1976; Li & Oldenburg 1996; Pilkington 1997). In such an approach, in-cell susceptibilities are obtained by minimizing an objective functional, generally consisting of a data misfit functional and a stabilizing functional (i.e., regularization term or model objective function). In principle, Tikhonov regularization theory can guarantee a unique solution, but the incorporation of general or site-specific a priori information is required to ensure that the solution is geologically reasonable. In Li & Oldenburg (1996), the authors utilized a stabilizing functional that includes a fidelity term, flatness terms in three spatial directions, and a depth-weighting function. The depth weighting is applied to counteract the spatial decay of the magnetic kernel function, so that the resultant model can capture the depth of the causative body. In addition, bound
constraints were also imposed. A similar approach was also used by Pilkington (1997). The concept of depth weighting was later generalized to distance weighting and sensitivity weighting functions to accommodate borehole data (Li & Oldenburg 2000). The algorithm described in Portniaguine & Zhdanov (2002) also involved the sensitivity-based weighting function and introduced an iterative weighting into the stabilizing functional to obtain a compact distribution of the recovered non-zero susceptibility. This category of magnetic susceptibility inversion techniques provide quantitative descriptions of the susceptibility distribution corresponding to geological structures; however, one has to extract the position of a causative body from resulting models afterwards, and this is not always an easy task.

The third category of inversion constructs the geometries or boundaries of a magnetic source by assuming some knowledge about magnetic susceptibility. For example, Wang & Hansen (1990) inverts for the positions of the vertices of a polyhedral body. In the basement inversion, the susceptibility contrast between the basin and basement is assumed to be known but the depth to the basement is recovered as a function of the horizontal location (Pedersen 1977; Pilkington & Crossley 1986). Thus far, few work is available that seeks to invert for the boundaries of general magnetic causative bodies in 3D. Towards this goal, Fullagar et al. (2008) develop a mixed parameterization in which both susceptibility and vertical interface position may be recovered. The reason for the lack of such a general interface inversion algorithm has been the difficulty in pre-determining the shapes and number of sources (i.e., topology) to ensure a parametrization consistent with geology. In fact, such information is precisely what we would like to extract from the magnetic inversion. The advantage, however, is that we do not necessarily need as much prior information except for the value of the magnetic susceptibility. Furthermore, the inverted boundaries are directly usable in interpretation or subsequent geological modeling. Therefore, we are motivated to develop a method that can directly delineate causative bodies beneath the surface without requiring too much prior information.

We propose a multiple level set method for inverting 3-D magnetic anomalous data which are induced by magnetization only, and this method automatically determines positions of causative bodies and does not involve any weighting function. The method is an extension of recently developed single level set methods for the inversion of gravity and gravity gradient data (Isakov et al. 2011, 2013; Lu et al. 2014; Lu & Qian 2015; Li et al. 2016). Mathematically, we formulate the inversion of magnetic anomalous data as the following inverse problem: recover the magnetic susceptibility distribution $\kappa$ in a subsurface domain $\Omega$, from the data $d$ measured on a measurement surface $\Gamma \in \mathbb{R}^3/\Omega$. To alleviate the non-uniqueness of the
inverse magnetic problem, we assume that the subsurface geological structure in domain $\Omega$ consists of $n$ uniform magnetic mass distributions surrounded by a homogeneous non-magnetic background such as host rocks, where each magnetic mass distribution has a known constant susceptibility and is supported on an unknown sub-domain so that the susceptibility contrast relative to the non-magnetic background satisfies

\[ \kappa(r) = \kappa_i, \quad r \in D_i; \quad \Omega_0 \cup (\cup_i D_i) = \Omega; \quad D_i \cap D_j = \emptyset, \quad (1) \]

where $\Omega_0$ is the background domain, and $D_i$ is the support of the $i$-th homogeneous material with susceptibility contrast $\kappa_i$. Such an assumption is reasonable due to the following reason: although $\kappa$ is generally not piecewise constant in practice, by the equivalent source principle, one can always find an averaged susceptibility $\kappa_i$ along with an average domain $D_i$ for each causative body in $\Omega$ so that $\kappa$ defined in (1) can reproduce the same magnetic data; in fact, there can exist infinitely many such models $\{\kappa_i, D_i\}_{i=1}^n$. Usually, a prior information about the model under consideration can help determine $\{\kappa_i\}_{i=1}^n$ so that only $\{D_i\}_{i=1}^n$ are to be determined. Consequently, we convert the original inverse problem into a domain inverse problem.

Since each unknown domain $D_i$ may take a variety of possible shapes, connected or disconnected, we may introduce multiple level set functions to parametrize all $n$ domains $D_i$ (DeCezaro et al. 2009) so that the domain inverse problem is reduced to a nonlinear optimization problem for multiple level set functions; such multiple level set formulation becomes essential when there are more than two different homogeneous mediums in $\Omega$. To compute rapidly the gradient of the nonlinear functional with respect to each of the multiple level set functions, we make use of the fact that the kernel function in the relation between field data $d$ and the susceptibility $\kappa$ decays rapidly off the diagonal so that matrix-vector multiplications for evaluating the gradients can be speeded up significantly.

The reason that we choose the level-set method for the inverse magnetic problem is the following. For the geometrical domain inverse problem under consideration, one needs to deal with closed irregular surfaces which are the boundary of an underlying domain. To describe such an irregular surface, one may introduce some surface parametrization so that one can carry out manipulation on such a surface in order to fit the given data. However, because such an irregular surface may change shapes or connectivities during nonlinear data-fitting process, we need to design a reliable and robust parameterization which is capable of changing shapes or connectivities automatically, and the level-set implicit parametrization (Osher & Sethian 1988) is exactly such a parametrization. We start with a continuous function which is defined everywhere in the whole computational domain, and we further require that this function be
positive inside a targeted domain and negative outside, which implies that the zero level-set
where the function is zero describes exactly the boundary of the targeted domain, and this
function is called the level-set function. A level-set implicit parametrization gives rise to many
advantages, such as we have a globally defined function to manipulate, and the changes of
genometry shapes and connectivities can be automatically taken care of due to the underlying
physical mechanism.

We remark that in the literature the level-set method (Osher & Sethian 1988) has been
widely used as a suitable and powerful tool for interfaces and shape-optimization problems
mainly due to its ability in automatic interface merging and topological changes. In terms
of non-geophysical inverse problems, the level-set method was first used for inverse obstacle
problems in Santosa (1996); since then it has been applied to a variety of inverse problems. In
Litman et al. (1998) it was used to reconstruct 2-D binary obstacles; furthermore, the level-
set method was used for inverse scattering problems to reconstruct geometry of extended
targets in Hou et al. (2004) and Dorn & Lesselier (2006), for electrical resistance tomography
in medical imaging in Ben Hadj Miled & Miller (2007), and for piece-wise constant surface
reconstruction in van den Doel et al. (2010). In terms of geophysical inverse problems, the
level-set method has also found its wide applications, and the following citations are by no
means complete. In Isakov et al. (2011), the level-set method was first applied to the gravity
data; in Papadopoulos et al. (2011) it was applied to identify uncertainties in the shape of
gophysical objects using temperature measurements; in Zheglova et al. (2013), Li & Leung
(2013) and Li et al. (2015) it was applied to travel-time tomography problems in different
settings.

The rest of this paper is organized as follows. In section 2, we present the forward model of
magnetic data. In section 3, we develop the multiple level-set based inversion of 3-D magnetic
data. In section 4, we present the key algorithm of the magnetic inversion along with some
implementation details. In section 5, intensive numerical examples are studied to illustrate
the effectiveness of our level-set algorithm; in addition, the issue of inherent non-uniqueness
is addressed and discussed. In Section 6, we apply our level-set inversion algorithm to a set of
field data acquired in mineral exploration and show that the method is effective and robust
under commonly encountered conditions in practical settings and that the inversion result is
consistent with existing algorithms.


2 FORWARD MODEL OF MAGNETIC DATA

We work with the anomalous magnetic flux density arising from subsurface magnetically susceptible material distributions subjected to Earth’s main field (i.e., inducing field). Supposing that there is no remanent magnetization and the demagnetization effect is negligible in comparison to the primary inducing magnetic field, and the data are acquired in free space (e.g. air), the linear model of magnetic anomaly data is given by the following equation (Blakely 1996):

\[
B(r) = \frac{1}{4\pi} \nabla_r \int_{\Omega} \kappa(\tilde{r}) B^0 \cdot \nabla_r \left( \frac{1}{|r - \tilde{r}|} \right) \, d\tilde{r}, \quad r \in \Gamma.
\]  

(2)

In equation (2), \( B(r) \) is the anomalous magnetic flux density observed above the subsurface, and \( B^0 \) is a constant vector which denotes the magnetic flux of the primary inducing field; both \( B(r) \) and \( B^0 \) have the units of nanotesla (nT). \( \kappa(\tilde{r}) \) is the magnetic susceptibility in the rock formations, which is dimensionless in the SI system. \( \Omega \subset \mathbb{R}^3 \) is a subsurface domain and \( \Gamma \subset \mathbb{R}^3 \setminus \Omega \) is the measurement surface. \( r \) and \( \tilde{r} \) denote spatial coordinates of observation and source locations, respectively. We use \( \nabla_r \) to denote the gradient with respect to the spatial coordinate \( r \), and \(|\cdot|\) denotes the vector \( l^2 \)-norm.

We can express \( B^0 \) as \( B^0 = B^0 \hat{B}_0 \), where \( B^0 \) denotes the strength of the inducing field and \( \hat{B}_0 \) denotes the corresponding direction. Plugging this expression into equation (2) and evaluating \( \nabla_r \) explicitly in the 3-D space yields an equivalent equation for the anomalous flux

\[
B(r) = \frac{1}{4\pi} B^0 \int_{\Omega} \kappa(\tilde{r}) \frac{1}{|r - \tilde{r}|^3} \left[ 3 \left( \hat{B}_0 \cdot (r - \tilde{r}) \right) \frac{(r - \tilde{r})}{|r - \tilde{r}|^2} - \hat{B}_0 \right] d\tilde{r}.
\]

Moreover, the majority of magnetic data are the measurements of total-field strength and subsequent processing yields the total-field anomaly, which is well approximated by the projection of the anomalous magnetic flux density onto the inducing field direction (Blakely 1996):

\[
d(r) = \hat{B}_0 \cdot B(r)
\begin{align*}
= \frac{1}{4\pi} B^0 \int_{\Omega} \kappa(\tilde{r}) \frac{1}{|r - \tilde{r}|^3} \left[ 3 \left( \hat{B}_0 \cdot (r - \tilde{r}) \right)^2 \frac{1}{|r - \tilde{r}|^2} - 1 \right] d\tilde{r}, \quad r \in \Gamma.
\end{align*}
\]

(3)

In this paper we use equation (3) as the forward model of magnetic anomaly data induced by magnetization only. The purpose of magnetic survey is to recover the susceptibility distribution \( \kappa(\tilde{r}) \) from the measurement data \( d(r) \) which is observed above the subsurface.
3 MAGNETIC INVERSION AND THE LEVEL SET FORMULATION

In this section we develop the method for 3-D inversion of magnetic data, where a continuous formulation based on equation (3) is adopted in the derivation. We propose a level set framework for the reconstruction of susceptibility distributions.

3.1 Data misfit

Denoting the observed survey data of anomalous magnetic flux by \( \{d^*_k\}_{k=1}^M \), and supposing that \( \{d_k\}_{k=1}^M \) is the predicted data from the forward model in equation (3), we propose the misfit function for data fitting as the following,

\[
E_d = \frac{1}{2M} \sum_{k=1}^{M} \left( \frac{d_k - d^*_k}{\sigma_k} \right)^2 ,
\]

where \( \sigma_k \) is the error standard deviation associated with the \( k \)-th datum. This is similar to a commonly used data misfit function in gravity and magnetic inversion (Li & Oldenburg 1996; Lelièvre & Oldenburg 2006; Lu & Qian 2015), and the model parameter (i.e. the susceptibility distribution) \( \kappa \) is recovered by minimizing this misfit function \( E_d \).

We study the Fréchet derivative of \( E_d \) by considering a perturbation on the predicted data \( \{d_k\}_{k=1}^M \): \( d_k \rightarrow d_k + \delta d_k \), which leads to the corresponding perturbation on \( E_d \):

\[
\delta E_d = \frac{1}{M} \sum_{k=1}^{M} \frac{d_k - d^*_k}{\sigma_k^2} \cdot \delta d_k .
\]

The predicted data \( d_k \) is related to the model parameter \( \kappa(\tilde{r}) \) by equation (3), so that we have

\[
\delta d_k = \frac{1}{4\pi} B^0 \int_{\Omega} \delta \kappa(\tilde{r}) \cdot K(r_k, \tilde{r}) \, d\tilde{r} ,
\]

where \( K(r, \tilde{r}) \) denotes the integral kernel:

\[
K(r, \tilde{r}) = \frac{1}{|r - \tilde{r}|^3} \left[ \frac{3 \left( \hat{B}_0 \cdot (r - \tilde{r}) \right)^2}{|r - \tilde{r}|^2} - 1 \right], \quad r \in \Gamma, \; \tilde{r} \in \Omega.
\]

Substituting equation (6) into equation (5), we get the equation relating \( \delta E_d \) to \( \delta \kappa \),

\[
\delta E_d = \frac{1}{4\pi} B^0 \int_{\Omega} \delta \kappa(\tilde{r}) \cdot \left[ \frac{1}{M} \sum_{k=1}^{M} \frac{d_k - d^*_k}{\sigma_k^2} \cdot K(r_k, \tilde{r}) \right] \, d\tilde{r} ,
\]

so that the Fréchet derivative of \( E_d \) with respect to \( \kappa(\tilde{r}) \) is obtained as the following

\[
\frac{\partial E_d}{\partial \kappa} = \frac{1}{4\pi} B^0 \cdot \frac{1}{M} \sum_{k=1}^{M} \frac{d_k - d^*_k}{\sigma_k^2} \cdot K(r_k, \tilde{r}) .
\]

A necessary condition for minimizing \( E_d \) is that the Fréchet derivative \( \partial E_d/\partial \kappa \) equals zero.
3.2 Level set formulations for susceptibility distribution

A principal difficulty with the inversion of magnetic data (and other geopotential data) is the inherent non-uniqueness (Li & Oldenburg 1996, 1998). By Gauss’s theorem, if the field distribution is known only on a bounding surface, there are infinitely many equivalent source distributions inside the boundary that can produce the known field.

To alleviate the inherent non-uniqueness, we assume that the subsurface geological structure in domain $\Omega$ consists of $n$ uniform magnetic mass distributions surrounded by a homogeneous non-magnetic background such as soil, where each magnetic mass distribution has a known constant susceptibility and is supported on an unknown sub-domain so that the susceptibility contrast relative to the non-magnetic background satisfies

$$\kappa(\tilde{r}) = 0, \quad \tilde{r} \in \Omega_0; \quad \kappa(\tilde{r}) = \kappa_i, \quad \tilde{r} \in D_i; \quad \Omega_0 \cup (\cup_i D_i) = \Omega; \quad D_i \cap D_j = \emptyset,$$

where $\Omega_0$ is the background domain, and $D_i$ is the support of the $i$-th homogeneous material with susceptibility contrast $\kappa_i$. We suppose that the values of $\kappa_i$’s are known as a part of a priori geologic information. Moreover the domain $D_i$ is not necessarily connected, which corresponds to the common occurrences of multiple separate causative bodies in a geologic setting. In this paper we adopt the level set method (Osher & Sethian 1988) to formulate the domain-wise homogeneous distribution of susceptibility. We use a single level set method to handle the case of multiple bodies having the same susceptibility value, and we propose a multiple level set method (DeCezaro et al. 2009) to deal with the case in which each causative body can have one of two distinct susceptibility values. The methodology can be generalized to the domain-wise homogeneous structure with arbitrarily many different susceptibility values, but the inherent non-uniqueness in the magnetic inversion problem prevents us to do so.

3.2.1 Single level set formulation

To start with, we assume that there is only one type of magnetic mineral in the subsurface structure, and the average susceptibility of the mineral is known as $\kappa_0$. Therefore the susceptibility distribution is given by

$$\kappa(\tilde{r}) = \kappa_0, \quad \tilde{r} \in D_0; \quad \kappa(\tilde{r}) = 0, \quad \tilde{r} \in \Omega \setminus D_0.$$

We use a level set formulation to express this distribution:

$$\kappa(\tilde{r}) = \kappa_0 \cdot H(\phi_0(\tilde{r})),$$

where $\phi_0(\tilde{r})$ is the level set function.
\[ \phi_0(\tilde{r}) = \begin{cases} > 0 & , \quad \tilde{r} \in D_0 \\ < 0 & , \quad \tilde{r} \in \Omega \setminus D_0 \end{cases} \]

and \( H(x) \) is the heaviside function

\[ H(x) = \begin{cases} 1 & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases} \]

The level set function \( \phi_0(\tilde{r}) \) parameterizes the unknown domain \( D_0 \), and the susceptibility distribution \( \kappa(\tilde{r}) \) is reconstructed by recovering \( \phi_0(\tilde{r}) \).

From equation (10) one can get

\[ \frac{\partial \kappa}{\partial \phi_0} = \kappa_0 \cdot H'(\phi_0) = \kappa_0 \cdot \delta_d(\phi_0), \tag{12} \]

where \( \delta_d(\phi_0) \) is the Dirac delta function (Bracewell 2000). Combining equation (12) and equation (8), one gets the Fréchet derivative of the misfit function \( E_d \) with respect to the level set function \( \phi_0 \):

\[ \frac{\partial E_d}{\partial \phi_0} = \frac{\partial E_d}{\partial \kappa} \cdot \frac{\partial \kappa}{\partial \phi_0} = \frac{1}{4\pi} \kappa_0 B^0 \cdot \delta_d(\phi_0) \cdot \frac{1}{M} \sum_{k=1}^{M} \frac{d_k - d_k^*}{\sigma_k^2} \cdot K(r_k, \tilde{r}). \tag{13} \]

Equation (13) relates the objective function to the new model parameter \( \phi_0 \), and a necessary condition for minimizing \( E_d \) is \( \partial E_d / \partial \phi_0 = 0 \).

3.2.2 Multiple level set formulation

For the case with two types of magnetic minerals in the subsurface, we are looking for a domain-wise homogeneous distribution of susceptibility with multiple compositions:

\[ \kappa(\tilde{r}) = \begin{cases} \kappa_1 & , \quad \tilde{r} \in D_1 \\ \kappa_2 & , \quad \tilde{r} \in D_2 \\ 0 & , \quad \tilde{r} \in \Omega \setminus (D_1 \cup D_2) \end{cases}, \tag{14} \]

where \( \kappa_1 \) and \( \kappa_2 \) are average susceptibilities of the minerals in \( D_1 \) and \( D_2 \) respectively. Supposing that the values of \( \kappa_1 \) and \( \kappa_2 \) are known, we use a multiple level set formulation (DeCezaro et al. 2009) to express the magnetic susceptibility:

\[ \kappa(\tilde{r}) = \kappa_1 \cdot H(\phi_1) \cdot (1 - H(\phi_2)) + \kappa_2 \cdot (1 - H(\phi_1)) \cdot H(\phi_2). \tag{15} \]

In formula (15), \( \phi_i \) is the level set function for \( D_i \) \((i = 1, 2)\),

\[ \phi_i(\tilde{r}) = \begin{cases} > 0 & , \quad \tilde{r} \in D_i \\ < 0 & , \quad \tilde{r} \in \Omega \setminus D_i \end{cases} \]

and \( H(x) \) is the heaviside function given in equation (11). Equation (15) is a robust formulation for the distribution of susceptibility in equation (14). In the inversion process there may exist
the region $E = \{ \tilde{r} \mid \phi_1(\tilde{r}) > 0, \phi_2(\tilde{r}) > 0 \}$, and equation (15) gives $\kappa(\tilde{r}) = 0$ in such a region and avoids the artificial value $\kappa(\tilde{r}) = \kappa_1 + \kappa_2$ which may be generated by a trivial level set formulation such as $\kappa(\tilde{r}) = \kappa_1 \cdot H(\phi_1) + \kappa_2 \cdot H(\phi_2)$. Moreover, formulation (15) can be simply generalized to handle a domain-wise homogeneous structure with more compositions (Tai & Chan 2004; DeCezaro et al. 2009).

From equation (15) one can compute the Fréchet derivatives of $\kappa(\tilde{r})$ with respect to the new model parameters $\phi_1$ and $\phi_2$,

$$\frac{\partial \kappa}{\partial \phi_1} = \delta_d(\phi_1) \cdot (\kappa_1 - (\kappa_1 + \kappa_2)H(\phi_2)),$$

(16)

$$\frac{\partial \kappa}{\partial \phi_2} = \delta_d(\phi_2) \cdot (\kappa_2 - (\kappa_1 + \kappa_2)H(\phi_1)),$$

(17)

where $\delta_d(.)$ denotes the Dirac delta function. Combining equations (16) and (17) with equation (8), one gets the Fréchet derivatives of the misfit function $E_d$ with respect to $\phi_1$ and $\phi_2$,

$$\frac{\partial E_d}{\partial \phi_1} = \frac{1}{4\pi} \delta_d(\phi_1) \cdot (\kappa_1 - (\kappa_1 + \kappa_2)H(\phi_2)) \cdot B^0 \cdot \frac{1}{M} \sum_{k=1}^{M} \frac{d_k - d_k^*}{\sigma_k^2} \cdot K(r_k, \tilde{r}),$$

(18)

$$\frac{\partial E_d}{\partial \phi_2} = \frac{1}{4\pi} \delta_d(\phi_2) \cdot (\kappa_2 - (\kappa_1 + \kappa_2)H(\phi_1)) \cdot B^0 \cdot \frac{1}{M} \sum_{k=1}^{M} \frac{d_k - d_k^*}{\sigma_k^2} \cdot K(r_k, \tilde{r}).$$

(19)

The necessary conditions for minimizing $E_d$ are $\partial E_d/\partial \phi_i = 0, (i = 1, 2)$, and the susceptibility $\kappa(\tilde{r})$ is recovered by using the new model parameters $\phi_1$ and $\phi_2$.

### 3.3 Additional regularization

Since the magnetic inversion problem is severely ill-posed, we introduce an additional regularization term into the objective function. For the single level set formulation the regularization term is defined as

$$E_r = \frac{1}{2} \int_{\Omega} |\nabla \phi_0|^2 \, d\tilde{r},$$

and the total objective function is formed by the linear combination of $E_d$ and $E_r$:

$$E_t = E_d + \alpha \cdot E_r$$

(20)

where $E_d$ is the misfit function given in equation (4), and $\alpha$ is a small constant that indicates the amount of regularization one needs. The inversion algorithm is to minimize the new objective function $E_t$. The regularization term $E_r$ tends to smoothen the shape of structure characterized by the zero level set $\{ \tilde{r} \mid \phi_0(\tilde{r}) = 0 \}$, and it helps to alleviate the severe ill-posedness in the inverse problem.

The Fréchet derivative of $E_r$ is
\[
\frac{\partial E_r}{\partial \phi_0} = -\Delta \phi_0,
\]
so that
\[
\frac{\partial E_t}{\partial \phi_0} = \frac{\partial E_d}{\partial \phi_0} - \alpha \Delta \phi_0
\]
where \(\frac{\partial E_d}{\partial \phi_0}\) is given by equation (13).

Since there are two level set functions for the multiple level set formulation (equation (15)), the regularization term is defined as
\[
E_r = \frac{1}{2} \int_{\Omega} |\nabla \phi_1|^2 \, d\tilde{r} + \frac{1}{2} \int_{\Omega} |\nabla \phi_2|^2 \, d\tilde{r}.
\]
Since the total objective function \(E_t\) is still in the form of equation (20), we have
\[
\frac{\partial E_t}{\partial \phi_i} = \frac{\partial E_d}{\partial \phi_i} - \alpha \Delta \phi_i, \quad i = 1, 2
\]
where \(\frac{\partial E_d}{\partial \phi_i}\) (\(i = 1, 2\)) are evaluated in equations (18) and (19), respectively. The Fréchet derivatives \(\frac{\partial E_t}{\partial \phi_i}\) illustrate the gradient descent directions of the total objective function.

### 3.4 Gradient descent method for minimization

We are looking for the model parameters \(\phi_i\) which minimize the total objective function \(E_t\), where \(i = 0\) for the single level set formulation and \(i = 1, 2\) for the multiple level set formulation. We use a gradient descent method for minimization so that we evolve the following gradient flow to the steady state to compute the minimizer,
\[
\frac{\partial \phi_i}{\partial t} = -\frac{\partial E_t}{\partial \phi_i}, \quad \text{in } \Omega
\]
\[
\frac{\partial \phi_i}{\partial n} = 0, \quad \text{on } \partial \Omega
\]
where \(\phi_i = \phi_i(\tilde{r}, t)\) with \(t\) being the artificial evolution time, and a natural boundary condition for \(\phi_i\) is imposed on the boundary \(\partial \Omega\). The gradient directions \(\frac{\partial E_t}{\partial \phi_i}\) are computed in subsection 3.3.

### 4 ALGORITHM AND NUMERICAL IMPLEMENTATION

In this section, we summarize the above-mentioned algorithm on magnetic inversion and describe the numerical implementation in details.
(i) Initialize the level set functions $\phi_i$, where $i = 0$ in the single level set formulation and $i = 1, 2$ in the multiple level set formulation.

(ii) Obtain the magnetic susceptibility $\kappa(\tilde{r})$ according to equation (10) or equation (15).

(iii) Compute the predicted data $\{d_k\}_{k=1}^{M}$ along the measurement surface $\Gamma$ according to equation (3).

(iv) Compute the gradient directions of the total objective function $E_t$. For the single level set formulation $\frac{\partial E_t}{\partial \phi_0}$ is given by equation (21), and for the multiple level set formulation $\frac{\partial E_t}{\partial \phi_i} (i = 1, 2)$ are given by equation (22).

(v) Evolve the level set functions $\phi_i$ according to the gradient flow (23).

(vi) Reinitialize the level set functions to maintain the signed distance property.

(vii) Go back to step 2 until the iteration converges.

In step 2, the Heaviside function $H(\phi)$ in the level set formulation is numerically approximated by

$$H_\epsilon(\phi) = \begin{cases} 0 & , \phi < -\epsilon \\ \frac{1}{2} + \frac{\phi}{2\epsilon} + \frac{1}{2\pi} \sin\left(\frac{\pi \phi}{\epsilon}\right) & , -\epsilon \leq \phi \leq \epsilon \\ 1 & , \epsilon < \phi \end{cases}$$

where $\epsilon$ controls the thickness of interface. Taking a zero $\epsilon$ leads to a sharp interface and taking a larger $\epsilon$ leads to a smoother structure.

In step 4, the Dirac delta function $\delta_d(\phi_i)$ appearing in the gradient directions $\frac{\partial E_t}{\partial \phi_i}$ is numerically approximated by $(\delta_d)_\tau(\phi_i) = \mathcal{I}_{T^i_\tau} \cdot |\nabla \phi_i|$ (Zhao et al. 1996; Lu & Qian 2015), where

$$\mathcal{I}_{T^i_\tau} = \begin{cases} 1 & , \tilde{r} \in T^i_\tau \\ 0 & , \tilde{r} \in \Omega \setminus T^i_\tau \end{cases}$$

with support $T^i_\tau = \{\tilde{r} \in \Omega : |\phi_i(\tilde{r})| \leq \tau\}$. The parameter $\tau$ controls the band-width of the numerical delta function, and the level set function $\phi_i$ is updated in the support region $T^i_\tau$. To have an effective updating we usually set $\tau = 0.5 \ast \min\{\Delta x, \Delta y, \Delta z\}$.

Then the gradient directions $\frac{\partial E_t}{\partial \phi_i}$ in (21) and (22) are evaluated by the following formulas,

$$\frac{\partial E_t}{\partial \phi_i} = F_i \cdot |\nabla \phi_i| - \alpha \Delta \phi_i$$

where $i = 0$ for the single level set formulation and $i = 1, 2$ for the multiple level set formul-
tion. From equations (13), (18) and (19) one can find that

\[ F_0 = \frac{1}{4\pi} \kappa_0 \cdot B^0 \cdot \mathcal{I}_{T^2} \cdot \frac{1}{M} \sum_{k=1}^{M} \frac{d_k - d_k^*}{\sigma_k^2} \cdot K(r_k, \tilde{r}), \]

\[ F_1 = \frac{1}{4\pi} (\kappa_1 - (\kappa_1 + \kappa_2)H_e(\phi_2)) \cdot B^0 \cdot \mathcal{I}_{T^3} \cdot \frac{1}{M} \sum_{k=1}^{M} \frac{d_k - d_k^*}{\sigma_k^2} \cdot K(r_k, \tilde{r}), \]

\[ F_2 = \frac{1}{4\pi} (\kappa_2 - (\kappa_1 + \kappa_2)H_e(\phi_1)) \cdot B^0 \cdot \mathcal{I}_{T^2} \cdot \frac{1}{M} \sum_{k=1}^{M} \frac{d_k - d_k^*}{\sigma_k^2} \cdot K(r_k, \tilde{r}), \]

where the predicted data \( d_k \) is computed in step 3 according to equation (3)

\[ d_k = \frac{1}{4\pi} B^0 \int_{\Omega} \kappa(\tilde{r}) \cdot K(r_k, \tilde{r}) \, d\tilde{r}. \]

To evaluate the integral and summation in the above equations, we develop a low-rank-matrix decomposition algorithm to speed up matrix-vector multiplications arising in the computation. The detailed formulations are described in Appendix A.

Then in step 5, the evolution equation (23) is reduced to the following form

\[ \frac{\partial \phi_i}{\partial t} = -F_i \cdot |\nabla \phi_i| + \alpha \Delta \phi_i, \]

which can be viewed as a Hamilton-Jacobi equation with artificial viscosity term. We simply apply forward differencing in the time direction and central differencing for the spatial derivative. Numerically, a CFL condition is enforced to maintain the stability of evolution,

\[ \Delta t \cdot \left( \frac{\max |F_i|}{\min \{\Delta x, \Delta y, \Delta z\}} + \frac{\alpha}{\min \{\Delta x^2, \Delta y^2, \Delta z^2\}} \right) < 1. \]

Since in our inversion algorithm the regularization weight \( \alpha \) is much smaller than \( \max |F_i| \), the CFL condition can be reduced as

\[ \Delta t \cdot \left( \frac{\max |F_i|}{\min \{\Delta x, \Delta y, \Delta z\}} \right) < 1. \]

In practice, we take the time step

\[ \Delta t = 0.5 \cdot \frac{\min \{\Delta x, \Delta y, \Delta z\}}{|F_0|} \]

in the single level set formulation, and

\[ \Delta t = 0.5 \cdot \frac{\min \{\Delta x, \Delta y, \Delta z\}}{\max \{|F_1|, |F_2|\}} \]

in the multiple level set formulation.

In step 6, the level set reinitialization (Sussman et al. 1994) is applied to maintain the signed distance property of \( \phi_i \). This is a standard technique in the level set method, and it can be viewed as a regularization on the model parameters \( \phi_i \) (Isakov et al. 2011; Li & Leung 2013; Li et al. 2015). Specifically, we solve the following system in an artificial time direction
ξ for several steps,

\[ \frac{\partial \Phi_i}{\partial \xi} + \text{sign}(\phi_i) \cdot (|\nabla \Phi_i| - 1) = 0, \]

\[ \Phi_i(\tilde{r}, \xi = 0) = \phi_i(\tilde{r}), \]

\[ \frac{\partial \Phi_i}{\partial n} \bigg|_{\partial \Omega} = 0, \]

where \( \text{sign}(\phi_i) = \frac{2}{\pi} \arctan \phi_i \) is the signum function (Qian & Leung 2004; Li & Leung 2013; Li et al. 2015). Since the structure characterized by the level set function is mainly determined by the zero level set, in practice there is no need to get the steady state solution. In our numerical implementation we only evolve this equation for two \( \Delta \xi \) steps and replace the original level set function \( \phi_i \) with the solution \( \Phi_i \), where \( i = 0 \) in the single level set formulation and \( i = 1, 2 \) in the multiple level set formulation.

5 SYNTHETIC NUMERICAL EXAMPLES

In this section we provide synthetic numerical examples to illustrate the performance of the inversion algorithm. The computational domain is set to be \( \Omega = [0, 1] \text{ km} \times [0, 1] \text{ km} \times [0, 0.5] \text{ km} \), which is uniformly discretized into \( 41 \times 41 \times 21 \) mesh grids. In our coordinate system, the \( x \)-axis corresponds to the geographic north, the \( y \)-axis corresponds to the geographic east, and the \( z \)-axis coincides with the depth direction. The magnetic anomalous data \( \{d_k^*\}_{k=1}^M \) are collected on the measurement surface \( \Gamma = [0, 1] \text{ km} \times [0, 1] \text{ km} \times \{z = -0.1 \text{ km}\} \), and there are \( 21 \times 21 = 441 \) observation points uniformly distributed on \( \Gamma \). To recover a compact structure with sharp interface, we use the numerical Heaviside function \( H_{\epsilon}(\phi) \) with \( \epsilon = 10^{-7} \) in the synthetic examples. Moreover the regularization weight \( \alpha \) in the total objective function \( E_t \) is selected in the order of \( 10^{-3} \cdot \left( \frac{1}{4\pi} B^0 \cdot \kappa \right)^2 \), and specific values will be given in the examples.

5.1 Single level set examples

Firstly we provide some examples of the single level set formulation; that is, there is only one type of magnetic mineral in the subsurface and one level set function \( \phi_0 \) is involved in the inversion algorithm. The initial guess of \( \phi_0 \) takes the following form

\[ \phi_{0, \text{initial}} = 1 - \sqrt{\frac{(x - 0.5)^2}{0.4^2} + \frac{(y - 0.5)^2}{0.4^2} + \frac{(z - 0.25)^2}{0.2^2}}, \]

which is approximately a signed distance function to an ellipsoid.
5.1.1 Two dykes

The model consists of two 3-D dykes buried in a non-susceptible half-space. Figure 1 (a) shows the exact model, where the susceptibility value $\kappa_0$ is given to be $\kappa_0 = 0.04$. Under an inducing field with a strength of $B^0 = 5 \times 10^4$ nT and a direction at inclination $I = 75^\circ$ and declination $D = 25^\circ$, the magnetic anomalous data are collected on the measurement surface $\Gamma = \{z = -0.1 \text{ km}\}$ as shown in Figure 1 (b). Figure 1 (c) displays the initial guess of the underlying structure in our inversion algorithm.

We set the regularization weight $\alpha = 25$ in this example, which is in the order of $10^{-3} \cdot \left(\frac{1}{4\pi}B^0 \cdot \kappa_0\right)^2$. Figure 1 (d) provides the recovered structure. The single ellipsoid in the initial guess separates into two disconnecting source bodies, and the locations of two dykes are well captured. In Figure 1 (e) and Figure 1 (f) we also plot the cross-sections of the structure at Easting = 0.5 km and Depth = 0.15 km respectively, where the dashed line indicates the exact model and the solid line plots the recovered structure. In these figures the inversion solution matches well with the exact model, where the shape and depth of two dykes are successfully recovered, though the solution loses sharp corners of the exact model due to regularization introduced in our inversion algorithm.

To further test the robustness of the algorithm, we have perturbed the measured magnetic data $\{d_k^*\}_{k=1}^M$ by 5% Gaussian noise with zero mean and repeated the inversion process. We use a unit standard deviation in the data misfit function (4), which means that we assume no knowledge of the noise level in the inversion algorithm. The results are shown in Figure 2, where Figure 2 (a) shows the measured data with 5% Gaussian noise, Figure 2 (b) provides the solution, Figure 2 (c) plots the cross-section at Easting = 0.5 km and Figure 2 (d) shows the cross-section at Depth = 0.15 km. Since the solution is similar to the recovered solution using the clean measurements as shown in Figure 1, our inversion algorithm is not sensitive to noise.

5.1.2 Four source bodies

In this example we test more source bodies in the subsurface. The model consists of two spheres and two cuboids buried in varying depths. Figure 3 (a) shows the exact model, where the susceptibility value $\kappa_0$ is given to be $\kappa_0 = 0.04$. The inducing field is in the direction of $I = 90^\circ$ with a strength of $B^0 = 5 \times 10^4$ nT, and the magnetic anomaly data are collected on the measurement surface $\Gamma = \{z = -0.1 \text{ km}\}$ as shown in Figure 3 (b). Figure 3 (c) displays the initial guess of the underlying structure which is the same as the initialization in the two-dyke example. The regularization weight $\alpha$ is again set to be $\alpha = 25$ in the total
objective function. Figure 3 (d) provides the recovered solution, where the single ellipsoid in
the initial guess evolves into four objects, and the shapes and locations of all source bodies
are successfully recovered.

To display the results more clearly we also provide the pictures of cross-sections in
Figure 4. Cross-sections of the underlying structure are displayed at Easting = 0.25 km,
Easting = 0.75 km, Depth = 0.2 km, Depth = 0.3 km, and Depth = 0.4 km, respectively,
where the dashed line indicates the exact structure and the solid line plots the recovered
solution. The reconstruction matches well with the exact model, and our level set algorithm
can automatically generate separated source bodies with varying depths to fit the given data.

5.2 Multiple level set examples

In this subsection we provide examples of the multiple level set formulation. There are two
types of magnetic minerals in the underlying subsurface and two level set functions \( \phi_1 \) and
\( \phi_2 \) are introduced in the inversion algorithm. Unless otherwise specified, we have the initial
guess of \( \phi_1 \) set to be

\[
\phi_{1,\text{initial}} = 1 - \sqrt{\frac{(x - 0.5)^2}{0.4^2} + \frac{(y - 0.75)^2}{0.15^2} + \frac{(z - 0.25)^2}{0.2^2}},
\]

and the initial guess of \( \phi_2 \) set to be

\[
\phi_{2,\text{initial}} = 1 - \sqrt{\frac{(x - 0.5)^2}{0.4^2} + \frac{(y - 0.25)^2}{0.15^2} + \frac{(z - 0.25)^2}{0.2^2}},
\]

so that the shape of the initial structure is composed by two ellipsoids.

5.2.1 Three source bodies with different susceptibilities

As shown in Figure 5 (a), the exact model consists of three cuboids with different suscepti-
bilities \( \kappa_1 = 0.04 \) and \( \kappa_2 = 0.08 \), where the gray color indicates the object with lower
susceptibility \( \kappa_1 = 0.04 \) and the dark color indicates the source body with higher suscep-
tibility \( \kappa_2 = 0.08 \). Figure 5(b) shows the magnetic data collected on the measurement sur-
face \( \Gamma = \{ z = -0.1 \text{ km} \} \), which is generated under an inducing field with a strength of
\( B^0 = 5 \times 10^4 \text{ nT} \) and a direction at the inclination \( I = 90^0 \). We are trying to recover this
model using our multiple level set algorithm.

The initial guess is displayed in Figure 5(c), where the gray color indicates \( \kappa_1 = 0.04 \)
and the dark color indicates \( \kappa_2 = 0.08 \). We choose the regularization weight \( \alpha = 50 \) in this
example, which is in the order of \( 10^{-3} \times \left( \frac{1}{15} B^0 \kappa_2 \right)^2 \). Figure 5 (d) shows the result of inversion.
The recovered solution successfully captures the shapes and locations of the underlying source
bodies. To display the result more clearly we also plot the cross-sections in Figure 6, where we provide slices of the structure at Easting \(= 0.2\) km, Easting \(= 0.7\) km, Depth \(= 0.25\) km, and Depth \(= 0.35\) km, respectively. Again the dashed line indicates the exact model and the solid line plots the recovered structure, and the gray color corresponds to \(\kappa_1 = 0.04\) and the dark color corresponds to \(\kappa_2 = 0.08\). Figure 6 (a) and Figure 6 (b) show that the varying depths of different source bodies are well located. Figure 6 (c) and Figure 6 (d) show that the recovered structure coincides with the exact model in shape.

To further test robustness of our multiple level set algorithm, we have perturbed the measurement data \(\{d_k^*\}_{k=1}^M\) by 5\% Gaussian noise with zero mean and repeated the inversion process. Again we assume no knowledge of the noise level and use a unit standard deviation in the inversion algorithm. The result is provided in Figure 7. Figure 7 (a) shows the measured magnetic data perturbed by 5\% Gaussian noise. Figure 7 (b) provides the recovered solution. Figures 7 (c)-7 (f) display the cross-sections of the structure at Easting \(= 0.2\) km, Easting \(= 0.7\) km, Depth \(= 0.25\) km, and Depth \(= 0.35\) km, respectively. We conclude that the result is quite similar to the recovered solution using the clean measurements as shown in Figure 5 and Figure 6, and the multiple level set algorithm is not sensitive to noise.

5.2.2 Four source bodies with different susceptibilities

In this example we test more source bodies with more sophisticated structures. The exact model is shown in Figure 8 (a), where the dark sphere and the dark dyke have the same susceptibility \(\kappa_2 = 0.08\), and the gray cuboids have susceptibility \(\kappa_1 = 0.04\). Under an inducing field with a strength of \(5 \times 10^4\) nT and a direction at \(I = 75^\circ\) and \(D = 25^\circ\), the magnetic anomaly data are collected on the measurement surface \(\Gamma = \{z = -0.1 \text{ km}\}\) as shown in Figure 8 (b). Figure 8 (c) shows the initial guess which is the same as the initialization of the three-source-body example. Again the regularization weight in the total objective function is set to be \(\alpha = 50\). We perform the multiple level set algorithm to invert for the underlying structure, and we provide the result in Figure 8 (d). Four separate source bodies are recovered, where the gray color indicates \(\kappa_1 = 0.04\) and the dark color indicates \(\kappa_2 = 0.08\). From the 3-D visualization one can find that the locations of different source bodies are well recovered. To display the shape of structure more clearly we also plot the cross-sections in Figure 9, where we display the slices at Easting \(= 0.75\) km, Northing \(= 0.8\) km, Northing \(= 0.375\) km, Depth \(= 0.15\) km, and Depth \(= 0.3\) km, respectively. Again the dashed line indicates the exact model and the solid line plots the recovered solution; the gray color corresponds to \(\kappa_1 = 0.04\) and the dark color corresponds to \(\kappa_2 = 0.08\). Our solution is reasonable and the recovered
structure gives helpful information of the underlying source bodies, though there are some distortions due to the complexity of the underlying structure. For example, in Figure 9 (a), the recovered structure near \{Easting = 0.75 km, Northing = 0.6 km\} is shallower than the exact source body, and this is partly because the magnetic anomaly data generated by the deeply buried source body is not significant comparing to its ambient data, where one can refer to Figure 8 (b) for the shape of measurement data.

5.2.3 Inherent non-uniqueness

We illustrate the limitation of our multiple level set algorithm. The exact model is shown in Figure 10(a), where two dykes are buried in the south-north and north-south directions both with a dipping angle of 45°, and the gray color indicates the source body with lower susceptibility \(\kappa_1 = 0.04\) and the dark color indicates the higher susceptibility \(\kappa_2 = 0.08\). Under an inducing field with a strength of \(5 \times 10^4\) nT and a direction at \(I = 75^0\) and \(D = 25^0\), the magnetic anomaly data are collected on the measurement surface \(\Gamma = \{z = -0.1\text{ km}\}\) as shown in Figure 10 (b). We set the regularization weight in the total objective function to be \(\alpha = 50\), and we perform inversion twice with different sets of initial guesses.

In the first experiment the initial guess of \(\phi_i (i = 1, 2)\) is given in equations (25) and (26) and displayed in Figure 11 (a). Figure 11 (b) gives the recovered solution. Figures 11 (c)-11 (f) display the cross-sections at Easting = 0.3 km, Easting = 0.7 km, Depth = 0.15 km, and Depth = 0.25 km, respectively. The recovered structure matches well with the exact model, and the shapes and locations of two different dykes are well recovered.

In the second experiment we interchange the initializations of \(\phi_1\) and \(\phi_2\) so that

\[
\phi_{1,\text{initial}} = 1 - \sqrt{\frac{(x - 0.5)^2}{0.4^2} + \frac{(y - 0.25)^2}{0.15^2} + \frac{(z - 0.25)^2}{0.2^2}},
\]

\[
\phi_{2,\text{initial}} = 1 - \sqrt{\frac{(x - 0.5)^2}{0.4^2} + \frac{(y - 0.75)^2}{0.15^2} + \frac{(z - 0.25)^2}{0.2^2}}.
\]

Figure 12 (a) displays the shape of the initial guess, where the gray color indicates \(\kappa_1 = 0.04\) and the dark color indicates \(\kappa_2 = 0.08\). Figure 12 (b) provides the recovered solution, and Figures 12 (c)-12 (f) display the cross-sections at Easting = 0.3 km, Easting = 0.7 km, Depth = 0.15 km, and Depth = 0.25 km, respectively. One finds that the recovered solution is different from the exact model as shown in Figure 10 (a); essentially the positions of two dykes with different susceptibilities are reversed in the recovered structure. The gray dyke with susceptibility \(\kappa_1 = 0.04\) is replaced by a smaller source body with higher susceptibility...
$\kappa_2 = 0.08$, and the dark dyke with susceptibility $\kappa_2 = 0.08$ is replaced by a larger source body with lower susceptibility $\kappa_1 = 0.04$.

We mention that this effect is due to the inherent non-uniqueness in the magnetic inversion problem. In this example the structures in Figures 11 (b) and 12 (b) have generated similar magnetic anomaly data on the measurement surface $\Gamma = \{ z = -0.1 \text{km} \}$, so that either of them can fit the data generated by the underlying structure. We may need more information of the magnetic mass distributions in the subsurface, and then we can choose the appropriate initial guess to recover the expected solution.

6 APPLICATION TO FIELD DATA

In this section we apply the level set inversion to a set of airborne magnetic data from mineral exploration. The data are a subset of a larger set acquired in a basin environment to identify magnetic sources associated with fault structures that may have the potential for hosting mineralization. The line spacing was approximately 200 m, with measurements every 30 m along the lines. The elevations of the observation varies slightly between from 569.6 m and 612.6 m over 4,000 m. The inclination of the inducing field was $I = 79^\circ$ and the declination was $D = 12^\circ$. This data set is studied by Davis & Li (2013) and the authors estimate the standard deviation of the data to be 2.5 nT. Given the geological setting, no remanent magnetization is expected and the data are consistent with the assumption of induced magnetization used in the forward modeling of our level set inversion.

To test the level set inversion algorithm, we first removed an IGRF field to obtain the total-field anomaly and then performed a susceptibility inversion-based regional residual separation to extract the anomalies deemed to be produced by a set of confined magnetic sources. Figures 13 (a) and (b) show respectively the extracted anomaly and the corresponding elevation variations. The data locations are indicated by white dots.

We perform a single level set inversion with the susceptibility value $\kappa_0 = 0.05$. We do not have petrophysical measurements to directly assign the susceptibilities of the magnetic sources in the inversion. Instead, we estimate this average value from the 3D susceptibility inversion presented by Davis & Li (2013).

The model domain of computation for the 3-D inversion is 4.9 km in easting by 4.35 km in northing by 1.8 km in depth, which is uniformly discretized into cubic cells of 50 m on each side. The data area is centered directly above the model domain. This results in a total of $99 \times 88 \times 37$ cells in the mesh. The initial guess shown in Figure 14(a) is a single oblate spheroid with the semimajor axis of 1.5 km and the semiminor axis of 0.6 km. We use a
numerical Heaviside function $H_\epsilon(\phi)$ with $\epsilon = \Delta x = 50 \text{ m}$, which leads to a smoother interface and produces more stable iterations from the inversion algorithm.

The optimal regularization parameters $\alpha$ is found to be $\alpha = 2.5$ in this inversion. The resultant data misfit value corresponds to a RMS difference of 6.25 nT, which is slightly higher for the estimated standard. However, examining the difference map shows that this misfit level is fine considering that the restrictive condition of isolated causative bodies with a single susceptibility value is being imposed. Figure 13 (c) and Figure 13 (d) display the predicted data and difference between observed and predicted data, respectively. We observe that the predicted data are a good representation of the observed data in Figure 13 (a). Meanwhile, large differences only occur along the east and north edges of the data map, where the presence of negative anomalies are notably misfit. This occurs because these partial anomalies are related to magnetic sources outside the computational domain and they cannot be reproduced by any magnetic objects with the specified susceptibility. The anomalies in the central part of the data map are well reproduced and the difference is consistent with the estimated standard deviation of 2.5 nT.

The inverted model is show in Figure 14(b), viewed from the southeast direction. We observe a main causative body striking in NW direction and three separate minor bodies to the north and south of the main body. These causative bodies have direct correspondence with the total-field anomalies present in the data in Figure 13(a). Figure 15 displays two cross-sections and one plane section of the recovered model. We observe a clear demarcation between the magnetic sources with the specified susceptibility and the zero-susceptibility background, and the spatial extent in horizontal and vertical directions are well defined. We remark that this result is highly consistent with that from the magnetic susceptibility inversion obtained by Davis & Li (2013). Thus, the level-set inversion is effective in the inversion of this field data set and would have provided the same general interpretation as that of magnetic susceptibility but with the added benefit of clearly defined boundaries.

7 CONCLUSION

In this paper we have developed a level set method for 3-D inversion of magnetic data. We have applied a single level set formulation for the model with one uniform susceptibility value for multiple causative bodies, and a multiple level set formulation for the model with distinct susceptibility values for different causative bodies. With the prior knowledge of magnetic susceptibility values, the level set algorithm takes full advantage of a priori information and alleviates the inherent non-uniqueness in the magnetic inversion problem. Moreover we have
imposed additional regularization to smoothen the shape of recovered structures, which has further reduced the ill-posedness of the inverse problem due to the availability of only a finite number of inaccurate data. The methodology has the potential to be generalized to study subsurface structures with multiple susceptibility values. The large number of numerical examples based on synthetic and field data sets have demonstrated the effectiveness of the method while elucidating the limitations.

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REFERENCES


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Figure 5. Three source bodies with different susceptibilities. (a) exact model; (b) magnetic anomaly data collected on the measurement surface; (c) initial guess; (d) solution. The gray color indicates $\kappa_1 = 0.04$ and the dark color indicates $\kappa_2 = 0.08$. 
Figure 6. Three source bodies with different susceptibilities, display of cross-sections. The dashed line indicates the exact model, the solid line plots the recovered structure; the gray color corresponds to \( \kappa_1 = 0.04 \) and the dark color corresponds to \( \kappa_2 = 0.08 \). (a) Easting = 0.2 km; (b) Easting = 0.7 km; (c) Depth = 0.25 km; (d) Depth = 0.35 km
Figure 7. Three source bodies with different susceptibilities, 5% Gaussian noise added to the measurement. (a) magnetic anomaly data with 5% Gaussian noise; (b) solution; (c)-(f) cross-sections at Easting = 0.2 km, Easting = 0.7 km, Depth = 0.25 km, and Depth = 0.35 km, respectively. The dashed line indicates the exact model and the solid line plots the recovered structure; the gray color corresponds to $\kappa_1 = 0.04$ and the dark color corresponds to $\kappa_2 = 0.08$. 
Figure 8. Four source bodies with different susceptibilities. (a) exact model; (b) magnetic anomaly data collected on the measurement surface; (c) initial guess; (d) solution. The gray color indicates $\kappa_1 = 0.04$ and the dark color indicates $\kappa_2 = 0.08.$
Figure 9. Four source bodies with different susceptibilities, display of cross-sections. The dashed line indicates the exact model, the solid line plots the recovered structure; the gray color corresponds to $\kappa_1 = 0.04$ and the dark color corresponds to $\kappa_2 = 0.08$. (a) Easting = 0.75 km; (b) Northing = 0.8 km; (c) Northing = 0.375 km; (d) Depth = 0.15 km; (e) Depth = 0.3 km
Figure 10. Inherent non-uniqueness. (a) Exact model. (b) magnetic anomaly data. In (a) the gray color indicates $\kappa_1 = 0.04$ and the dark color indicates $\kappa_2 = 0.08$. 
Figure 11. Inherent non-uniqueness, solution using the first set of initial guess. (a) Initial guess; (b) recovered solution; (c)-(f) cross-sections at Easting = 0.3 km, Easting = 0.7 km, Depth = 0.15 km and Depth = 0.25 km, respectively. The dashed line indicates the exact model, the solid line plots the recovered structure; the gray color corresponds to $\kappa_1 = 0.04$ and the dark color corresponds to $\kappa_2 = 0.08$. 
Figure 12. Inherent non-uniqueness, solution using the second set of initial guess. (a) Initial guess; (b) solution; (c)-(f) cross-sections at Easting = 0.3 km, Easting = 0.7 km, Depth = 0.15 km and Depth = 0.25 km, respectively. The dashed line indicates the exact model, the solid line plots the recovered structure; the gray color corresponds to $\kappa_1 = 0.04$ and the dark color corresponds to $\kappa_2 = 0.08$. 
Figure 13. Application to field data. (a) Observed total-field anomaly data; (b) location of measurements: the contour map shows the elevation, and the white dots display the location of measurements; (c) predicted data; (d) difference between observed and predicted data.
Figure 14. Application to field data, solution. (a) initial guess of inversion; (b) recovered structure.
Figure 15. Cross-sections of the field data inversion. (a) Cross-section at Easting = 1000 m; (b) cross-section at Northing = 1800 m; (c) cross-section at Elevation = 15 m.
We develop a low-rank-matrix decomposition algorithm to evaluate the integral and summation arising in the magnetic-inversion problem. Without loss of generality, we provide the detailed formulations for the following integral and summation:

\[ P(\mathbf{r}) = \int_{\Omega} \kappa(\tilde{\mathbf{r}}) \cdot K(\mathbf{r}, \tilde{\mathbf{r}}) \, d\tilde{\mathbf{r}}, \quad \mathbf{r} \in \Gamma \]  
(A.1)

\[ F(\tilde{\mathbf{r}}) = \mathcal{I}_{T_\tau}(\tilde{\mathbf{r}}) \cdot \sum_{k=1}^{M} \frac{d_k - d^*_k}{\sigma_k^2} \cdot K(\mathbf{r}_k, \tilde{\mathbf{r}}), \quad \tilde{\mathbf{r}} \in \Omega \]  
(A.2)

where \( \mathcal{I}_{T_\tau} \) is given by equation (24):

\[ \mathcal{I}_{T_\tau}(\tilde{\mathbf{r}}) = \begin{cases} 1, & \tilde{\mathbf{r}} \in T_\tau \\ 0, & \tilde{\mathbf{r}} \in \Omega \setminus T_\tau \end{cases} \]

with support \( T_\tau = \{ \tilde{\mathbf{r}} \in \Omega : |\phi(\tilde{\mathbf{r}})| \leq \tau \}. \) The integral (A.1) arises in the computation of the predicted data \( d(\mathbf{r}); \) the summation (A.2) arises in the computation of gradient directions.

To describe the numerical integration, we take \( \Omega = [0, X] \times [0, Y] \times [0, Z], \) which is uniformly discretized into \( N = n_x \times n_y \times n_z \) points \( \{ \tilde{\mathbf{r}}_j \}_{j=1}^{N} \) with grid size \( \tilde{\mathbf{h}} \) in each direction. Thus the integrals (A.1) can be discretized as follows:

\[ P(\mathbf{r}_k) \approx \tilde{\mathbf{h}}^3 \cdot \sum_{j=1}^{N} \kappa(\tilde{\mathbf{r}}_j) K(\mathbf{r}_k, \tilde{\mathbf{r}}_j), \quad k = 1, 2, \ldots, M; \]  
(A.3)

Rewriting equations (A.2) and (A.3) in matrix form, we have

\[ \mathbf{P} = \tilde{\mathbf{h}}^3 \cdot \mathbf{K} \kappa^T, \]  
(A.4)

\[ \mathbf{F} = \mathbf{d}^T \mathbf{K} \mathbf{D}, \]  
(A.5)

where we have defined

\[ \mathbf{P} = [P(\mathbf{r}_1), \ldots, P(\mathbf{r}_M)]^T, \]
\[ \mathbf{d} = \left[ \frac{d(\mathbf{r}_1) - d^*(\mathbf{r}_1)}{\sigma_1^2}, \ldots, \frac{d(\mathbf{r}_M) - d^*(\mathbf{r}_M)}{\sigma_M^2} \right]^T, \]
\[ \kappa = [\kappa(\tilde{\mathbf{r}}_1), \ldots, \kappa(\tilde{\mathbf{r}}_N)], \]
\[ \mathbf{F} = [F(\tilde{\mathbf{r}}_1), \ldots, F(\tilde{\mathbf{r}}_N)], \]

and

\[ \mathbf{K} = [K(\mathbf{r}_k, \tilde{\mathbf{r}}_j)]_{M \times N}, \]
$$\mathbf{D} = \text{diag}\{\mathcal{I}_{T_{\tau}}(\tilde{r}_1), \ldots, \mathcal{I}_{T_{\tau}}(\tilde{r}_N)\}.$$ 

Due to the large matrix \( \mathbf{K} = [K(r_k, \tilde{r}_j)]_{M \times N} \), matrix-vector multiplications in equations (A.4) and (A.5) are of superlinear complexity

$$\mathcal{O}(MN) = \mathcal{O}(N^{5/3}),$$

which are expensive when \( N \) is large. Recalling that the integration kernel \( K(r_k, \tilde{r}_j) \) is given by equation (7) and the magnitude of \( K(r_k, \tilde{r}_j) \) decays rapidly when the distance \( |r_k - \tilde{r}_j| \) increases, we are motivated to partition \( \mathbf{K} \) into submatrices such that each submatrix is expected to have low-rank structure, based on which a fast algorithm can be developed to speed up relevant matrix-vector multiplications.

We first split the \( N = n_x \times n_y \times n_z \) mesh points \( \{\tilde{r}_j\}_{j=1}^N \) into \( n_z \) sets

$$\{\tilde{r}_i^j \mid 1 \leq l \leq \tilde{M}\}, \quad i = 1, \ldots, n_z,$$

with the \( z \)-coordinate of \( \tilde{r}_i^j \) being \( (i-1)\tilde{h} \), where \( \tilde{M} \) denotes \( n_x \times n_y \). According to this, we partition \( \mathbf{K} \) into \( n_z \) submatrices \( [\mathbf{K}^1, \mathbf{K}^2, \ldots, \mathbf{K}^{n_z}] \) where

$$\mathbf{K}^i = [K(r_k, \tilde{r}_i^j)]_{M \times \tilde{M}}.$$ 

To illustrate the partition more clearly, we provide a particular example. In our numerical implementation, the computational domain \( \Omega \) is taken to be \( \Omega = [0, 1] \text{km} \times [0, 1] \text{km} \times [0, 0.5] \text{km} \) and discretized into \( N = 41 \times 41 \times 21 = 35301 \) mesh points with grid size \( \tilde{h} = 0.025 \); the measurement surface \( \Gamma \) is set as \( \Gamma = [0, 1] \text{km} \times [0, 1] \text{km} \times \{z = -0.1 \text{km}\} \) and there are \( M = 21 \times 21 = 441 \) observation points uniformly distributed on \( \Gamma \). Thus the kernel matrix \( \mathbf{K} \) is of size \( 441 \times 35301 \), and it is partitioned into \( n_z = 21 \) \( 441 \times 41^2 \) submatrices \( [\mathbf{K}^1, \mathbf{K}^2, \ldots, \mathbf{K}^{21}] \), columns of which are indexed by mesh points with the \( z \)-coordinate 0, 0.025, 0.05, \ldots, 0.5, respectively.

As \( \Gamma \) is parallel to the top of \( \Omega \), the distance between \( \Gamma \) and the slice \( z = (i-1)\tilde{h} \) increases as \( i \) increases; so that entries in \( \mathbf{K}^i \) decay rapidly with \( i \) increasing, and therefore the numerical rank of \( \mathbf{K}^i \) decreases with \( i \) as well.

Next, we apply singular value decompositions (SVDs) to find a low-rank-matrix decomposition of \( \mathbf{K}^i \) for \( 1 \leq i \leq n_z \). Starting with \( \mathbf{K}^1 \), we first compute its standard SVD:

$$\mathbf{K}^1 = \mathbf{U}^1 \mathbf{S}^1 \mathbf{V}^1^T,$$

where \( \mathbf{U}^1 \) is an \( M \times M \) unitary matrix, \( \mathbf{V}^1 \) is an \( \tilde{M} \times \tilde{M} \) unitary matrix, and \( \mathbf{S}^1 = (\text{diag}\{\eta_s\})_{M \times \tilde{M}} \) is an \( M \times \tilde{M} \) diagonal matrix with its diagonal entries being the singular values \( \eta_s \) in descend-
ing order. Then we choose a threshold $\epsilon_{SVD}$, set those singular values less than $\epsilon_{SVD}$ to be 0, and obtain the following truncated SVD:

$$K^1 \approx \tilde{U}^1 \tilde{S}^1 (\tilde{V}^1)^T.$$ 

Here $\tilde{S}^1 = \text{diag}\{\eta_s\}_{1 \leq s \leq \tau_1}$ is a $\tau_1 \times \tau_1$ diagonal matrix, $\tilde{U}^1$ and $\tilde{V}^1$ are the first $\tau_1$ columns of $U^1$ and $V^1$, respectively, and $\tau_1$ is the number of preserved singular values which indicates the numerical rank of $K^1$. Similar procedures can be executed for the remaining matrices $K^i$ to obtain their truncated SVDs:

$$K^i \approx \tilde{U}^i \tilde{S}^i (\tilde{V}^i)^T,$$

where the unitary matrices $\tilde{U}^i$ and $\tilde{V}^i$ have the size $M \times \tau_i$ and $\tilde{M} \times \tau_i$ respectively, and the $\tau_i \times \tau_i$ diagonal matrix $\tilde{S}^i$ consists of the first $\tau_i$ singular values, for $i = 2, \cdots, n_z$. Although it is expensive to compute the truncated SVDs of $K^i$, this computational procedure is executed only once as a preprocessing step and we store those matrices $[\tilde{U}^i, \tilde{S}^i, \tilde{V}^i]$ so that we can reload them whenever they are needed.

Based on these truncated SVDs, we have the following evaluations for formulas (A.4) and (A.5):

$$P = \tilde{h}^3 \cdot [K^1, \cdots, K^{n_z}] \kappa^T = \tilde{h}^3 \cdot \sum_{i=1}^{n_z} K^i (\kappa_i)^T$$

$$\approx \tilde{h}^3 \cdot \sum_{i=1}^{n_z} (\tilde{U}^i (\tilde{S}^i (\kappa_i \tilde{V}^i)^T))) ;$$

(A.6)

and

$$F_i = d^T K^i D_i \approx ((d^T \tilde{U}^i) \tilde{S}^i) (D_i^T \tilde{V}^i)^T, \quad 1 \leq i \leq n_z$$

(A.7)

$$F = [F_1, F_2, \cdots, F_{n_z}].$$

(A.8)

In the above, for $1 \leq i \leq n_z$, we have defined

$$\kappa_i = [\kappa(\tilde{r}_1), \kappa(\tilde{r}_2), \cdots, \kappa(\tilde{r}_{\tilde{M}})],$$

$$D_i = \text{diag} \{I_{T_r} (\tilde{r}_l^i)\}_{1 \leq l \leq \tilde{M}}.$$

Since the diagonal entry $I_{T_r} (\tilde{r}_l^i)$ in $D_i$ is 0 when $\tilde{r}_l^i \in \Omega \setminus T_r$, we perform the multiplication of $D_i^T$ and $\tilde{V}^i$ at first to avoid some unnecessary calculations in equation (A.7).

To sum up, we mention that $O(\tilde{M}) = O(M) = O(N^{2/3})$, therefore by equations (A.6), (A.7) and (A.8), the computational complexity is reduced from $O(N^{5/3})$ to
\[ \sum_{i=1}^{n_z} \mathcal{O}(\tau_i \tilde{M}) = \mathcal{O}(N^{2/3} \sum_{i=1}^{n_z} \tau_i). \]

In practice, according to the accuracy requirement, one can adjust the threshold $\epsilon_{SVD}$ to make the sum of $\tau_i$ smaller and to further improve the performance of this low-rank-matrix decomposition algorithm.