Abstract. Gaussian beams are approximate solutions to hyperbolic partial differential equations that are concentrated on a curve in space-time. In this paper, we present a method for computing the stationary in time wave field that results from steady air flow over topography as a superposition of Gaussian beams. We derive the system of equations that governs these mountain waves as a linearization of the basic equations of fluid dynamics and show that this system is well-posed. Furthermore, we show that the approximate Gaussian beam stationary solution is close to a true time-dependent solution of the linearized system.

Key words. orographic waves, geometric optics, ray tracing, superpositions of Gaussian beams

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1. Introduction. Mountain waves are stationary atmospheric waves generated by steady air flow over topography. These waves can propagate far in the vertical direction and are in part responsible for transporting momentum to the top of the troposphere. Since these waves exist on a scale that is smaller than the scales resolved by general circulation models, their effect on the large-scale model flow is not directly resolved; it has to be parametrized and incorporated as a correction in simulations of Earth’s atmosphere. In this paper, we explore a new method for calculating the stationary wave field generated by steady wind flow over mountains—the method of superposition of Gaussian beams.

Gaussian beams are approximate asymptotically valid solutions to hyperbolic partial differential equations which are concentrated on a single curve through the domain. The existence of such solutions has been known to the pure mathematics community since sometime in the 1960s, and these solutions have been used to obtain results on propagation of singularities in hyperbolic PDEs (see [5] and [9]). An integral superposition of these solutions can be used to define a more general solution that is not necessarily concentrated on a single curve. We will show that one can use these solutions as building blocks to assemble more general solutions, and, in particular, we will use a Gaussian beam superposition to model mountain waves. Gaussian beams are closely related to geometric optics or ray tracing. Ray tracing as a method for modeling mountain waves was explored by Broutman, Rottman, and Eckermann in [1] and [2]. A common problem with ray tracing is that the resulting solution does not exist globally. Generally, this breakdown of the solution occurs when nearby rays cross, resulting in a caustic where ray tracing incorrectly predicts that the amplitude...
of the solution is infinite. This blowup can be corrected and the solution can be extended past caustics, once they are identified, by Maslov’s method. This technique was also explored by Broutman, Rottman, and Eckermann in [1]; however, as pointed out in [2], caustics can occur anywhere along the ray, and their correction in numerical ray tracing is nontrivial. Gaussian beams have the advantage over geometric optics of being accurate approximations near all points on a ray. Hence, superpositions of Gaussian beams provide a global solution of the PDE without having to identify and correct for caustics.

In this paper, we will derive the equations for mountain waves from the basic equations of fluid dynamics and show how one obtains a full description of the atmosphere, including the wind velocity, density, and pressure, using a superposition of Gaussian beams. In the appendix, we show that the equations that govern mountain waves are well-posed in the sense of Hadamard, and we provide a description of the general construction of Gaussian beams. We will also show that the stationary Gaussian beam solution is an approximate solution to the full time-dependent formulation of the mountain wave equations.

In the geophysical applications, Gaussian beam superpositions have been used to model the seismic wave field [3] and for seismic migration [4]. The computational methods in these papers are based on ray-centered coordinates which prove to be computationally inefficient. More recently, inspired by our work, an Eulerian computational approach was proposed in [7] which overcomes some of these difficulties.

2. Mountain waves.

2.1. Time-dependent formulation. The equations relating the velocity, \( \vec{u} = (u, v, w) \), density, \( \rho \), and pressure, \( P \), in three dimensions, \( x = (x_1, x_2, x_3) \), are

\[
\frac{D}{Dt} \vec{u} + \frac{1}{\rho} \nabla_x P + \nabla_x (g x_3) = 0, \\
\rho_t + \nabla_x \cdot (\rho \vec{u}) = 0,
\]

where \( \frac{D}{Dt} = \partial_t + \vec{u} \cdot \nabla_x \). These two relations are the mathematical formulation of conservation of momentum and conservation of mass. They give us four equations for the five unknown functions. In order to complete the system, we will use the polytropic gas law \( P = A(S) \rho^\gamma \), where \( \gamma \) is a constant and \( A \) is some function of the entropy, \( S \). Since \( \frac{D}{Dt} S = 0 \), we have

\[
\frac{D}{Dt} (P \rho^{-\gamma}) = 0.
\]

We will linearize this system of equations about a steady background flow solution which depends only on the height \( x_3 \):

\[
\vec{u}_0(x_3) = (u_0(x_3), v_0(x_3), 0), \quad \rho_0(x_3), \quad \text{and} \quad P_0(x_3).
\]

Since this background state needs to satisfy the system of equations, from the third component of the conservation of momentum equation, we have the hydrostatic relationship \( \partial_{x_3} P_0 = -g \rho_0 \). We treat mountain waves as a perturbation of this state. To carry out the perturbation, we linearize the system about the background steady state and obtain the linear system of equations for the perturbation quantities (we will abuse notation here and denote them by \( (\vec{u}, \rho, P) \)):

\[
\frac{D}{Dt} \vec{u} + w \frac{\partial \vec{u}_0}{\partial x_3} + \frac{1}{\rho_0} \nabla_x P - \frac{\rho}{\rho_0^2} \nabla_x P_0 = 0,
\]
We will use \( \overline{\rho} \) from the previous equation. The topography, \( h(x') \), enters the equations as a boundary condition for the vertical velocity, \( w(t, x', 0) = \overline{u}_0(0) \cdot \nabla_x h(x') \), where \( x' = (x_1, x_2) \) and \( \overline{u}_0 = (u_0(x_1), v_0(x_1)) \).

In order to apply existence and uniqueness results to this hyperbolic system, it is convenient to write it in symmetric form. After some elementary row operations, this system of equations can be written as

\[
(A_0 \frac{\partial}{\partial t} + B_1 \frac{\partial}{\partial x_1} + B_2 \frac{\partial}{\partial x_2} + B_3 \frac{\partial}{\partial x_3} + C) \begin{bmatrix} \overline{u} \\ \rho \\ P \end{bmatrix} = 0,
\]

where, with \( \alpha = \frac{\rho_0}{\gamma T_0^2} + \frac{1}{\gamma T_0 \rho_0} \) and \( \beta = \frac{\rho_0}{\gamma T_0} \),

\[
A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\beta \\ 0 & 0 & 0 & \alpha \end{pmatrix}, \quad B_1 = \begin{pmatrix} u_0 & 0 & 0 & 0 \\ 0 & u_0 & 0 & 0 \\ 0 & 0 & -\beta u_0 & 0 \\ \frac{1}{\rho_0} & 0 & 0 & \beta u_0 \end{pmatrix},
\]

\[
B_2 = \begin{pmatrix} v_0 & 0 & 0 & 0 \\ 0 & v_0 & 0 & 0 \\ 0 & 0 & v_0 & -\beta v_0 \\ 0 & 0 & 0 & \beta v_0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\rho_0} \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
C = \begin{pmatrix} 0 & \partial_{x_3} u_0 & 0 & 0 \\ 0 & \partial_{x_3} v_0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_{x_3} P_0 \\ 0 & 0 & \alpha \partial_{x_3} P_0 & 0 \end{pmatrix}.
\]

Note that, since \( \alpha > \beta^2 \), \( A_0 \) is a positive definite matrix and that the eigenvalues of \( B_3 \) are \( \{0, 0, 0, -1/\rho_0, 1/\rho_0\} \). Thus, Theorem A.4 in section A.2 guarantees that a solution to this problem exists and is unique, provided that we specify initial data and suitable boundary conditions. An adequate choice for the boundary condition that satisfies the requirements of Theorem A.4 is

\[
(2.2) \quad M \begin{bmatrix} \overline{u} \\ \rho \\ P \end{bmatrix} \bigg|_{x_3=0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ \rho \end{bmatrix} \bigg|_{x_3=0} = \begin{bmatrix} \overline{u}_0(0) \cdot \nabla_x h(x') \end{bmatrix}.
\]

In other words, we need only specify the boundary values of the vertical wind velocity component.

2.2. Approximate time-independent solutions. Since we are looking for stationary wave solutions in the far field, it is convenient to rescale the space variables. We let \( \bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) = (x_1/R, x_2/R, x_3/R) \), where \( R \) is the large far field parameter. We will use \( \nabla \) to denote the gradient with respect to \( (\bar{x}_1, \bar{x}_2, \bar{x}_3) \), \( \nabla' \) the gradient with respect to \( (\bar{x}_1, \bar{x}_2) \), and \( \nabla_{x'} \) the gradient with respect to \( (x_1, x_2) \). We now look for...
We can find a unique solution to (2.1), (2.5), and solutions to it of the form linearized time-dependent system, called the geometric optics ansatz. Note that solutions of this form will satisfy the difference between this time-dependent solution and an asymptotic solution of (2.4)

\[ H = \gamma R \rho \] where \( \gamma R \rho \) is an extension of \( \gamma R \rho \) on the order of 1.

Consider the system

\[
\begin{align*}
\left( \frac{1}{R} L_1 + C_1' \right) \left[ \begin{array}{c}
\bar{u}' \\
\rho' \\
P'
\end{array} \right] &= 0
\end{align*}
\]

and solutions to it of the form

\[
\begin{align*}
\left[ \begin{array}{c}
\bar{u}' \\
\rho' \\
P'
\end{array} \right] &= e^{iR\phi} \left( \left[ \begin{array}{c}
\bar{u}_1 \\
\rho_1 \\
P_1
\end{array} \right] + \frac{1}{R} \left[ \begin{array}{c}
\bar{u}_2 \\
\rho_2 \\
P_2
\end{array} \right] + \cdots \right),
\end{align*}
\]

called the geometric optics ansatz. Note that solutions of this form will satisfy the linearized time-dependent system,

\[
\begin{align*}
\left( A_0 \frac{\partial}{\partial t} + B_1 \frac{\partial}{\partial x_1} + B_2 \frac{\partial}{\partial x_2} + B_3 \frac{\partial}{\partial x_3} + C \right) \left[ \begin{array}{c}
\bar{u}' \\
\rho' \\
P'
\end{array} \right] &= F',
\end{align*}
\]

for some \( F' \) which is on the order of \( 1/P_0 + 1/R^2 \). Now using the results of section A.2, we can find a unique solution to (2.1), (\( \bar{u}, \rho, P \)), with initial and boundary data given by

\[
\begin{align*}
\left[ \begin{array}{c}
\bar{u}' \\
\rho' \\
P'
\end{array} \right]_{t=0} &= \left[ \begin{array}{c}
u' \\
v' \\
w' + H_e h
\end{array} \right]
\text{ and } M \left[ \begin{array}{c}
u \\
v \\
w
\end{array} \right]_{x_3=0} = \left[ \begin{array}{c} 0 \\
0 \\
\bar{u}_0 \cdot \nabla x' h
\end{array} \right],
\end{align*}
\]

where \( H_e \) is an extension of \( \bar{u}_0 \cdot \nabla x' h - w'|_{x_3=0} \) in the \( x_3 \)-direction. Taking the difference between this time-dependent solution and an asymptotic solution of (2.4) and using the bounds and notations of section A.2, we have

\[
\left\| \left[ \begin{array}{c}
\bar{u}' \\
\rho' \\
P'
\end{array} \right] - \left[ \begin{array}{c}
\bar{u}' \\
\rho' \\
P'
\end{array} \right]_{As} \right\| \leq C_T \left( \| F' \| + \| w' \|_{x_3=0} - \bar{u}_0 \cdot \nabla x' h \right).
Hence, if \((u', \rho', P')\) is an approximate solution that approximates the boundary data well, then it will be close to a true solution of (2.1) in the sense of the inequality above.

We proceed by constructing an approximate solution of (2.4). Substituting the ansatz (2.5) into the system of equations (2.4), we see that to top order in \(R\),

\[
(iL_1 \phi + C'_1) \begin{bmatrix} \tilde{u}_1 \\ \rho_1 \\ P_1 \end{bmatrix} = 0,
\]

where \(L_1 \phi\) denotes the matrix \(\phi \tilde{x}_1 B_1 + \phi \tilde{x}_2 B_2 + \phi \tilde{x}_3 B_3\). In order for the above equation to hold, we need to choose \(\phi\), so that the determinant of the matrix \((iL_1 \phi + C'_1)\) vanishes and we need to choose \((\tilde{u}_1, \rho_1, P_1)\) to belong to the null space of this matrix. Note that this immediately shows us why we need to keep some terms of order 1 in \(C'_1\). If we were to omit them, then the third equation would imply that \(\tilde{u}_0 \cdot \nabla \phi = 0\).

This is a rather strong requirement on the phase function, and it will not permit us to approximate the mountain profile well. Thus, we assume that \(\tilde{u}_0 \cdot \nabla \phi \neq 0\) and compute

\[
\det \left[ \begin{array}{cccc}
\frac{\partial x_1}{\rho_0} & \frac{\partial x_2}{\rho_0} & \frac{\partial x_3}{\rho_0} & 0 \\
\frac{i\partial x_2}{\rho_0} \tilde{u}_0 \cdot \nabla \phi & \frac{i\partial x_2}{\rho_0} \tilde{u}_0 \cdot \nabla \phi & \frac{i\partial x_1}{\rho_0} \tilde{u}_0 \cdot \nabla \phi & 0 \\
0 & \partial x_3 \rho_0 + \frac{\gamma P_0}{\rho_0} & i\tilde{u}_0 \cdot \nabla \phi & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right] = \frac{i}{\rho_0} \left( \tilde{u}_0 \cdot \nabla \phi \right) \left[ -gp_0 \left( \partial x_3 \rho_0 + \frac{\gamma P_0}{\rho_0} \right) |\nabla' \phi|^2 - \rho_0^2 (\tilde{u}_0 \cdot \nabla \phi)^2 |\nabla \phi|^2 \right].
\]

Defining the buoyancy frequency \(N^2(x_3) = g \partial x_3 (\log P_0^{1/\gamma} \rho_0^{-1})\) and using \(\partial x_3 \rho_0 = -gp_0\), for the determinant to vanish we need

\[
\frac{i}{\rho_0} \left( \tilde{u}_0 \cdot \nabla \phi \right) \left[ N^2 |\nabla' \phi|^2 - (\tilde{u}_0 \cdot \nabla \phi)^2 |\nabla \phi|^2 \right] = 0.
\]

Thus \(\phi\) must satisfy the eikonal-type equation,

\[
|D \phi| |\nabla \phi| - N |\nabla' \phi| = 0,
\]

for \(D = (u_0 \partial x_1 + v_0 \partial x_2)\). Provided that \(\nabla' \phi \neq 0\) and the relation above holds, a simple calculation verifies that the null space of \((iL_1 \phi + C'_1)\) is 1 dimensional and that it is spanned by

\[
\tilde{a}_1 = \begin{bmatrix}
-\phi x_3 \phi x_1 \\
-\phi x_3 \phi x_3 \\
\Vert \nabla \phi \Vert^2 \\
(I D \phi) \phi x_3 \phi x_3 \\
\end{bmatrix}.
\]

Since the null space is 1 dimensional, \((u_1, v_1, w_1, \rho_1, P_1) = b_1 \tilde{a}_1\) for some scalar function \(b_1(\tilde{x})\).

The next order terms give the equation

\[
(iL_1 \phi + C'_1) \begin{bmatrix} \tilde{a}_2 \\ \rho_2 \\ P_2 \end{bmatrix} = - (L_1 b_1 + b_1 C'_1) \tilde{a}_1 - b_1 (L_1 \tilde{a}_1).
\]

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This equation gives us an equation for \( b_1 \) in a rather backward fashion. In order for us to be able to solve for \((\tilde{u}_2, \rho_2, P_2)\), the terms on the right-hand side must add up to a vector in the range of \((iL_1 \phi + C_1')\). One can easily compute that the cokernel of \((iL_1 \phi + C_1')\) is spanned by
\[
\begin{bmatrix}
-\phi_x \phi_y \\
\phi_x^2 \\
|\nabla \phi|^2 \\
\frac{\phi_x \phi_y}{|\nabla \phi|^2} \\
(D\phi) \phi \rho
\end{bmatrix}.
\]
Taking the inner product between this vector and \((2.7)\), we obtain an equation for \( b_1 \). Thus, if we solve the resulting equation for \( b_1 \) and \((2.6)\) for \( \phi \), then
\[
e^{iR\phi(x)} b_1(x) \tilde{a}_1(x)
\]
would give us an approximate solution of \((2.4)\).

2.3. Gaussian beam solution. In the traditional geometric optics approach, one would try to solve \((2.6)\) in all of space for a real valued phase function \( \phi \). However, in the spirit of Gaussian beams, we will look for a complex valued function \( \phi \) which satisfies this equation to high order on only one curve through space. The imaginary part will be chosen in such a way so that \((2.5)\) is an approximate solution to \((2.4)\). For details and a proof that such a construction is possible, see section A.1.

We proceed by looking at the null bicharacteristics for the eikonal equation \((2.6)\). Let the bicharacteristics be given by \((X(s), \xi(s))\) and define
\[
\begin{align*}
\hat{D} &= (\xi_1 u_0 + \xi_2 v_0), \\
\hat{D}' &= (\xi_1 \partial_{\xi_2} u_0 + \xi_2 \partial_{\xi_2} v_0), \\
D_0 &= (u_0(0) \xi_1 + v_0(0) \xi_2), \\
p(X, \xi) &= |\hat{D}| \xi - N|\xi'|.
\end{align*}
\]
The null bicharacteristics satisfy
\[
\begin{align*}
\dot{X} &= \frac{\partial(p(X, \xi)}{\partial \xi} = \begin{bmatrix}
\frac{\partial_1 u_0 + \xi_1}{|\xi|} - \frac{\partial_1 \xi_1}{|\xi|^2} \\
\frac{\partial_2 u_0 + \xi_2}{|\xi|} - \frac{\partial_2 \xi_2}{|\xi|^2} \\
\frac{\partial_3 u_0 + \xi_3}{|\xi|} - \frac{\partial_3 \xi_3}{|\xi|^2} \\
\frac{\partial_4 u_0 + \xi_4}{|\xi|} - \frac{\partial_4 \xi_4}{|\xi|^2}
\end{bmatrix}, \\
X(0) &= y = \begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix}, \\
\dot{\xi} &= -\frac{\partial(p(X, \xi)}{\partial X} = \begin{bmatrix}
0 \\
0 \\
-\hat{D}'|\xi| + \frac{\partial \xi'|\partial_{\xi_3} N}{|\hat{D}|} \\
\frac{\partial_3 \xi_3}{|\xi|} - \frac{\partial_3 \xi_3}{|\xi|^2}
\end{bmatrix}, \\
\xi(0) &= \eta = \begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3 \\
\eta_4
\end{bmatrix},
\end{align*}
\]
where “\("\)" signifies differentiation with respect to \( s \). The sign of \( \xi_3(0) \) is chosen to give upward propagating waves, and the parameter \( s \) is chosen so that it increases as \( X_3 \) increases. Note that not all choices for \( \gamma' = (\eta_1, \eta_2) \) will lead to a real valued \( \xi_3(0) \). Thus, we are assuming that \( \gamma' \) is chosen so that \( N(0) > |D_0| > 0 \).

The phase function, \( \phi \), satisfies
\[
\phi = \hat{D}\xi, \quad \phi(0) = y_1 \eta_1 + y_2 \eta_2.
\]

Similarly the Hessian of \( \phi \), \( H = QY^{-1} \), evolves according to
\[
\begin{align*}
\dot{Y} &= CQ + BY, \\
Q &= -B^T Q - AY, \\
Y(0) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
Q(0) &= \begin{bmatrix} i \epsilon & 0 & 0 & \epsilon \\ 0 & i \epsilon & \epsilon & 0 \\ \epsilon & \epsilon & i \epsilon & 0 \\ 0 & \epsilon & 0 & i \epsilon \end{bmatrix},
\end{align*}
\]
with $\epsilon$ a parameter controlling the width of the beam and
\[
(A)_{j,k} = \frac{\partial^2 p_m}{\partial x_j \partial x_k}, \quad (B)_{j,k} = \frac{\partial^2 p_m}{\partial \xi_j \partial \xi_k}, \quad (C)_{j,k} = \frac{\partial^2 p_m}{\partial y_j \partial y_k}, \quad (H)_{j,k} = \frac{\partial^2 \phi}{\partial x_j \partial x_k}.
\]
The $*$'s in the initial condition for $Q$ are determined by the requirement that
\[
Q(0) = Q(0)^T \quad \text{and} \quad Q(0)\dot{\xi}(0) = \dot{\xi}.
\]
Finally, we have the transport equation
\[
\dot{b}_1 = - \left[ \frac{\xi^2 (u_0^2 H_{11} + 2u_0 v_0 H_{12} + v_0^2 H_{22})}{2D|\xi|} - \frac{2u_0 (H_{11} \xi_1 + H_{12} \xi_2 + H_{13} \xi_3) + 2v_0 (H_{12} \xi_1 + H_{22} \xi_2 + H_{23} \xi_3)}{2D|\xi|} - \frac{\dot{\xi}^2 (H_{11} + H_{22} + H_{33}) - N^2 (H_{11} + H_{22})}{2D|\xi|} + \frac{2 \dot{\xi}_0 (\log |\xi|) \dot{\xi}_2}{2|\xi|} \right] b_1
\]
with some initial data $b_1(0; \eta')$.

We let $O$ be a tubular neighborhood of the $x$-projection of the bicharacteristic and define the phase and amplitude functions through a Taylor expansion in $O$. With $\phi = \phi(s(\mathcal{X}))$, $\xi = \xi(s(\mathcal{X}))$, $b_1 = b_1(s(\mathcal{X}))$, and so on, we write the Gaussian beam solution as
\[
\psi_y(\bar{x}) b_1 \begin{bmatrix} -\xi_1 \xi_3 & -\xi_2 \xi_3 & |\xi'|^2 \\ \frac{|\eta|^2}{D_0|\eta_3|} \end{bmatrix} e^{iR(\phi + \xi \cdot (\bar{x} - \mathcal{X}) + \frac{1}{2} |(\bar{x} - \mathcal{X}) \cdot H(\bar{x} - \mathcal{X})|)}
\]
where $\psi_y$ is a $C^\infty$-function supported in $O$ and equal to 1 in a neighborhood of $\mathcal{X}$. Looking at $\bar{x}_3 = 0$, which corresponds to $s = 0$, we see that the Gaussian beam is given by
\[
\psi_y(\bar{x}') b_1(0; \eta') \begin{bmatrix} -\eta_1 \eta_3 & -\eta_2 \eta_3 & |\eta'|^2 \\ \frac{|\eta|^2}{D_0|\eta_3|} \end{bmatrix} e^{iR(\bar{x}' \cdot \eta' + \frac{1}{2} |\bar{x}' - y'|^2)}
\]
at the boundary, with $y' = (y_1, y_2)$ and $\bar{x}' = (\bar{x}_1, \bar{x}_2)$. This is not the form of the boundary condition (2.2); however, as we will see in the next section, this solution can be used as a building block to build solutions that do satisfy the boundary condition.

### 2.4. Boundary conditions and superposition of Gaussian beams

As described in the previous section, for given $(y', \eta')$, provided that $N(0) > |D_0| > 0$, we can construct a Gaussian beam of the form (2.9) with the initial condition for $b_1$ given by
\[
b_1(0; \eta') = \frac{1}{|\eta'|^2} \mathcal{F} \left[ \bar{u}_0 \cdot \nabla_{x'} \mathcal{h} \right],
\]
where $\mathcal{F}$ denotes the Fourier transform from $x'$ to $\eta'$:
\[
\bar{u}_{GB}(\bar{x}; y, \eta) = \psi_y(\bar{x}) b_1(\eta') \begin{bmatrix} -\xi_1 \xi_3 & -\xi_2 \xi_3 & |\xi'|^2 \\ \frac{|\eta|^2}{D_0|\eta_3|} \end{bmatrix} e^{iR(\phi + \xi \cdot (\bar{x} - \mathcal{X}) + \frac{1}{2} |(\bar{x} - \mathcal{X}) \cdot H(\bar{x} - \mathcal{X})|)}.
\]
We claim that the Gaussian beam superposition

\[
\frac{Re}{2\pi^2} \int_{\mathbb{R}^2} \int_{N(0)>|\mathcal{D}_0|>0} u_{GB}(\vec{x}; y', \eta') \ dy'dy'
\]

is a solution to (2.4). Certainly, differentiation in \(\vec{x}\) commutes with these integrals, and so this expression is an asymptotic solution to (2.4). For the boundary condition, we need only look at vertical velocity, since all other quantities are in the null space of the boundary condition matrix \(M\). Now evaluating the Gaussian beam superposition at \(\bar{x}_3 = 0\) and integrating we have for the vertical velocity

\[
\frac{Re}{2\pi^2} \int_{\mathbb{R}^2} \int_{N(0)>|\mathcal{D}_0|>0} \psi_y(\vec{x}')|\eta'|^2 b_1(0; \eta') e^{iR(y' \cdot \vec{x}' + \frac{i}{2} |\vec{x}' - y'|^2)} \ dy'dy' = C \int_{N(0)>|\mathcal{D}_0|>0} \mathcal{F}[\vec{u}_0 \cdot \nabla_{x'} h] e^{i\eta' \cdot \vec{x}'} \ d\eta',
\]

where \(C\) is a constant which approaches 1 as \(Re \to \infty\) and depends on the support of \(\psi_y\). How well the superposition (2.11) satisfies the boundary condition (2.2) depends on how much of the support of the Fourier transform of \(\vec{u}_0 \cdot \nabla_{x'} h(x')\) is contained inside \(N(0) > |\mathcal{D}_0|\).

We make a further remark that will be useful in numerical simulations. If instead of (2.10) we let the initial condition for \(b_1\) be given by

\[
b_1(0; y', \eta') = \frac{\lambda(y')}{|\eta'|^2} \mathcal{F}[\vec{u}_0 \cdot \nabla_{x'} h],
\]

for some smooth compactly supported function \(\lambda\), then the vertical velocity would be approximately given by

\[
\frac{\lambda(\vec{x}')}{2\pi} \int_{N(0)>|\mathcal{D}_0|>0} \mathcal{F}[\vec{u}_0 \cdot \nabla_{x'} h] e^{i\eta' \cdot \vec{x}'} \ d\eta'\]

as \(Re \to \infty\). To show this, we let

\[
Q(\vec{x}') = \frac{1}{2\pi} \int_{N(0)>|\mathcal{D}_0|>0} \mathcal{F}[\vec{u}_0 \cdot \nabla_{x'} h] e^{iR \eta' \cdot \vec{x}'} \ d\eta'
\]

and compute for a Gaussian beam superposition with the modified \(b_1\) at \(\bar{x}_3 = 0:\)

\[
\left| \frac{Re}{2\pi^2} \int_{\mathbb{R}^2} \int_{N(0)>|\mathcal{D}_0|>0} u_{GB}|_{\bar{x}_3=0} \ dy'dy' - \lambda(\vec{x}') Q(\vec{x}') \right| 
\leq \left| \frac{Re}{\pi} \int_{\mathbb{R}^2} \psi_y(\vec{x}') \lambda(y') e^{-\frac{Re}{2} |\vec{x}' - y'|^2} Q(\vec{x}') \ dy' - \lambda(\vec{x}') Q(\vec{x}') \right| 
\leq \left| \frac{Re}{\pi} \int_{\mathbb{R}^2} \left[ \psi_y(\vec{x}') \lambda(y') - \lambda(\vec{x}') \right] e^{-\frac{Re}{2} |\vec{x}' - y'|^2} Q(\vec{x}') \ dy' \right| 
\leq \frac{1}{\pi} |Q(\vec{x}')| \int_{\mathbb{R}^2} \left| \psi_{\vec{x} + z'/\sqrt{Re}}(\vec{x}') \lambda \left( \frac{\vec{x}' + z'}{\sqrt{Re}} \right) - \lambda(\vec{x}') \right| e^{-\frac{1}{2} |z'|^2} \ dz'.
\]

Now expanding \(\lambda(\vec{x}' + z'/\sqrt{Re})\) about \(\vec{x}'\) in a Taylor series, we see that the expression that multiplies \(|Q(\vec{x}')|\) is bounded from above by

\[
\frac{1}{\pi} \int_{\mathbb{R}^2} \left| \left( \psi_{\vec{x} + z'/\sqrt{Re}}(\vec{x}') - 1 \right) \lambda(\vec{x}') \right| + \left| \frac{1}{\sqrt{Re}} z' \cdot \nabla_{\vec{x}'} \lambda(\vec{x}') \right| e^{-\frac{1}{4} |z'|^2} \ dz',
\]

which goes to 0 as \(Re \to \infty\).
3. Numerical results. In order to compare the numerical results obtained through the method of superposition of Gaussian beams to the numerical results obtained by other authors (see [11] and [1]), we need to make the same assumptions and use the same mountain profile and background flow. The basic approximations that these authors make are the hydrostatic, Boussinesq, and incompressible approximations. They also assume that the buoyancy frequency is constant. In essence this changes the approximate system (2.4) by deleting some of the entries in the coefficient matrices which are presumably small:

\[
B_1'' = \begin{pmatrix}
\frac{1}{\rho_0} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\rho_0} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\rho_0} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\rho_0} & 0 \\
0 & 0 & 0 & 0 & \rho_0
\end{pmatrix},
\]

\[
B_2'' = \begin{pmatrix}
v_0 & 0 & 0 & 0 & 0 \\
0 & v_0 & 0 & 0 & 0 \\
v_0 & 0 & \frac{1}{\rho_0} & 0 & 0 \\
n_0 & 0 & 0 & \frac{1}{\rho_0} & 0 \\
0 & 0 & 0 & 0 & \rho_0
\end{pmatrix},
\]

\[
B_3'' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
C_1'' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\frac{1}{\rho_0} & \frac{1}{\rho_0} & \frac{1}{\rho_0} & \frac{1}{\rho_0} & \frac{1}{\rho_0}
\end{pmatrix},
\]

\[
C_2'' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Note that the results of Theorem A.4 apply to this reduced system as well.

Carrying out the same analysis as we did in section 2.2, we find that the eikonal equation in this case is

\[
|D\phi||\phi_{x_3}| - N|\nabla'\phi| = 0,
\]

the null space of \((iL_1'\phi + C_1'')\) is spanned by

\[
\tilde{a}_1 = \begin{pmatrix}
-\phi_{x_3} \phi_{x_3} \\
-\phi_{x_3} \phi_{x_2} \\
|\nabla'\phi|^2 \\
(iL_1\phi)_{x_3} (\phi_{x_3})^2 \\
(D\phi)\phi_{x_3}\rho_0
\end{pmatrix},
\]

and the transport equation for \(b_1\) is

\[
\dot{b}_1 = -\left(\frac{\phi_{x_3}^2 D^2\phi + 4\phi_{x_3} D\phi \phi_{x_3} \phi_{x_3} + \phi_{x_3} (D\phi)^2}{2D\phi|\phi_{x_3}|} - N^2 \nabla'\phi \right) b_1.
\]

We now look at two cases with different background wind flows. We carry out the calculations for both the reduced system of equations above (Examples 1a and 1b) and the system derived in section 2.2 (Examples 2a and 2b). In all cases, the transport equation is initialized with

\[
b_1(0; y', \eta') = \lambda(y')|\eta'|^2 \mathcal{F}[\tilde{u}_0 \cdot \nabla\bar{x} h(x')],
\]

where the mountain profile, \(h\), is of the form

\[
h = \frac{h_0}{(1 + |x'|^2 a^2)^{3/2}},
\]

with \(h_0 = 100m\) and \(a = 20km\), and the cut-off function \(\lambda(y')\) is given by

\[
\lambda(y') = \frac{1}{2} - \frac{1}{\pi} \arctan \left( \frac{|y'| - 80}{2} \right).
\]
This makes the initial condition for $b_1$ essentially compactly supported in $y'$ and thus the integration in $y'$ much simpler. As described in section 2.4, how close the Gaussian beam superposition is to a true time-dependent solution depends on how well the boundary data is approximated. Since the Fourier transform of this particular choice for $h$ is largely contained inside $N(0) > |\hat{D}_0|$ and the cut-off function $\lambda$ is such that $\lambda(x') h(x') \approx h(x')$, the Gaussian beam superposition will approximate the boundary data well.

For the full system as derived in section 2.2, we need the background pressure and density. For the pressure profile, we use

$$P_0(x_3) = c_1(c_2 - \delta x_3)^\tau,$$

with $(c_1, c_2, \delta, r) = (101290, 1.0002, 2.2528 \times 10^{-5}, 5.256)$. This model for the pressure is valid in the troposphere [8]. The density is obtained from the pressure using the hydrostatic relationship

$$\partial_{x_3} P_0 = -g \rho_0.$$

To compare with the results obtained in [11], we also compute the vertical displacement quantity, $\eta$, defined implicitly through the equation

$$\vec{u}_0 \cdot \nabla_{x'} \eta = w.$$

Upon examining the ray tracing system (2.8) and the other equations for the phase, we note that they are independent of $y'$. Thus, to carry out the computations for the phase, we need only compute Gaussian beams for the location $y' = (0, 0)$. For beams that do not originate from $(0, 0)$, we can simply use translation. The equation for the amplitude depends on $y'$, but it does so in such a way that recalculation is unnecessary. The computations were carried out using MATLAB; more specifically, its “ODE23” function was used to solve the system of ODEs.

The specifics of the numerical tests are as follows. We fix the large parameter, $R = 1000$, and the initial beam size, $\epsilon = 5 \cdot 10^{-6}$. The $\eta'$ integral in (2.11) is carried out in polar coordinates for $10^{-7} \leq |\eta'| \leq 5 \cdot 10^{-5}$, using on the order of 8000 points. However, since not all of the rays remain within the computation domain and some of them turn around, as $z$ increases, the superposition is carried out over fewer and fewer Gaussian beams. At $z = 6$ km the number of beams is about 3000. The $y'$ integral is carried over the support of the mountain using about 50 points. More specific details about each one of the tests is given below.

3.1. Example 1a. We consider a background wind profile of the form

$$\vec{u}_0 = U_0[\cos(\pi x_3/H), \sin(\pi x_3/H)]$$

and use the reduced system of equations to compute the stationary wave field. Figure 3.1 shows numerical results obtained for a particular case with $U_0 = 10 m/s$, $H = 12 km$, and $N = 0.0113 s^{-1}$. Each of the computed quantities was multiplied by $\exp(x_3/2H_p)$, $H_p = 7.5 km$, as in [1], to correct for the decrease in density with height. The results obtained are almost identical to the results obtained in [1]; see their Figure 3.

3.2. Example 1b. For this example, we also use the reduced system of equations, but instead we use a background wind profile given by

$$\vec{u}_0 = [U_0, \Lambda x_3]$$

and compare with the results obtained in [11]. The specific values for the wind are $U_0 = 10 m/s$ and $\Lambda = 3 \times 10^{-3} s^{-1}$. The buoyancy frequency is $N = 0.01 s^{-1}$. The
Fig. 3.1. Results obtained using a superposition of Gaussian beams. The contour intervals (CI) and the minimum and maximum values for the plotted quantities are (a) CI = 0.05, (−0.39, 0.36), (b) CI = 0.004, (−0.034, 0.034), (c) CI = 0.05, (−0.23, 0.22), and (d) CI = 0.002, (−0.019, 0.012), which are all in good agreement with the minimum and maximum values obtained in [1].

Fig. 3.2. Results obtained using a superposition of Gaussian beams. The minimum and maximum values for η are −0.0817 and 0.186 times the mountain height \( h_0 \).

results are shown in Figure 3.2. Qualitatively, the results are similar to the results shown by Shutts in Figure 2 of [11], and the Gaussian beam superposition minimum (−0.0817\( h_0 \)) and maximum (0.186\( h_0 \)) compare well with those obtained by Shutts, (−0.085\( h_0 \)) and (0.213\( h_0 \)).

3.3. Example 2a. In this example, we look at the same wind profile as in Example 1a, but instead of the reduced equations we use the system derived in section 2.2.
a) Perturbation quantities at $x_3 = 2.1 km$

![Graphs showing perturbation quantities at $x_3 = 2.1 km$.](image)

b) Perturbation quantities at $x_3 = 6.3 km$

![Graphs showing perturbation quantities at $x_3 = 6.3 km$.](image)

**Fig. 3.3.** Perturbation quantities computed using a superposition of Gaussian beams for the system of equations derived in section 2.2. Each graph has 15 equally spaced contour intervals from the minimum to the maximum values, which are listed at the top of each graph.

The results obtained are shown in Figure 3.3. The first set of graphs shows the perturbation quantities $(u, v, w, \rho, P)$ at $x_3 = 2.1 km$, and the second shows them at $x_3 = 6.3 km$. Qualitatively, the wave field is similar to the field obtained through the reduced system.

### 3.4. Example 2b.

In this example, we look at the same wind profile as in Example 1b, but instead of the reduced equations we use the system derived in section 2.2. The results are shown in Figure 3.4. The minimum and maximum values are higher than those computed in Example 1b; however, this is expected, as in Example 1b no corrections were made for the decrease in density with height. If we multiply each of the quantities by the same exponential growth factor as in Example 1a, then the two
sets of minimums and maximums are in good agreement.

4. Conclusion. We have shown that the method of superposition of Gaussian beams can be used to model the perturbations of the air velocity, density, and pressure that result from steady air flow over topography. The stationary in time solution obtained through this method is globally valid and is close to a true time-dependent solution of the linearized system of equations that governs mountains waves. This method has the advantage over previous methods that it provides a global solution without the need to identify and correct for caustics. As mentioned by other authors, this correction is not easily done in numerical simulations. The numerical results obtained by superposition of Gaussian beams are in good agreement with the results obtained in previous studies by other authors.

Appendix A.

A.1. Construction of Gaussian beams. This section is very closely based on section 2.1 in [9]. For the sake of completeness, however, we review the construction of Gaussian beams here.

Consider the PDE
\begin{equation}
P \left( x_0, x', \frac{\partial}{\partial x_0}, \nabla x' \right) u = 0 \quad \text{for} \quad x = (x_0, x') \in \mathbb{R} \times \mathbb{R}^n,
\end{equation}

where $P$ is a partial differential operator of order $m$. We would like to construct an asymptotically valid solution, $u(x; k)$, that is concentrated on a single smooth curve, $\gamma$. That is to say, $u(x; k)$ is small away from $\gamma$ and $\|Pu(x; k)\| = O(k^{-M})$ for some appropriate norm and fixed integer $M$. Of course, this is not possible for all partial differential operators. The restrictions that are necessary for such a solution to exist will become clear in what follows.

The construction of Gaussian beams begins with the geometric optics ansatz
\begin{equation}
u(x) = e^{ik\phi(x)} \left[ a_0(x) + \frac{a_1(x)}{k} + \cdots + \frac{a_N(x)}{k^N} \right].
\end{equation}
The functions $a_0, \ldots, a_N$ and $\phi$ are all assumed to be smooth, $k$ is the large asymptotic parameter, and $N$ is a fixed number. The requirements on the phase function $\phi$ are slightly different from those of traditional geometric optics. We will require that $\phi$ is real valued on $\gamma$, but away from this curve, $\phi$ can be complex valued with the restriction that the imaginary part of the Hessian of $\phi$ is positive definite on planes perpendicular to $\gamma$. This will make $u$ look like a Gaussian distribution with variance $1/k$ on planes perpendicular to $\gamma$ and hence gives the name to these asymptotic solutions.

The first step is to substitute the ansatz into the PDE:

$$Pu = k^m p_m(x, \nabla \phi)a_0(x)e^{ik\phi(x)} + O(k^{m-1}).$$

Here $p_m(x, \xi)$ is the principal symbol of $P$ and is assumed to be real valued. Recall that this allows us to write $P$ as $p_m(x, i\nabla)$ plus derivatives of order lower than $m$.

We must choose $\phi$ in such a way so that the contribution of the first term is order less than $k^m$. The idea of the construction is to choose the Taylor series of $\phi$ on $\gamma$ in such a way that $p_m$ vanishes to high order on $\gamma$. In combination with the decay from the imaginary part of $\phi$, we will make the contribution of the first term small.

To accomplish this, we must have $p_m$ and a number of its derivatives vanishing on $\gamma$. We require that

$$p_m = 0, \quad \frac{\partial p_m}{\partial x_i} = 0, \quad \frac{\partial^2 p_m}{\partial x_i \partial x_j} = 0, \quad \ldots$$

on the curve $\gamma$. We let $x(s)$ parametrize this curve and define $\xi(s) = \nabla \phi(x(s))$. Using the summation convention and expanding the first two equations,

(A.3) \hspace{1cm} p_m = 0,

(A.4) \hspace{1cm} \frac{\partial p_m}{\partial x_i} + \frac{\partial p_m}{\partial \xi_l} \frac{\partial^2 \phi}{\partial x_i \partial x_l} = 0,

we can make the following conclusion about the curve $x(s)$.

**Theorem A.1.** The construction of an asymptotic solution of the form (A.2) with the restrictions

(i) $x(s)$ is a smooth curve,

(ii) $\phi$ is real on $x(s)$,

(iii) $\text{Im}[\frac{\partial^2 \phi}{\partial x_i \partial x_j}]$ is positive definite on $\dot{x}^+(s)$

is possible only if (up to a rescaling of $s$)

(a) $\dot{x}(s) = \frac{\partial p_m}{\partial \xi}(x(s), \xi(s))$, $\dot{\xi}(s) = -\frac{\partial p_m}{\partial x}(x(s), \xi(s))$,

(b) $p_m(x(0), \xi(0)) = 0$,

(c) $\frac{\partial p_m}{\partial \xi} \neq 0$.

The curve $(x(s), \xi(s))$ in phase space is referred to as a null bicharacteristic of the operator $P$.

**Proof.** Examining (A.4) and noting that $p_m$ is real, we have

$$\frac{\partial p_m}{\partial \xi_l} \text{Im} \left[ \frac{\partial^2 \phi}{\partial x_i \partial x_l} \right] = 0.$$ 

Thus $\frac{\partial p_m}{\partial \xi}$ is in the null space of $\text{Im}[\frac{\partial^2 \phi}{\partial x_i \partial x_j}]$, and hence $\frac{\partial p_m}{\partial \xi}$ cannot belong to $\dot{x}^+$. Therefore, for some constant $\alpha$ we must have

$$\dot{x}(s) = \alpha \frac{\partial p_m}{\partial \xi}.$$
Substituting for $\dot{x}(s)$ in (A.4) and using the compatibility condition

$$\dot{\xi}_i(s) = \frac{\partial^2 \phi(x(s))}{\partial x_i \partial x_j} \dot{x}_j(s),$$

we get

$$\dot{\xi}_i = -\alpha \frac{\partial p_m}{\partial x}.$$

Finally, since $p_m = 0$ and $\dot{p}_m = 0$, the curve $(x(s), \xi(s))$ is a null bicharacteristic of the equation. Since we require $x(s)$ to be a smooth path, $\frac{\partial p_m}{\partial \xi} \neq 0$.

Next, we look at the second order derivative of $p_m$. We have to satisfy

$$\partial^2 p_m \partial x_i \partial x_j = 0$$

on $x(s)$:

$$\frac{\partial^2 p_m}{\partial x_i \partial x_j} + \frac{\partial^2 p_m}{\partial x_i \partial \xi_k} \frac{\partial^2 \phi}{\partial x_k \partial x_j} + \frac{\partial^2 p_m}{\partial x_j \partial \xi_k} \frac{\partial^2 \phi}{\partial x_i \partial x_k} + \frac{\partial^2 p_m}{\partial \xi_i \partial \xi_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0.$$

Recognizing that $\frac{\partial p_m}{\partial \xi_k} \frac{\partial^3 \phi}{\partial x_i \partial x_j} = \frac{d}{ds} \frac{\partial^2 \phi}{\partial x_i \partial x_j}$ and defining

$$A_{i,j} = \frac{\partial^2 p_m}{\partial x_i \partial x_j}, \quad B_{i,j} = \frac{\partial^2 p_m}{\partial x_i \partial \xi_j}, \quad C_{i,j} = \frac{\partial^2 p_m}{\partial \xi_i \partial \xi_j}, \quad H_{i,j} = \frac{\partial^2 \phi}{\partial x_i \partial x_j},$$

the last equation becomes

$$(A.5) \quad A + B^T H + H B + H C H + H = 0.$$ 

This equation is known as a matrix Ricatti equation, and, since it is nonlinear, the solution might not exist for all $s$. In our case, however, the choice of initial data will guarantee that a solution exists for all $s$.

In order to satisfy the requirements that we want to impose on $\phi$, the Hessian of $\phi$ must satisfy the following:

(i) $H(s) = H^T(s)$;

(ii) $H(s) \dot{x}(s) = \dot{\xi}(s)$;

(iii) $\text{Im}[H(s)]$ is positive definite on $\dot{x}^\perp$.

The initial condition $H(0)$ must also satisfy these conditions, and we choose it as follows. Without loss of generality, assume that $\dot{x}_0(0) = 1$. Choose $\text{Im}[H(0)]_{i,j}$ for $1 \leq i, j \leq n$ to be a symmetric positive definite $n \times n$ matrix and choose the rest of $H(0)$ so that (i) and (ii) hold. This procedure determines all of $H(0)$, and, since $\dot{x}_0(0) = 1$, all three conditions will be satisfied for $s = 0$. We obtain a global solution of (A.5) in the following way.

**Theorem A.2.** Let $N(s)$ and $Y(s)$ be the (global) solutions of

$$(A.6) \quad \dot{Y}(s) = CN + BY, \quad \dot{N}(s) = -B^T N - A Y$$

with the initial conditions ($I$ is the identity matrix)

$$(A.7) \quad Y(0) = I, \quad N(0) = H(0).$$

We obtain the following results:
(a) \( (\dot{x}(s), \dot{\xi}(s)) = (Y(s)\dot{x}(0), N(s)\dot{\xi}(0)) \);
(b) \( Y(s) \) is invertible for all \( s \);
(c) \( H(s) = N(s)Y^{-1}(s) \) satisfies (A.5) and conditions (i), (ii), and (iii) above.

Proof. (a) Differentiating \( \dot{x} \) and \( \dot{\xi} \) with respect to \( s \), we see that they satisfy (A.6).

Noting that \( Y(s)\dot{x}(0) \) and \( N(s)\dot{\xi}(0) \) also satisfy (A.6) and that \( \dot{x}(0) = Y(0)\dot{x}(0) \) and \( \dot{\xi}(0) = N(0)\dot{\xi}(0) \), we have that \( \dot{x}(s) = Y(s)\dot{x}(0) \) and \( \dot{\xi}(s) = N(s)\dot{\xi}(0) \) for all \( s \), by uniqueness for ODEs.

(b) Let \( \psi_1(s) = (y^1(s), \eta^1(s)) \) and \( \psi_2(s) = (y^2(s), \eta^2(s)) \) be two vector valued solutions of (A.6). Define
\[
\sigma(\psi_1, \psi_2) = y^2 \cdot \eta^1 - y^1 \cdot \eta^2.
\]

By differentiating this expression in \( s \) and using the fact that \( A = A^\top \) and \( C = C^\top \), we see that \( \sigma \) is constant for solutions of (A.6).

Suppose that \( Y(s) \) fails to be invertible for some \( s = s_0 \). Then there exists a nonzero \( c \in \mathbb{C}^{n+1} \) such that \( Y(s_0)c = 0 \). Certainly, \( \psi(s) = (Y(s)c, N(s)c) \) and \( \overline{\psi(s)} \) are two vector valued solutions to (A.6), and so we have
\[
0 = \overline{Y(s_0)c \cdot N(s_0)c} - Y(s_0)c \cdot \overline{N(s_0)c} = \sigma(\psi(s_0), \overline{\psi(s_0)})
\]
\[
= \sigma(\psi(0), \overline{\psi(0)}) = c \cdot H(0)c - c \cdot H(0)c
\]
\[
= 2\sigma \cdot \Im[H(0)]c.
\]

Since \( \Im[H(0)] \) is positive definite on \( \dot{x}(0)^\perp \), \( c = \alpha \dot{x}(0) \). By part (a), \( \alpha \dot{x}(s_0) = \alpha Y(s_0)\dot{x}(0) = Y(s_0)c = 0 \), and since \( \dot{x}(s) \) is nonvanishing, \( \alpha = 0 \). This is a contradiction, since \( c \) was assumed to be nonzero. Thus, \( Y(s) \) is invertible for all \( s \).

(c) By substituting \( N(s)Y^{-1}(s) \) into (A.5) instead of \( H(s) \), one directly verifies that it satisfies the equation.

(c.i) Since \( H^\top \) also satisfies (A.5) and \( H(0) = H^\top(0) \), it is clear that \( H(s) = H^\top(s) \).

(c.ii) From parts (a) and (b) we deduce that \( \dot{x}(0) = Y^{-1}(s)\dot{x}(s) \), and so \( \dot{\xi}(s) = N(s)Y^{-1}(s)\dot{x}(s) = H(s)\dot{x}(s) \).

(c.iii) With the definitions in part (b), for an arbitrary \( c \in \mathbb{C}^{n+1} \) we have that
\[
2i\sigma \cdot \Im[H(0)]c = \sigma(\psi(0), \overline{\psi(0)}) = \sigma(\psi(s), \overline{\psi(s)})
\]
\[
= \overline{Y(s)c \cdot N(s)c} - Y(s)c \cdot \overline{N(s)c} = \overline{Y(s)c \cdot H(s)Y(s)c} - Y(s)c \cdot \overline{H(s)Y(s)c}
\]
\[
= 2i\overline{Y(s)c} \cdot \Im[H(s)]Y(s)c.
\]

Since \( Y(s)c \in \dot{x}^\perp(s) \) implies \( c \in \dot{x}^\perp(0) \), we have that \( \Im[H(s)] \) is positive definite on \( \dot{x}^\perp(s) \).

This concludes the difficult part of the construction. For higher order derivatives of \( p_m \) on \( \gamma \), we note that the equations that must be solved are linear first order inhomogeneous ODEs for the third and higher order derivatives of \( \phi \). The solutions to these equations will exist for all \( s \), and thus we can make \( p_m \) vanish to any prescribed order on \( \gamma \).

We now look to \( Pu \) to determine the amplitude functions \( a_0, \ldots, a_N \). Once again substituting the ansatz into the equation and collecting powers of \( k \), we have
\[
Pu = \sum_{j=-m}^{N-m} k^{-j}c_j(x)e^{ik\phi(x)}, \tag{A.8}
\]
where the $c_j$’s are defined as
\[
c_{-m} = p_m(x, \nabla \phi)a_0, \\
c_{-m+1} = \frac{1}{i} \frac{\partial p_m}{\partial \xi_i} \frac{\partial a_0}{\partial x_0} + \frac{1}{2i} \frac{\partial^2 p_m}{\partial \xi_i \partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} a_0 \\
+ p_{m-1}(x, \nabla \phi)a_0 + p_m(x, \nabla \phi)a_1 \\
\equiv L a_0 + p_m(x, \nabla \phi)a_1, \\
c_{-m-r+1} = L a_r + p_m(x, \nabla \phi)a_{r+1} + g_r
\]
and $g_r$ is a known function of $\phi, a_0, \ldots, a_{r-1}$. Just as in the case for $\phi$, we will solve these equations for the amplitude functions so that the $c_j$’s vanish on $\gamma$.

If $p_m(x, \nabla \phi)$ vanishes to order $R$ on $\gamma$ and we recursively solve
\[
La_0 = 0, \quad \frac{\partial}{\partial x_i} La_0 = 0, \quad \ldots
\]
up to derivatives of order $R - 2$, $c_{-m+1}$ will vanish to order $R - 2$. Note that on $\gamma$,
\[
La_r = \frac{1}{i} \frac{\partial a_r}{\partial s} + \left[ \frac{1}{2i} \frac{\partial^2 p_m}{\partial \xi_i \partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + p_{m-1} \right] a_r,
\]
and so the equations that we are trying to solve are linear ODEs and they will have solutions for all $s$. Also, note that we cannot solve for derivatives of $a_0$ of order higher than $R - 2$ because those equations require derivatives of $\phi$ of order higher than $R$.

The rest of the amplitude functions are constructed in a similar way. The equations that they satisfy are linear inhomogeneous ODEs, and so their solutions will exist for all $s$ as well. Through this process, we get $a_r$ and its derivatives of order up to $R - 2(r + 1)$ on $\gamma$, making $c_j$ vanish to order $R - 2(m + j)$ on $\gamma$. We remark here that number $N$ in the ansatz (A.2) is closely related to $R$. In fact, $2 \leq R - 2N \leq 3$. This expression comes from a balance between having enough information to determine $a_N$ on $\gamma$ and recognizing that it is useless to have $c_{-m+N}$ vanish to anything beyond order 1 (otherwise the remainder terms would contribute more than it to the whole sum).

To extend the phase and amplitude functions beyond $\gamma$, we use a Taylor series. Using $x_0(s) = s$, we define these functions by
\[
\phi(x) = \sum_{|\alpha|=0}^R \frac{1}{\alpha!} \frac{\partial^\alpha \phi(x(x_0))}{\partial x^\alpha} (x - x(x_0))^\alpha, \\
a_r(x) = \sum_{|\alpha|=0}^{R-2r-2} \frac{1}{\alpha!} \frac{\partial^\alpha a_r(x(x_0))}{\partial x^\alpha} (x - x(x_0))^\alpha,
\]
where $\alpha$ is a multi-index. Clearly these functions are smooth. Note that since the Taylor series of the phase and amplitude functions depend continuously on the initial data, the asymptotic solution given by (A.2) will depend continuously on the initial data.

The final step is to multiply each amplitude by a smooth function supported on $U$ and equal to 1 near $\gamma$. The set $U$ is chosen so that $U \cap \{|x_0| < T\}$ is compact and that $\text{Im}[\phi(x)] > a d^2(x, \gamma)$ for $x \in \{|x_0| < T\} \cap U$. Here $d(x, \gamma)$ is the distance function from $x$ to $\gamma$, and $a$ is some positive fixed constant.
It remains to show that this construction gives us an asymptotic solution. The key result is contained within the following lemma.

**Lemma A.3.** Assume that \( c(x) \) vanishes to order \( S - 1 \) on \( \gamma \), \( \text{supp}(c) \cap \{|x_0| < T\} \) is compact, and \( \text{Im}[\phi(x)] \geq \text{ad}^2(x, \gamma) \) on \( \text{supp}(c) \cap \{|x_0| < T\} \). Then

\[
\int_{|x_0| < T} |c(x)e^{ik\phi(x)}|^2 dx \leq Ck^{-S-n/2}.
\]

**Proof.** Let \( z \) be \( k \)-independent local coordinates for the curve \( \gamma \) such that the curve is traced out by \( z_0 = s, z_i = 0 \) and such that \( \text{ad}^2(x(z), \gamma) \geq z_1^2 + \cdots + z_n^2 \). Rescaling these coordinates so that \( y_0 = z_0 \) and \( y_i = k^{1/2}z_i \), we obtain that \( kd^2(x(y), \gamma) \geq y_1^2 + \cdots + y_n^2 \).

By making this change of variables in the integral, we have that the new integrand is bounded by a constant times

\[
k^{-S-n/2} |k^{S/2}c(x(y_0, k^{-1/2}y'))|^2 e^{-2a|y'|^2}.
\]

Since \( c \) vanishes to order \( S - 1 \), \( |k^{S/2}c(x(y_0, k^{-1/2}y'))|^2 \) is bounded on \( \{|y_0| < T\} \) as \( k \to \infty \). Hence,

\[
\int_{|x_0| < T} |c(x)e^{ik\phi(x)}|^2 dx \leq Ck^{-S-n/2}.
\]

Finally, we estimate the Sobolev \( s \)-norm of \( Pu \) on \( |x_0| < T \) with the help of the lemma. Since differentiation of (A.8) will either lower the order to which \( c_j \) vanishes on \( \gamma \) or multiply \( c_j \) by \( k \), repeated application of the lemma shows that

\[
\|Pu\|_s \leq Ck^{m+s-(R+1)/2-n/4}.
\]

Thus \( \|Pu\|_s = O(k^{-M}) \) for an appropriate choice of \( R \).

**A.2. Well posedness of the symmetric hyperbolic system.** In this section, we prove that the time-dependent problem is well-posed. Before we begin, we introduce the following notations which will be used in this section:

- \( I = [0, T], \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \mathbb{R}_+; \)
- \( \| \cdot \| \) denotes the \( L^2 \) norm in \((t, x) \in I \times \mathbb{R}_+^n; \)
- \( | \cdot | \) denotes the \( L^2 \) norm in \( x \in \mathbb{R}_+^n; \)
- \( \langle \cdot, \cdot \rangle \) denotes the vector dot product in \( \mathbb{R}^k; \)
- “boundary conditions” refer to both the \( x_n = 0 \) boundary and the \( t = 0 \) boundary;
- the Einstein convention of summing over repeated indexes, i.e., \( B_j \partial_{x_j} = \sum_{j=1}^n B_j \partial_{x_j}; \)

**Theorem A.4.** The symmetric hyperbolic system

\[
A_0 \partial_t u + B_j \partial_{x_j} u + Cu \equiv Lu = F(t, x) \quad \text{in } I \times \mathbb{R}_+^n,
\]

(A.9)

\[
M(u - h) = 0 \quad \text{for } x_n = 0,
\]

\[
u = f \quad \text{for } t = 0,
\]

where

- \( A_0(x), B_j(x) \) are \( k \times k \) symmetric and \( \langle A_0(x)u, u \rangle \geq d|u|^2, d > 0, \)
- \( A_0(x), B_j(x), \) and \( C(x) \) are smooth and bounded along with their derivatives,
- \( F, h, \) and \( f \) are \( C^\infty \)-functions in \( L^2, H^1, \) and \( H^1, \) respectively,
- \( B_n \) is of constant rank near \( x_n = 0, \)
• \( M(x') \) is a projection and \( M(x')\mathbb{R}^k \) is a nonnegative subspace for \( B_n(x') \), that is, \( \langle B_n(x')M(x')w, M(x')w \rangle \geq 0 \) for all \( w \) and \( x' \),
• \( M(x')\mathbb{R}^k \) is maximal in the sense that it is not properly contained in any other nonnegative subspace for \( B_n(x') \), and
• \( h(0, x') = Mf(x', 0) \),

is well-posed in the sense that a unique solution exists and its norm is controlled by the norms of the initial data:

\[
\|u\| \leq C_T (\|f\| + \|f\|_{H^1} + \|h\|_{H^1}).
\]

Before we prove this theorem, we need a result which is proved in [10].

**Lemma A.5.** Suppose that

• \( A_0(x) \) is symmetric positive definite,
• \( B_j(x)'s \) are symmetric and \( B_n \) is of constant rank near \( x_n = 0 \),
• \( N(x') \) is a maximal nonpositive subspace for \( B_n(x') \), and
• \( A_0, B_j, \) and \( C \) are assumed to be smooth and bounded along with their derivatives, and \( G \) is in \( L^2 \).

Then there exists a weak solution to

\[
A_0\partial_t u + B_j\partial_{x_j} u + Cu \equiv Lu = G(t, x) \quad \text{in} \quad I \times \mathbb{R}^n_+,
\]

\[
(A.10)
\]

\[
u \in N(t, x') \quad \text{for} \quad x_n = 0,
\]

\[
u = 0 \quad \text{for} \quad t = 0;
\]

i.e., there exists a \( u \in L^2 \) such that for all piecewise continuously differentiable \( v \in C([0, T], C_0(\mathbb{R}^n_+)) \), \( v \in N^*(x') = [B_n(x')N(x')]^\perp \), and \( v(T) = 0 \),

\[
(L^* v, u) = \langle v, G \rangle.
\]

**Proof.** The adjoint operator \( L^* \) is given by

\[
L^* \equiv -A_0\partial_t - (\partial_{x_j} B_j) - B_j\partial_{x_j} + C^*.
\]

Let

\[
Z^* \equiv -\frac{L + L^*}{2} = -\frac{C + C^*}{2} + B_j\partial_{x_j}.
\]

Since \( A_0 \) is a symmetric positive definite matrix, \( \langle A_0 u, u \rangle \geq d|u|^2 \) with \( d > 0 \) and \( A_0 \) = \( S^2 \) for some \( S \). Now, for piecewise continuously differentiable \( v \in C([0, T], C_0(\mathbb{R}^n_+)) \), \( v \in N^* \), and \( v(T) = 0 \), we compute

\[
\partial_t |Sv(T - t)|_{L^2(\mathbb{R}^n_+)}^2 \leq 2\langle v(T - t), Z^* v(T - t) \rangle_{L^2(\mathbb{R}^n_+)}
\]

\[
= 2\text{Re}(\langle A_0 v, v \rangle + 2\text{Re}(\langle -\partial_{x_j} (B_j v) + C^* v, v \rangle - 2 \int_{x_n = 0} (B_n v, v) dx')
\]

\[
\leq 2\text{Re}(L^* v, v),
\]

as \( \langle B_n v, v \rangle \geq 0 \) for \( v \in N^* \). Furthermore, as \( Z^* \) is a bounded operator,

\[
\partial_t |Sv(T - t)|^2 \leq \frac{2}{d^{1/2}} |L^* v(T - t)| |Sv(T - t)| + 2\frac{C^Z}{d} |Sv(T - t)|^2,
\]

\[
\partial_t |Sv(T - t)| \leq C \{ |L^* v(T - t)| + |Sv(T - t)| \}.
\]
By Gronwall’s inequality, we obtain the estimate

$$\sup_{t \in I} |Sv(t)| \leq C_T \left[ |Sv(T)| + \int_0^T |L^*v(T - s)| \, ds \right].$$

Let

$$B = \{ v \mid v \in C(I; C_0(\mathbb{R}^n_+)), \text{v piecewise } C^1, v \in N^*, v(T) = 0 \}$$

and let $R = L^*B$. The estimate (A.11) shows that $L^*$ maps $B \to R$ one-to-one, since for $w_1, w_2 \in B$ and $L^*w_1 = L^*w_2$ we have

$$\sup_{t \in I} |S(w_1 - w_2)| \leq C \int_0^T |L^*w_1(s) - L^*w_2(s)| \, ds = 0.$$ 

Consequently, for a fixed $G \in L^2(I \times \mathbb{R}^n_+)$ we can define a linear functional on $R$ by

$$l(r) = l(L^*w) = \int_I \int_{\mathbb{R}^n_+} \langle w, G \rangle \equiv (w, G).$$

Now

$$|l(r)| \leq \|w\| \|G\| \leq C \|Sv\| \|G\| \leq C \|L^*w\| \|G\| = C \|r\| \|G\|,$$

and so $l$ is a bounded linear functional on $R$. As $B$ is a subspace of $L^2$, the Hahn–Banach theorem allows us to extend $l$ to all of $L^2$, and the Riesz representation theorem guarantees the existence of a function $u \in L^2$ such that

$$l(w) = (w, u).$$

For $v \in B$,

$$(L^*v, u) = l(L^*v) = (v, G).$$

Hence, $u$ is a weak solution of (A.10).

The classical results of Lax and Phillips in [6, section 4] can almost be directly applied to obtain that $u$ is a semistrong solution of (A.10). We restate the relevant theorem (Theorem 2.1) from their paper and a brief outline of the proof.

**Lemma A.6.** If $u$ satisfies the equation $Lu = G$ and the boundary conditions in the weak sense, then $u$ also satisfies the equation and the boundary conditions in the semistrong sense. That is, there exist $u_\epsilon$’s such that

- $u_\epsilon$ is continuously differentiable away from the boundaries,
- $B_nu_\epsilon$ is continuously differentiable up to $x_n = 0$ and orthogonal to $N^*(x')$ at $x_n = 0$,
- $A_0u_\epsilon$ is continuously differentiable up to $t = 0$ and $u_\epsilon = 0$ at $t = 0$,
- $u_\epsilon \to u$ in $L^2$,
- there exist functions $g_\epsilon \in L^2$ such that $Lu_\epsilon = g_\epsilon$ weakly,
- $g_\epsilon \to G$ in $L^2$,
- first-order derivatives in $(t, x')$ variables of $u_\epsilon$ are in $L^2$, and
- $\partial_{x_n}B_nu_\epsilon \in L^2$. 

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Proof. The proof begins by partitioning the domain into a region close to the boundary (both the \( t = 0 \) and \( x_n = 0 \) boundaries) and a region away from the boundary. Then \( u \) is broken up by a partition of unity into two pieces. Away from the boundary we obtain the result by mollifying \( u \) in all variables.

Near the boundary, we extend the weak solution for \( t \in [-T,T] \) by defining it to be equal to 0 for \( t < 0 \). Also, we extend \( G \) in the same way and consider \( L \) for all \( t \in [-T,T] \), since it is independent of \( t \). Certainly, the extended \( u \) is a weak solution of \( Lu = G \) in \([ -T,T ] \) \( \times \mathbb{R}_n^+ \). We then obtain a regularity result in the \( x_n \)-direction for the components of \( u \) which are not annihilated by \( B_n \) (see Lemma 2.1 in [6]), which states that \( B_n u \) is a continuous function of \( x_n \) taking values in \( H^{-1} \). Also, we obtain that \( B_n u = 0 \) at \( x_n = 0 \) in the distribution sense.

Next, we look at the \((t,x')\) variables. Consider the shifted mollifier \( \tilde{j}_\epsilon \) in the \((t,x')\) coordinates given by the convolution kernel

\[
\tilde{j}_\epsilon(t,x') = \frac{1}{\epsilon^n} j \left( \frac{t}{\epsilon} - 2, \frac{x'}{\epsilon} \right),
\]

where \( j \in C^\infty(\mathbb{R}^n) \), \( j \geq 0 \), supp\( (j) \subset B_1(0) \), and \( |j|_{L^1} = 1 \). Note that this particular choice of mollifier ensures that \( \tilde{j}_\epsilon v \in N^* \) and \( \tilde{j}_\epsilon^* v = 0 \) for \( t = T \) whenever \( v \) enjoys these properties. Using this property, the definition of the shifted mollifier, and the regularity of \( u \) in the \( x_n \)-direction, one can prove that \( u_\epsilon = \tilde{j}_\epsilon u \) satisfies all of the conditions required to make \( u \) a semistrong solution.

We now have all of the tools to prove that the solution to (A.10) is unique. Suppose that there are two solutions. Denote their difference by \( w \). This difference satisfies (A.10) with \( G \equiv 0 \). We can use the above to find a sequence of functions \( w_\epsilon \) converging to \( w \) and \( Lw_\epsilon \) converging to \( G \equiv 0 \). Rederiving the estimate (A.11) for \( L \), we find that

\[
\sup_{t \in I} |Sw_\epsilon| \leq C_T \int_0^T |Lw_\epsilon(s)| \, ds.
\]

In the limit, this shows that \( w \equiv 0 \). Hence the solution is unique.

Proof of Theorem A.4. We reduce to the \( f = 0 \) and \( h = 0 \) case by subtracting the following \( H^1 \) function \( u_0 \) from \( u \):

\[
u_0(x,t) = \frac{x_n \chi(t)}{t + x_n} Mf(x) + \frac{t \chi(x_n)}{t + x_n} h(x',t) + \chi(t) M^\perp f(x),\]

where \( \chi \) is a nonnegative \( C^\infty_0(\mathbb{R}) \)-function such that \( \chi(s) = 1 \) for \( |s| < 1 \) and \( \chi(s) = 0 \) for \( |s| > 2 \). The function \( u_0 \) is such that \( u_0|_{t=0} = f(x) \) and \( Mu_0|_{x_n=0} = h(t,x') \). We now apply the above existence arguments to obtain a solution \( w \) for \( G = F - Lu_0 \) and \( N(x') = [M(x') \mathbb{R}^k]^\perp \). Hence \( u = w + u_0 \) is solution to (A.9). Uniqueness follows, since the difference of two solutions solves (A.10) with \( f = h = F = 0 \). Using the approximations \( w_\epsilon \) for \( w \), we have

\[
\|u\| \leq \|w\| + \|u_0\| \leq \lim_{\epsilon \to 0} \|w_\epsilon\| + C (|f| + |h|) \leq T \lim_{\epsilon \to 0} \sup_{t \in I} \|w_\epsilon(t)\| + C (|f| + |h|)
\]

\[
\leq C_T \lim_{\epsilon \to 0} \int_0^T |g_\epsilon(s)| \, ds + C (|f| + |h|) \leq C_T \lim_{\epsilon \to 0} \|g_\epsilon\| + C (|f| + |h|)
\]

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\[
\leq C_T (\| F - Lu_0 \| + | f | + | h |) \\
\leq C_T (\| F \| + | f |_{H^1} + | h |_{H^1}).
\]

Therefore, the hyperbolic system (A.9) is well-posed. \(\square\)

REFERENCES


