



Min–max formulas and other properties of certain classes of nonconvex effective Hamiltonians

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Abstract This paper is the first attempt to systematically study properties of the effective Hamiltonian \overline{H} arising in the periodic homogenization of some coercive but nonconvex Hamilton–Jacobi equations. Firstly, we introduce a new and robust decomposition method to obtain min–max formulas for a class of nonconvex \overline{H} . Secondly, we analytically and numerically investigate other related interesting phenomena, such as “quasi-convexification” and breakdown of symmetry, of \overline{H} from other typical nonconvex Hamiltonians. Finally, in the appendix, we show that our new method and those a priori formulas from the periodic setting can be used to obtain stochastic

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homogenization for the same class of nonconvex Hamilton–Jacobi equations. Some conjectures and problems are also proposed.

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1 Introduction

1.1 Overview

Let us describe the periodic homogenization theory of Hamilton–Jacobi equations. For each $\varepsilon > 0$, let $u^\varepsilon \in C(\mathbb{R}^n \times [0, \infty))$ be the viscosity solution to

$$\begin{cases} u_t^\varepsilon + H(Du^\varepsilon) - V\left(\frac{x}{\varepsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u^\varepsilon(x, 0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \quad (1.1)$$

Here, the Hamiltonian $H(p) - V(x)$ is of separable form with $H \in C(\mathbb{R}^n)$, which is coercive (i.e., $\lim_{|p| \rightarrow \infty} H(p) = +\infty$), and $V \in C(\mathbb{R}^n)$, which is \mathbb{Z}^n -periodic. The initial data $g \in \text{BUC}(\mathbb{R}^n)$, the set of bounded, uniformly continuous functions on \mathbb{R}^n .

It was proven by Lions et al. [29] that u^ε converges to u locally uniformly on $\mathbb{R}^n \times [0, \infty)$ as $\varepsilon \rightarrow 0$, and u solves the effective equation

$$\begin{cases} u_t + \bar{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \quad (1.2)$$

The effective Hamiltonian $\bar{H} \in C(\mathbb{R}^n)$ is determined in a nonlinear way by H and V through the cell problems as following. For each $p \in \mathbb{R}^n$, it was shown in [29] that there exists a unique constant $\bar{H}(p) \in \mathbb{R}$ such that the following cell problem has a continuous viscosity solution

$$H(p + Dv) - V(x) = \bar{H}(p) \quad \text{in } \mathbb{T}^n, \quad (1.3)$$

where \mathbb{T}^n is the n -dimensional flat torus $\mathbb{R}^n/\mathbb{Z}^n$.

Although there is a vast literature on homogenization of Hamilton–Jacobi equations in different settings after [29], characterizing the shape of \bar{H} remains largely open even in basic situations. Let us summarize quickly what is known in the literature about \bar{H} . It is not hard to see that \bar{H} is coercive thanks to the coercivity of H . If one assumes furthermore that H is convex, then \bar{H} is also convex and the graph of \bar{H} can contain some flat parts (i.e., $\{\bar{H} = \min \bar{H}\}$ has interior points). See [29] and the works of Concorde [12, 13]. Furthermore, in this convex setting, we have the following representation formula, thanks to the results of Contreras et al. [14], and Gomes [23],

$$\bar{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{T}^n} (H(p + D\phi(x)) - V(x)). \quad (1.4)$$

Note that the above representative formula still holds if H is quasiconvex (level-set convex), in which case \overline{H} is also quasiconvex. More interestingly, in case $n = 2$, $H(p) = |p|^2$ and $V \in C^\infty(\mathbb{T}^2)$, a deep result of Bangert [9] says that the level curve $\{\overline{H} = c\}$ for every $c > -\min V$ must contain line segments (i.e., not strictly convex) unless V is a constant function. Bangert’s result relies on detailed information about the structure of Aubry–Mather sets in two dimensions [8]. See also Jing et al. [26] for discussion regarding locations of line segments of the level curves for Mañé type Hamiltonians.

The first numerical computation of effective Hamiltonians is due to Qian [38] based on the so called *big-T method*, that is, $\overline{H}(p) = -\lim_{t \rightarrow \infty} \frac{w(x,t)}{t}$, where $w(x, t)$ is the unique viscosity solution to

$$\begin{cases} w_t + H(Dw) - V(x) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ w(x, 0) = p \cdot x & \text{on } \mathbb{R}^n. \end{cases}$$

For other numerical schemes, we refer to Gomes and Oberman [25], Falcone andorro [19], Achdou et al. [1], Oberman et al. [37], Luo et al. [33] and the references therein.

It is worth mentioning that cell problem (1.3) and representation formula (1.4) appear also in weak KAM theory (see E [17], Evans and Gomes [18], Fathi [20] for the convex case, and Cagnetti et al. [11] for the nonconvex case). In fact, a central goal of the weak KAM theory is to find information of underlying dynamical system encoded in the effective Hamiltonian. See also [34, 35].

In the case where H is nonconvex, to identify the shape of \overline{H} is highly nontrivial even in the one dimensional space. This was settled only very recently by Armstrong et al. [7], and Gao [22]. One fundamental feature obtained is the “*quasi-convexification*” phenomenon, that is, the effective Hamiltonian \overline{H} becomes quasiconvex (level-set convex) when the oscillation of V is large enough. See Sect. 3 for more precise statements. In multi-dimensional spaces, Armstrong et al. [6] obtained a qualitative shape of \overline{H} for a representative case where $H(p) = (|p|^2 - 1)^2$. Other than [6], very little is known about finer properties of nonconvex \overline{H} in multi-dimensional spaces, partly due to the extreme complexity of dynamics associated with nonconvex Hamiltonians. Furthermore, as far as the authors know, there is no numerical study of \overline{H} in this case.

Let us also call attention to an extrinsic way to study \overline{H} via inverse problems. See Luo et al. [32].

1.2 Main results

Reducing a complex quantity to relatively simpler objects is a very natural and common idea in mathematics. For a class of nonconvex Hamiltonians H , we introduce a new decomposition method to obtain min–max type representation formulas for \overline{H} . These formulas consist of effective Hamiltonians of quasiconvex Hamiltonians which are presumably less challenging to analyze. The most general statement is given by inductive formulas (Theorem 2.6). Two specific (but important) cases of Theorem 2.6 are provided in Theorem 2.1 and Lemma 2.5. One immediate corollary is the evenness

of \overline{H} associated with a certain class of radially symmetric Hamiltonians, which is otherwise not obvious at all. Given the vast variety of nonconvex functions, our surgical approach is only a preliminary step toward understanding the shape of nonconvex \overline{H} . In Sect. 2.4, we present some natural obstacles to decompose a nonconvex \overline{H} . In particular, there is a connection between “non-decomposable” and loss of evenness.

As another interesting application, the method and the representation formulas are robust enough that we are also able to prove stochastic homogenization for the same class of nonconvex H in the appendix. For instance, Theorem 4.1 includes the result in [6] as a special case with a much shorter proof. The detailed discussion on this (including a brief overview of stochastic homogenization) is left to the appendix. We would like to point out that a priori identification of shape of \overline{H} is currently the only available way to tackle homogenization of nonconvex Hamilton–Jacobi equations in general stationary ergodic setting.

In Sect. 3, we provide various numerical computations of \overline{H} in multi-dimensional spaces for general radially symmetric Hamiltonians and a double-well type Hamiltonian. These provide insights on how the changes of potential energy V affect the changes in shape of effective Hamiltonian \overline{H} . The important “quasi-convexification” phenomenon is observed in multi-dimensional cases as well. Nevertheless, verifying it rigorously seems to be quite challenging. Interesting connections between decomposition, loss of evenness and quasi-convexification are demonstrated in Sect. 2.4 and Remark 4. Several open problems are provided based on the numerical evidences we have in this section.

2 Min–max formulas

2.1 Basic case

The setting is this. Let $H = H(p) : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous, coercive Hamiltonian such that

(H1) $\min_{\mathbb{R}^n} H = 0$ and there exists a bounded domain $U \subset \mathbb{R}^n$ such that

$$\{H = 0\} = \partial U.$$

(H2) $H(p) = H(-p)$ for all $p \in \mathbb{R}^n$.

(H3) There exist $H_1, H_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ such that H_1, H_2 are continuous and

$$H = \max\{H_1, H_2\}.$$

Here, H_1 is coercive, quasiconvex, even ($H_1(p) = H_1(-p)$ for all $p \in \mathbb{R}^n$), $H_1 = H$ in $\mathbb{R}^n \setminus U$ and $H_1 < 0$ in U . The function H_2 is even, quasiconcave, $H_2 = H$ in U , $H_2 < 0$ in $\mathbb{R}^n \setminus U$ and $\lim_{|p| \rightarrow \infty} H_2(p) = -\infty$.

It is easy to see that any H satisfying (H1)–(H3) can be written as $H(p) = \lambda|F(p)|$ for some $\lambda > 0$ and even, coercive quasiconvex function F with $\min_{\mathbb{R}^n} F < 0$. Below is the first decomposition result.

Theorem 2.1 *Let $H \in C(\mathbb{R}^n)$ be a Hamiltonian satisfying (H1)–(H3). Let $V \in C(\mathbb{T}^n)$ be a potential energy with $\min_{\mathbb{T}^n} V = 0$.*

Assume that \overline{H} is the effective Hamiltonian corresponding to $H(p) - V(x)$. Assume also that \overline{H}_i is the effective Hamiltonian corresponding to $H_i(p) - V(x)$ for $i = 1, 2$. Then

$$\overline{H} = \max \{ \overline{H}_1, \overline{H}_2, 0 \}.$$

In particular, \overline{H} is even.

We would like to point out that the evenness of \overline{H} will be used later and is not obvious at all although H is even. See the discussion in Sect. 2.4 for this subtle issue.

Proof We proceed in few steps.

Step 1. It is straightforward that $0 \leq \overline{H}(p) \leq H(p)$ for all $p \in \mathbb{R}^n$. Indeed, for each fixed $p \in \mathbb{R}^n$, the corresponding cell problem is (1.3). Pick $y \in \mathbb{T}^n$ such that $\min_{\mathbb{T}^n} v = v(y)$. By the definition of viscosity supersolutions to (1.3), we get

$$H(p) \geq H(p) - V(y) \geq \overline{H}(p).$$

On the other hand, as (1.3) holds in the almost everywhere sense, we take ess sup of its sides to imply

$$\overline{H}(p) = \operatorname{ess\,sup}_{x \in \mathbb{T}^n} (H(p + Dv(x)) - V(x)) \geq \operatorname{ess\,sup}_{x \in \mathbb{T}^n} (-V(x)) = 0.$$

In particular,

$$\overline{H}(p) = 0 \quad \text{for all } p \in \partial U. \quad (2.1)$$

Besides, as $H_i \leq H$, we get $\overline{H}_i \leq \overline{H}$. Therefore,

$$\overline{H} \geq \max \{ \overline{H}_1, \overline{H}_2, 0 \}. \quad (2.2)$$

It remains to prove the reverse inequality of (2.2) in order to get the conclusion.

Step 2. Fix $p \in \mathbb{R}^n$. Assume now that $\overline{H}_1(p) \geq \max \{ \overline{H}_2(p), 0 \}$. In particular, $\overline{H}_1(p) \geq 0$. We will show that $\overline{H}_1(p) \geq \overline{H}(p)$.

Since H_1 is quasiconvex and even, we use the inf-max representation formula for \overline{H}_1 (see [4, 16, 36]) to get that

$$\begin{aligned} \overline{H}_1(p) &= \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{T}^n} (H_1(p + D\phi(x)) - V(x)) \\ &= \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{T}^n} (H_1(-p - D\phi(x)) - V(x)) \\ &= \inf_{\psi \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{T}^n} (H_1(-p + D\psi(x)) - V(x)) = \overline{H}_1(-p). \end{aligned}$$

Thus, \overline{H}_1 is even. Let $v(x, -p)$ be a solution to the cell problem

$$H_1(-p + Dv(x, -p)) - V(x) = \overline{H}_1(-p) = \overline{H}_1(p) \quad \text{in } \mathbb{T}^n. \quad (2.3)$$

Let $w(x) = -v(x, -p)$. For any $x \in \mathbb{T}^n$ and $q \in D^+w(x)$, we have $-q \in D^-v(x, -p)$ and hence, in light of (2.3) and the quasiconvexity of H_1 (see [10]),

$$\overline{H}_1(p) = H_1(-p - q) - V(x) = H_1(p + q) - V(x).$$

We thus get $H_1(p + q) = \overline{H}_1(p) + V(x) \geq 0$ as $\overline{H}_1(p) \geq 0$, and therefore, $H(p + q) = H_1(p + q) \geq 0$ in light of (H3). This yields that w is a viscosity subsolution to

$$H(p + Dw) - V(x) = \overline{H}_1(p) \quad \text{in } \mathbb{T}^n.$$

Hence, $\overline{H}(p) \leq \overline{H}_1(p)$.

Step 3. Assume now that $\overline{H}_2(p) \geq \max\{\overline{H}_1(p), 0\}$. By using similar arguments as those in the previous step (except that we use $v(x, p)$ instead of $v(x, -p)$ due to the quasiconcavity of H_2), we can show that $\overline{H}_2(p) \geq \overline{H}(p)$.

Step 4. Assume that $\max\{\overline{H}_1(p), \overline{H}_2(p)\} < 0$. We now show that $\overline{H}(p) = 0$ in this case. Thanks to (2.1) in Step 1, we may assume that $p \notin \partial U$.

For $\sigma \in [0, 1]$ and $i = 1, 2$, let $\overline{H}^\sigma, \overline{H}_i^\sigma$ be the effective Hamiltonians corresponding to $H(p) - \sigma V(x)$, $H_i(p) - \sigma V(x)$, respectively. It is clear that

$$0 \leq \overline{H}^1 = \overline{H} \leq \overline{H}^\sigma \quad \text{for all } \sigma \in [0, 1]. \quad (2.4)$$

By repeating Steps 2 and 3 above, we get

$$\text{For } p \in \mathbb{R}^n \text{ and } \sigma \in [0, 1], \text{ if } \max\{\overline{H}_1^\sigma(p), \overline{H}_2^\sigma(p)\} = 0, \text{ then } \overline{H}^\sigma(p) = 0. \quad (2.5)$$

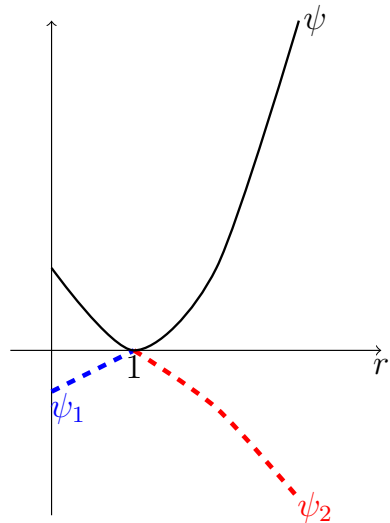
We only consider the case $p \notin \overline{U}$ here. The case $p \in U$ is analogous. Notice that

$$H(p) = H_1(p) = \overline{H}_1^0(p) > 0 \quad \text{and} \quad \overline{H}_1(p) = \overline{H}_1^1(p) < 0.$$

By the continuity of $\sigma \mapsto \overline{H}_1^\sigma(p)$, there exists $s \in (0, 1)$ such that $\overline{H}_1^s(p) = 0$. Note furthermore that, as $p \notin \overline{U}$, $\overline{H}_2^s(p) \leq H_2(p) < 0$. These, together with (2.4) and (2.5), yield the desired result. \square

Remark 1 We emphasize that Step 4 in the above proof is important. It plays the role of a “patching” step, which helps glue \overline{H}_1 and \overline{H}_2 together. Nevertheless, if the answer to Question 3 in Appendix is positive, there is an alternative approach to get the same representation formula of \overline{H} without the evenness assumption (H2). The evenness of H_2 is only used to get the evenness of \overline{H} . The representation formula of \overline{H} still holds true without the evenness of H in U and of H_2 .

Fig. 1 Graphs of ψ , ψ_1 , ψ_2



Assumptions (H1)–(H3) are general and a bit complicated. A simple situation where (H1)–(H3) hold is a radially symmetric case where $H(p) = \psi(|p|)$, and $\psi \in C([0, \infty), \mathbb{R})$ satisfying

$$\begin{cases} \psi(0) > 0, \psi(1) = 0, \lim_{r \rightarrow \infty} \psi(r) = +\infty, \\ \psi \text{ is strictly decreasing in } (0, 1) \text{ and is strictly increasing in } (1, \infty). \end{cases} \quad (2.6)$$

Let $\psi_1, \psi_2 \in C([0, \infty), \mathbb{R})$ be such that

$$\begin{cases} \psi_1 = \psi \text{ on } [1, \infty), \text{ and } \psi_1 \text{ is strictly increasing on } [0, 1], \\ \psi_2 = \psi \text{ on } [0, 1], \psi_2 \text{ is strictly decreasing on } [1, \infty), \\ \text{and } \lim_{r \rightarrow \infty} \psi_2(r) = -\infty. \end{cases} \quad (2.7)$$

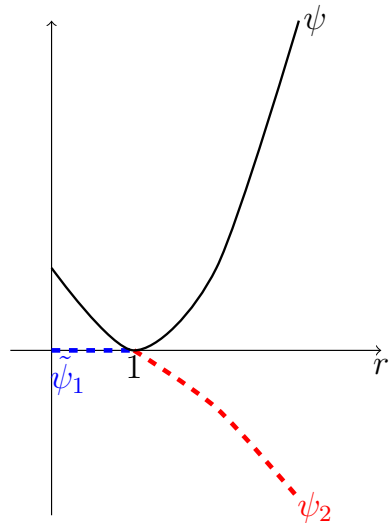
See Fig. 1. Set $H_i(p) = \psi_i(|p|)$ for $p \in \mathbb{R}^n$, and for $i = 1, 2$. It is clear that (H1)–(H3) hold provided that (2.6) and (2.7) hold.

An immediate consequence of Theorem 2.1 is

Corollary 2.2 *Let $H(p) = \psi(|p|)$, $H_i(p) = \psi_i(|p|)$ for $i = 1, 2$ and $p \in \mathbb{R}^n$, where ψ, ψ_1, ψ_2 satisfy (2.6) and (2.7). Let $V \in C(\mathbb{T}^n)$ be a potential energy with $\min_{\mathbb{T}^n} V = 0$.*

Assume that \overline{H} is the effective Hamiltonian corresponding to $H(p) - V(x)$. Assume also that \overline{H}_i is the effective Hamiltonian corresponding to $H_i(p) - V(x)$ for $i = 1, 2$. Then

$$\overline{H} = \max \{ \overline{H}_1, \overline{H}_2, 0 \}.$$

Fig. 2 Graphs of ψ , $\tilde{\psi}_1$, ψ_2 

Remark 2 A special case of Corollary 2.2 is when

$$H(p) = \psi(|p|) = (|p|^2 - 1)^2 \quad \text{for } p \in \mathbb{R}^n,$$

which was studied first by Armstrong, Tran and Yu [6]. The method here is much simpler and more robust than that in [6].

By using Corollary 2.2 and approximation, we get another representation formula for \overline{H} which will be used later.

Corollary 2.3 Assume that (2.6) and (2.7) hold. Set

$$\tilde{\psi}_1(r) = \max\{\psi_1, 0\} = \begin{cases} 0 & \text{for } 0 \leq r \leq 1, \\ \psi(r) & \text{for } r > 1. \end{cases}$$

Let $H(p) = \psi(|p|)$, $\tilde{H}_1(p) = \tilde{\psi}_1(|p|)$ and $H_2(p) = \psi_2(|p|)$ for $p \in \mathbb{R}^n$. Let $V \in C(\mathbb{T}^n)$ be a potential energy with $\min_{\mathbb{T}^n} V = 0$.

Assume that \overline{H} , $\overline{\tilde{H}}_1$, \overline{H}_2 are the effective Hamiltonian corresponding to $H(p) - V(x)$, $\tilde{H}_1(p) - V(x)$, $H_2(p) - V(x)$, respectively. Then

$$\overline{H} = \max \left\{ \overline{\tilde{H}}_1, \overline{H}_2 \right\}.$$

See Fig. 2 for the graphs of ψ , $\tilde{\psi}_1$, ψ_2 .

When the oscillation of V is large enough, we have furthermore the following result.

Corollary 2.4 *Let $H \in C(\mathbb{R}^n)$ be a coercive Hamiltonian satisfying (H1)–(H3), except that we do not require H_2 to be quasiconcave. Assume that*

$$\operatorname{osc}_{\mathbb{T}^n} V = \max_{\mathbb{T}^n} V - \min_{\mathbb{T}^n} V \geq \max_{\overline{U}} H = \max_{\mathbb{R}^n} H_2.$$

Then

$$\overline{H} = \max \left\{ \overline{H}_1, -\min_{\mathbb{T}^n} V \right\}.$$

In particular, \overline{H} is quasiconvex in this situation.

It is worth noting that the result of Corollary 2.4 is interesting in the sense that we do not require any structure of H in U except that $H > 0$ there.

Proof Without loss of generality, we assume that $\min_{\mathbb{T}^n} V = 0$. Choose a quasiconcave function $H_2^+ \in C(\mathbb{R}^n)$ such that

$$\begin{cases} \{H = 0\} = \{H_2^+ = 0\} = \partial U, \\ H \leq H_2^+ \text{ in } U, \text{ and } \max_{\overline{U}} H = \max_{\mathbb{R}^n} H_2^+, \\ \lim_{|p| \rightarrow \infty} H_2^+(p) = -\infty. \end{cases}$$

Denote $H^+ \in C(\mathbb{R}^n)$ as

$$H^+(p) = \max\{H, H_2^+\} = \begin{cases} H_1(p) & \text{for } p \in \mathbb{R}^n \setminus U, \\ H_2^+(p) & \text{for } p \in \overline{U}. \end{cases}$$

Also denote by \overline{H}^+ and \overline{H}_2^+ the effective Hamiltonians associated with $H^+(p) - V(x)$ and $H_2^+(p) - V(x)$, respectively. Apparently,

$$\max\{\overline{H}_1, 0\} \leq \overline{H} \leq \overline{H}^+. \quad (2.8)$$

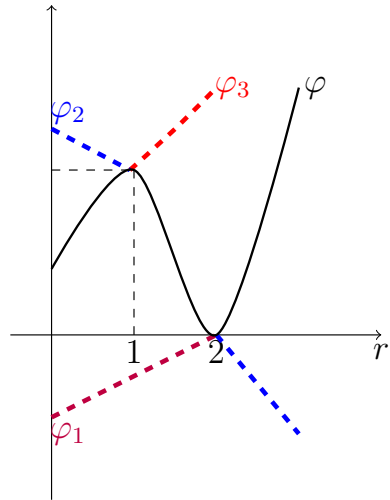
On the other hand, by Lemma 2.1, the representation formula for \overline{H}^+ is

$$\overline{H}^+ = \max\{\overline{H}_1, \overline{H}_2^+, 0\} = \max\{\overline{H}_1, 0\}, \quad (2.9)$$

where the second equality is due to

$$\overline{H}_2^+ \leq \max_{\mathbb{R}^n} H_2^+ - \max_{\mathbb{T}^n} V = \max_{\overline{U}} H - \max_{\mathbb{R}^n} V \leq 0.$$

We combine (2.8) and (2.9) to get the conclusion. \square

Fig. 3 Graphs of $\varphi, \varphi_1, \varphi_2, \varphi_3$ 

2.2 A more general case

We first extend Theorem 2.1 as following. To avoid unnecessary technicalities, we only consider radially symmetric cases from now on. See Fig. 3. The results still hold true for general Hamiltonians (without the radially symmetric assumption) under corresponding appropriate conditions, which are similar to (H1)–(H3).

Let $H : \mathbb{R}^n \rightarrow \mathbb{R}$ be such that

(H4) $H(p) = \varphi(|p|)$ for $p \in \mathbb{R}^n$, where $\varphi \in C([0, \infty), \mathbb{R})$ such that

$$\begin{cases} \varphi(0) > 0, \varphi(2) = 0, \lim_{r \rightarrow \infty} \varphi(r) = +\infty, \\ \varphi \text{ is strictly increasing on } [0, 1] \text{ and } [2, \infty), \text{ and is strictly decreasing on } [1, 2]. \end{cases}$$

(H5) $H_i(p) = \varphi_i(|p|)$ for $p \in \mathbb{R}^n$ and $1 \leq i \leq 3$, where $\varphi_i \in C([0, \infty), \mathbb{R})$ such that

$$\begin{cases} \varphi_1 = \varphi \text{ on } [2, \infty), \varphi_1 \text{ is strictly increasing on } [0, 2], \\ \varphi_2 = \varphi \text{ on } [1, 2], \varphi_2 \text{ is strictly decreasing on } [0, 1] \text{ and } [2, \infty), \\ \lim_{r \rightarrow \infty} \varphi_2(r) = -\infty, \\ \varphi_3 = \varphi \text{ on } [0, 1], \varphi_3 \text{ is strictly increasing on } [1, \infty), \text{ and } \varphi_3 > \varphi \text{ in } (1, \infty). \end{cases}$$

Lemma 2.5 Let $H(p) = \varphi(|p|)$, $H_i(p) = \varphi_i(|p|)$ for $1 \leq i \leq 3$ and $p \in \mathbb{R}^n$, where $\varphi, \varphi_1, \varphi_2, \varphi_3$ satisfy (H4)–(H5). Let $V \in C(\mathbb{T}^n)$ be a potential energy with $\min_{\mathbb{T}^n} V = 0$.

Assume that \overline{H} is the effective Hamiltonian corresponding to $H(p) - V(x)$. Assume also that \overline{H}_i is the effective Hamiltonian corresponding to $H_i(p) - V(x)$ for $1 \leq i \leq 3$. Then

$$\begin{aligned}\overline{H} &= \max \{0, \overline{H}_1, \overline{K}\} \\ &= \max \{0, \overline{H}_1, \min \{\overline{H}_2, \overline{H}_3, \varphi(1) - \max_{\mathbb{T}^n} V\}\}.\end{aligned}$$

Here \overline{K} is the effective Hamiltonian corresponding to $K(p) - V(x)$ for $K : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as

$$K(p) = \min \{\varphi_2(|p|), \varphi_3(|p|)\} = \begin{cases} \varphi(|p|) & \text{for } |p| \leq 2, \\ \varphi_2(|p|) & \text{for } |p| \geq 2. \end{cases}$$

In particular, both \overline{H} and \overline{K} are even.

Proof Considering $-K(-p)$, thanks to the representation formula and evenness from Theorem 2.1,

$$\overline{K} = \min \left\{ \overline{H}_2, \overline{H}_3, \varphi(1) - \max_{\mathbb{T}^n} V \right\}.$$

Define $\tilde{\varphi}_2 = \min \{\varphi_2, \varphi(1)\}$. Let $\tilde{H}_2(p) = \tilde{\varphi}_2(|p|)$ and \tilde{H}_2 be the effective Hamiltonian corresponding to $\tilde{H}_2(p) - V(x)$. Then, thanks to Corollary 2.3, we also have that

$$\overline{K} = \min \left\{ \tilde{H}_2, \overline{H}_3 \right\}. \quad (2.10)$$

Our goal is then to show that $\overline{H} = \max \{0, \overline{H}_1, \overline{K}\}$. To do this, we again divide the proof into few steps for clarity. *Readers should notice that the proof below does not depend on the quasiconvexity of \overline{H}_3 . It only uses the fact that $\overline{H}_3 \geq \overline{H}$. This is essential to prove the most general result, Theorem 2.6.*

Step 1. Clearly $0 \leq \overline{H} \leq H$. This implies further that

$$\overline{H}(p) = 0 \quad \text{for all } |p| = 2. \quad (2.11)$$

We furthermore have that $\overline{K}, \overline{H}_1 \leq \overline{H}$ as $K, H_1 \leq H$. Thus,

$$\overline{H} \geq \max \{0, \overline{H}_1, \overline{K}\} \quad (2.12)$$

We now show the reverse inequality of (2.12) to finish the proof.

Step 2. Fix $p \in \mathbb{R}^n$. Assume that $\overline{H}_1(p) \geq \max \{0, \overline{K}(p)\}$. Since H_1 is quasiconvex, we follow exactly the same lines of Step 2 in the proof of Theorem 2.1 to deduce that $\overline{H}_1(p) \geq \overline{H}(p)$.

Step 3. Assume that $\overline{K}(p) \geq \max \{0, \overline{H}_1(p)\}$. Since K is not quasiconvex or quasiconcave, we cannot directly copy Step 2 or Step 3 in the proof of Theorem 2.1. Instead, there are two cases that need to be considered.

Firstly, we consider the case that $\overline{K}(p) = \overline{\tilde{H}}_2(p) \leq \overline{H}_3(p)$. Let $v(x, p)$ be a solution to the cell problem

$$\tilde{H}_2(p + Dv(x, p)) - V(x) = \overline{\tilde{H}}_2(p) \geq 0 \quad \text{in } \mathbb{T}^n. \quad (2.13)$$

Since \tilde{H}_2 is quasiconcave, for any $x \in \mathbb{T}^n$ and $q \in D^+v(x, p)$, we have

$$\tilde{H}_2(p + q) - V(x) = \overline{\tilde{H}}_2(p) \geq 0,$$

which gives that $\tilde{H}_2(p + q) \geq 0$ and hence $\tilde{H}_2(p + q) \geq H(p + q)$. Therefore, $v(x, p)$ is a viscosity subsolution to

$$H(p + Dv(x, p)) - V(x) = \overline{\tilde{H}}_2(p) \quad \text{in } \mathbb{T}^n.$$

We conclude that $\overline{K}(p) = \overline{\tilde{H}}_2(p) \geq \overline{H}(p)$.

Secondly, assume that $\overline{K}(p) = \overline{H}_3(p) \leq \overline{\tilde{H}}_2(p)$. Since $\varphi_3 \geq \varphi$, $\overline{H}_3(p) \geq \overline{H}(p)$. Combining with $\overline{H}(p) \geq \overline{K}(p)$ in (2.12), we obtain $\overline{K}(p) = \overline{H}(p)$ in this step.

Step 4. Assume that $0 > \max \{ \overline{H}_1(p), \overline{K}(p) \}$. Our goal now is to show $\overline{H}(p) = 0$. Thanks to (2.11) in Step 1, we may assume that $|p| \neq 2$.

For $\sigma \in [0, 1]$ and $i = 1, 2$, let \overline{H}^σ , \overline{H}_1^σ , \overline{K}^σ be the effective Hamiltonians corresponding to $H(p) - \sigma V(x)$, $H_1(p) - \sigma V(x)$, $K(p) - \sigma V(x)$, respectively. It is clear that

$$0 \leq \overline{H}^1 = \overline{H} \leq \overline{H}^\sigma \quad \text{for all } \sigma \in [0, 1]. \quad (2.14)$$

By repeating Steps 2 and 3 above, we get

$$\text{For } p \in \mathbb{R}^n \text{ and } \sigma \in [0, 1], \text{ if } \max \{ \overline{H}_1^\sigma(p), \overline{K}^\sigma(p) \} = 0, \text{ then } \overline{H}^\sigma(p) = 0. \quad (2.15)$$

We only consider the case $|p| < 2$ here. The case $|p| > 2$ is analogous. Notice that

$$H(p) = K(p) = \overline{K}^0(p) > 0 \quad \text{and} \quad \overline{K}(p) = \overline{K}^1(p) < 0.$$

By the continuity of $\sigma \mapsto \overline{K}^\sigma(p)$, there exists $s \in (0, 1)$ such that $\overline{K}^s(p) = 0$. Note furthermore that, as $|p| < 2$, $\overline{H}_1^s(p) \leq H_1(p) < 0$. These, together with (2.14) and (2.15), yield the desired result. \square

2.3 General cases

By using induction, we can obtain min–max (max–min) formulas for \overline{H} in case $H(p) = \varphi(|p|)$ where φ satisfies some certain conditions described below. We consider two cases corresponding to Figs. 4 and 5.

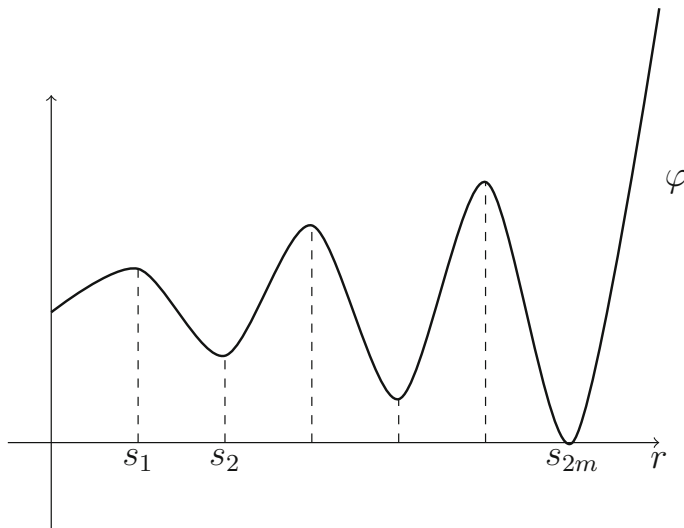


Fig. 4 Graph of φ in first general case

In this first general case corresponding to Fig. 4, we assume that

(H6) $\varphi \in C([0, \infty), \mathbb{R})$ satisfying

$$\begin{cases} \text{there exist } m \in \mathbb{N} \text{ and } 0 = s_0 < s_1 < \cdots < s_{2m} < \infty = s_{2m+1} \text{ such that} \\ \varphi \text{ is strictly increasing in } (s_{2i}, s_{2i+1}), \text{ and is strictly decreasing in } (s_{2i+1}, s_{2i+2}), \\ \varphi(s_0) > \varphi(s_2) > \cdots > \varphi(s_{2m}), \text{ and } \varphi(s_1) < \varphi(s_3) < \cdots < \varphi(s_{2m+1}) = \infty. \end{cases}$$

For $0 \leq i \leq m$,

- let $\varphi_{2i} : [0, \infty) \rightarrow \mathbb{R}$ be a continuous, strictly increasing function such that $\varphi_{2i} = \varphi$ on $[s_{2i}, s_{2i+1}]$ and $\lim_{s \rightarrow \infty} \varphi_{2i}(s) = \infty$. Also $\varphi_{2i} \geq \varphi_{2i+2}$.
- let $\varphi_{2i+1} : [0, \infty) \rightarrow \mathbb{R}$ be a continuous, strictly decreasing function such that $\varphi_{2i+1} = \varphi$ on $[s_{2i+1}, s_{2i+2}]$ and $\lim_{s \rightarrow \infty} \varphi_{2i+1}(s) = -\infty$. Also $\varphi_{2i+1} \leq \varphi_{2i+3}$.

Define

$$H_{m-1}(p) = \max \{ \varphi(|p|), \varphi_{2m-2}(|p|) \} = \begin{cases} \varphi(|p|) & \text{for } |p| \leq s_{2m-1}, \\ \varphi_{2m-2}(|p|) & \text{for } |p| > s_{2m-1} \end{cases}$$

and

$$k_{m-1}(s) = \min \{ \varphi(s), \varphi_{2m-1}(s) \} = \begin{cases} \varphi(s) & \text{for } s \leq s_{2m}, \\ \varphi_{2m-1}(s) & \text{for } s > s_{2m}. \end{cases}$$

Denote \overline{H}_{m-1} , \overline{H}_m , \overline{K}_{m-1} , $\overline{\Phi}_j$ as the effective Hamiltonians associated with the Hamiltonians $H_{m-1}(p) - V(x)$, $\varphi(|p|) - V(x)$, $k_{m-1}(|p|) - V(x)$ and $\varphi_j(|p|) - V(x)$ for $0 \leq j \leq 2m$, respectively.

The following is our main decomposition theorem in this paper.

Theorem 2.6 Assume that (H6) holds for some $m \in \mathbb{N}$. Then

$$\overline{H}_m = \max \left\{ \overline{K}_{m-1}, \overline{\Phi}_{2m}, \varphi(s_{2m}) - \min_{\mathbb{T}^n} V \right\}, \quad (2.16)$$

and

$$\overline{K}_{m-1} = \min \left\{ \overline{H}_{m-1}, \overline{\Phi}_{2m-1}, \varphi(s_{2m-1}) - \max_{\mathbb{T}^n} V \right\}. \quad (2.17)$$

In particular, \overline{H}_m and \overline{K}_{m-1} are both even.

Again, we would like to point out that the evenness of \overline{H}_m and \overline{K}_{m-1} is far from being obvious although H_m and K_m are both even. See the discussion in Sect. 2.4 for this subtle issue.

Proof We prove by induction. When $m = 1$, the two formulas (2.16) and (2.17) follow from Lemma 2.5 and Theorem 2.1.

Assume that (2.16) and (2.17) hold for $m \in \mathbb{N}$. We need to verify these equalities for $m + 1$. Using similar arguments as those in the proof Lemma 2.5, noting the statement in italic right above Step 1, we first derive that

$$\overline{K}_m = \min \left\{ \overline{H}_m, \overline{\Phi}_{2m+1}, \varphi(s_{2m+1}) - \max_{\mathbb{T}^n} V \right\}.$$

Then again, by basically repeating the proof of Lemma 2.5, we obtain

$$\overline{H}_{m+1} = \max \left\{ \overline{K}_m, \overline{\Phi}_{2m+2}, \varphi(s_{2m+2}) - \min_{\mathbb{T}^n} V \right\}.$$

□

Remark 3 (i) By approximation, we see that representation formulas (2.16) and (2.17) still hold true if we relax (H6) a bit, that is, we only require that φ satisfies

$$\begin{cases} \varphi \text{ is increasing in } (s_{2i}, s_{2i+1}), \text{ and is decreasing in } (s_{2i+1}, s_{2i+2}), \\ \varphi(s_0) \geq \varphi(s_2) \geq \cdots \geq \varphi(s_{2m}), \text{ and } \varphi(s_1) \leq \varphi(s_3) \leq \cdots < \varphi(s_{2m+1}) = \infty. \end{cases}$$

(ii) According to Corollary 2.4, if $\text{osc}_{\mathbb{T}^n} V = \max_{\mathbb{T}^n} V - \min_{\mathbb{T}^n} V \geq \varphi(s_{2m-1}) - \varphi(s_{2m})$, then \overline{H} is quasiconvex and

$$\overline{H}_m = \max \left\{ \overline{\Phi}_{2m}, \varphi(s_{2m}) - \min_{\mathbb{T}^n} V \right\}.$$

The second general case corresponds to the case where $H(p) = -k(|p|)$ for all $p \in \mathbb{R}^n$ as described in Fig. 5. After changing the notations appropriately, we obtain similar representation formulas as in Theorem 2.6. We omit the details here.

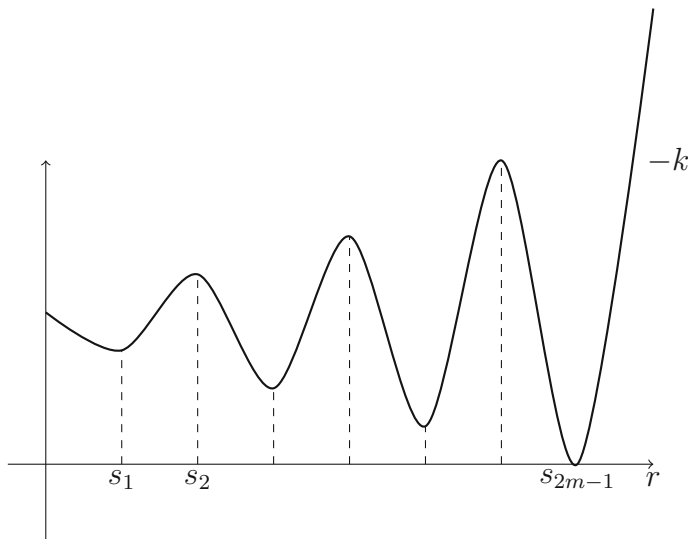


Fig. 5 Graph of $-k$ in the second general case

2.4 “Non-decomposable” and breakdown of symmetry

A natural question is whether we can extend Theorem 2.6 to other nonconvex H , i.e., there exist quasiconvex/concave H_i ($1 \leq i \leq m$) such that \bar{H} is given by a “decomposition” formula (e.g., min-max type) involving \bar{H}_i , $\min V$ and $\max V$:

$$\bar{H} = G(\bar{H}_1, \dots, \bar{H}_m, \min V, \max V). \quad (2.18)$$

for any $V \in C(\mathbb{T}^n)$. Here \bar{H} and \bar{H}_i are effective Hamiltonians associated with $H - V$ and $H_i - V$. Note that for quasiconvex/concave function F , using the inf-max formula (1.4), it is easy to see that the effective Hamiltonians associated with $F(p) - V(x)$ and $F(p) - V(-x)$ are the same. Hence if such a “decomposition” formula indeed exists for a specific nonconvex H , effective Hamiltonians associated with $H(p) - V(x)$ and $H(p) - V(-x)$ have to be identical as well. In particular, if H is an even function, this is equivalent to saying that \bar{H} is even too, which leads to the following question.

Question 1 *Let $H \in C(\mathbb{R}^n)$ be a coercive and even Hamiltonian, and $V \in C(\mathbb{T}^n)$ be a given potential. Let \bar{H} be the effective Hamiltonian associated with $H(p) - V(x)$. Is it true that \bar{H} is also even? In general, we may ask what properties of the original Hamiltonian will be preserved under homogenization.*

Even though that this is a simple and natural question, it has not been studied much in the literature as far as the authors know. We give below some answers and discussions to this:

- If H is quasiconvex, the answer is “yes” due to the inf-max formula

$$\bar{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{R}^n} (H(p + D\phi(x)) - V(x))$$

as shown in the proof of Lemma 2.1.

- For genuinely nonconvex H , if \overline{H} can be written as a min–max formula involving effective Hamiltonians of even quasiconvex (or quasiconcave) Hamiltonians, then \overline{H} is still even (e.g., see Corollary 2.2, Lemma 2.5, and Theorem 2.6).
- However, in general, the evenness is lost as presented in Remark 1.2 in [32]. Let us quickly recall the setting there. We consider the case $n = 1$, and choose $H(p) = \varphi(|p|)$ for $p \in \mathbb{R}$, where φ satisfies (H8) (see Fig. 7 below) with $m_1 = \frac{1}{3}$ and $M_1 = \frac{1}{2}$. Fix $s \in (0, 1)$, and set $V_s(x) = \min\{\frac{x}{s}, \frac{1-x}{1-s}\}$ for $x \in [0, 1]$. Extend V to \mathbb{R} in a periodic way. Then \overline{H} is not even unless $s = \frac{1}{2}$. In particular, this implies that a decomposition formula for \overline{H} does not exist. Also, see Fig. 12 below for loss of evenness when the Hamiltonian is of double-well type.
- It is extremely interesting if we can point out some further general requirements on H and V in the genuinely nonconvex setting, under which \overline{H} is even. The interplay between H and V plays a crucial role here (see Remark 4 for intriguing observations).

Some related discussions and interesting applications of evenness can also be found in [40].

3 Quasi-convexification phenomenon in multiple-dimensional spaces

Intuitively, homogenization, a nonlinear averaging procedure, makes the effective Hamiltonian less nonconvex. The question is how to describe this in a rigorous and systematic way. Some special cases have been handled in Remark 3. In this section, we look at more generic and important situations: general radially symmetric Hamiltonians and a typical double-well Hamiltonian. These two types of Hamiltonians more or less capture essential features of nonconvexity. In some sense, quasi-convexification represents a scenario where there is no genuine decomposition of \overline{H} . Due to the difficulty in rigorous analysis, we focus more on numerical computations. The Lax–Friedrichs based big-T method is used to compute the effective Hamiltonian.

3.1 Radially symmetric Hamiltonians

Assume that $H(p) = \varphi(|p|)$ for all $p \in \mathbb{R}^n$, where $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is a given function. The following is quite a general condition on φ .

(H7) $\varphi \in C([0, \infty), \mathbb{R})$ satisfying that $\lim_{s \rightarrow \infty} \varphi(s) = +\infty$, $\varphi(0) = \min \varphi = 0$ and

$$\begin{cases} \text{there exist } m \in \mathbb{N} \text{ and } 0 = s_0 < s_1 < \cdots < s_{2m} < \infty = s_{2m+1} \text{ such that} \\ \varphi \text{ is strictly increasing in } (s_{2i}, s_{2i+1}), \\ \text{and is strictly decreasing in } (s_{2i+1}, s_{2i+2}). \end{cases}$$

It is clear that (H7) is more general than (H6). In fact, any coercive function $\psi \in C([0, \infty), \mathbb{R})$ can be approximated by φ satisfying (H7).

Denote by

$$\begin{cases} M_i = \varphi(s_{2i-1}) & \text{for } 1 \leq i \leq m, \\ m_j = \varphi(s_{2j}) & \text{for } 1 \leq j \leq m. \end{cases}$$

We propose the following conjecture.

Conjecture 1 Assume that (H7) holds. Let $H(p) = \varphi(|p|)$ for all $p \in \mathbb{R}^n$, and $V \in C(\mathbb{T}^n)$ be a given potential function. Let \bar{H} be the effective Hamiltonian corresponding to $H(p) - V(x)$. If

$$\operatorname{osc}_{\mathbb{T}^n} V = \max_{\mathbb{T}^n} V - \min_{\mathbb{T}^n} V \geq \max_{i,j} (M_i - m_j),$$

then the effective Hamiltonian \bar{H} is quasiconvex.

When $n = 1$, the above conjecture was proven in [7] based on some essentially one dimensional approaches. See [24] for a related result on the vanishing discount problem. In multi-dimensional spaces, this conjecture seems quite challenging in general. Let us now consider a basic situation, which we believe is an important step toward proving Conjecture 1.

(H8) $\varphi \in C([0, \infty), \mathbb{R})$ such that there exist $0 < s_1 < s_2 < \infty$ satisfying

$$\begin{cases} \varphi(0) = 0 < \varphi(s_2) = m_1 < \varphi(s_1) = M_1 < \lim_{s \rightarrow \infty} \varphi(s) = +\infty, \\ \varphi \text{ is strictly increasing on } [0, s_1] \text{ and } [s_2, \infty) \\ \text{and } \varphi \text{ is strictly decreasing on } [s_1, s_2]. \end{cases}$$

See Fig. 7. For this particular case, the conjecture says that \bar{H} is quasiconvex if

$$\operatorname{osc}_{\mathbb{T}^n} V \geq M_1 - m_1.$$

This is clear in terms of numerical results (Fig. 6).

Numerical Example 1 Let $n = 2$. We consider the following setting

$$\begin{aligned} H(p) &= \min \left\{ 4\sqrt{p_1^2 + p_2^2}, 2 \left| \sqrt{p_1^2 + p_2^2} - 1 \right| + 1 \right\} \quad \text{for } p = (p_1, p_2) \in \mathbb{R}^2, \\ V(x) &= S \cdot (1 + \sin(2\pi x_1))(1 + \sin(2\pi x_2)) \quad \text{for } x = (x_1, x_2) \in \mathbb{T}^2. \end{aligned}$$

The constant S serves as the scaling parameter to increase or decrease the effect of the potential energy V . For this specific case, $M_1 - m_1 = 2 - 1 = 1$ and $\operatorname{osc}_{\mathbb{T}^2} V = 4S$. So the threshold value is $S = 0.25$.

The Lax–Friedrichs based big-T method is used to compute this effective Hamiltonian. The computational x domain $[0, 1]^2$ is discretized by 401×401 mesh points

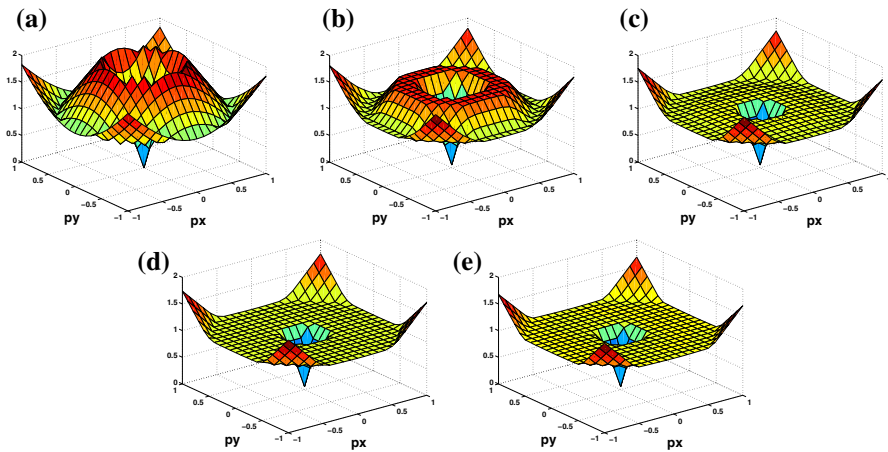


Fig. 6 **a** The original Hamiltonian: $S = 0$, **b** $S = 0.125$, **c** $S = 0.250$, **d** $S = 0.30$, **e** $S = 0.50$

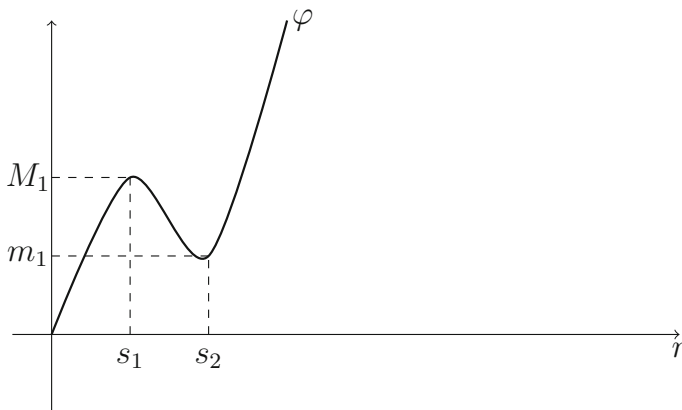


Fig. 7 Graphs of φ

and the p domain $[-1, 1]^2$ is sampled by 21×21 mesh points. The initial condition for the big-T method is taken to be $\cos(2\pi x_1) \sin(2\pi x_2)$. See Fig. 6.

However, we are only able to rigorously verify this for level sets above m_1 . This partially demonstrate the “quasi-convexification” since the nonconvexity of the original H appears on level sets between m_1 and M_1 . Denote $H_1 = \max\{H, m_1\}$ and

$$H_2(p) = \begin{cases} H(p) & \text{for } |p| \leq s_1, \\ \max\{M_1, H(p)\} & \text{for } |p| \geq s_1. \end{cases} \quad (3.1)$$

Note that H_1 is a “decomposable” nonconvex function from Remark 3 and H_2 is quasiconvex. Precisely speaking,

Theorem 3.1 Assume that (H8) holds. Let $H(p) = \varphi(|p|)$ for all $p \in \mathbb{R}^n$, and $V \in C(\mathbb{T}^n)$ be a given potential energy such that

$$\operatorname{osc}_{\mathbb{T}^n} V \geq M_1 - m_1.$$

Let \overline{H} be the effective Hamiltonian corresponding to $H(p) - V(x)$. Then for any $\mu \geq m_1$, the level set

$$\{\overline{H} \leq \mu\}$$

is convex.

Proof Without loss of generality, we assume that $\min_{\mathbb{T}^n} V = 0$. Let \overline{H}_1 be the effective Hamiltonian of $H_1 - V$. Clearly \overline{H}_1 is quasiconvex by Remark 3. So it suffices to show that for every $\mu > m_1$,

$$\overline{H}(p) = \mu \quad \text{if and only if} \quad \overline{H}_1(p) = \mu.$$

Since $\overline{H} \leq \overline{H}_1$, we only need to show that for fixed $p \in \mathbb{R}^n$, if $\overline{H}_1(p) > m_1$, then $\overline{H}(p) = \overline{H}_1(p)$. In fact, let $v(x, p)$ be a solution to

$$H_1(p + Dv) - V = \overline{H}_1(p) > m_1 \quad \text{in } \mathbb{T}^n.$$

Note that $\overline{H}_1(p) + V > m_1$. It is straightforward that v is also a solution to

$$H(p + Dv) - V = \overline{H}(p) \quad \text{in } \mathbb{T}^n.$$

Thus, $\overline{H}_1(p) = \overline{H}(p)$. The proof is complete. \square

Remark 4 Here is an interesting transition between, min–max decomposition, evenness and quasi-convexification when $n = 1$. Assume $\min_{\mathbb{T}^n} V = 0$.

- If $c_0 = \max_{\mathbb{T}^n} V < M_1 - m_1$, it is not hard to obtain a representation formula for \overline{H} (“conditional decomposition”):

$$\overline{H} = \min \{ \overline{H}_1, \overline{H}_2 \}.$$

Here, \overline{H}_1 falls into the category of item (i) in Remark 3. \overline{H}_2 is the effective Hamiltonian associated with the quasiconvex H_2 in (3.1). In particular, \overline{H} is **even but not quasiconvex**. The shape of \overline{H} is qualitatively similar to that of H . It is not clear to us whether this decomposition formula holds when $n \geq 2$. The key is to answer Question 3 in the Appendix.

- If $c_0 = \max_{\mathbb{T}^n} V = M_1 - m_1$, \overline{H} is **both even and quasiconvex**.
- If $c_0 = \max_{\mathbb{T}^n} V > M_1 - m_1$, for $n = 1$ and $V_s(x) = c_0 \max\{\frac{x}{s}, \frac{1-x}{1-s}\}$ for $x \in [0, 1]$ (extend periodically), \overline{H} is **quasiconvex but loses evenness**. More precisely, by adapting Step 1 in the proof of Theorem 1.4 in [32], we can show that the level set $\{\overline{H} \leq \mu\}$ is not even for any $\mu \in (M_1 - c_0, m_1)$ if $s \neq \frac{1}{2}$. Hence the above formula or decomposition no longer holds.

See also Fig. 8 below for numerical computations of a specific example.

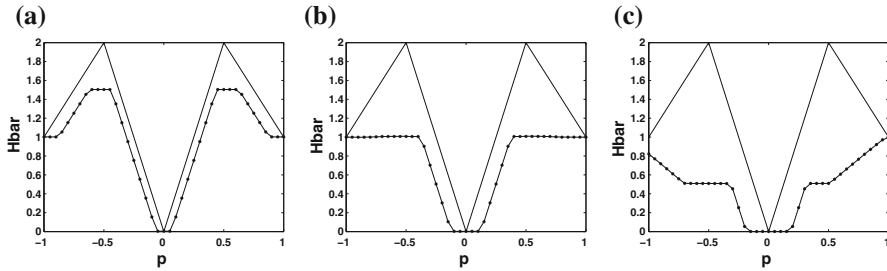


Fig. 8 1-D case. **a** $S = 0.5$, **b** $S = 1.0$, **c** $S = 1.50$

Numerical Example 2 Let $n = 1$. We consider the following setting

$$H(p) = \min \{4|p|, 2||p| - 1| + 1\}, \quad \text{for } p \in \mathbb{R},$$

$$V(x) = S \cdot \min \left\{ 3x, \frac{3}{2}(1-x) \right\}, \quad \text{for } x \in [0, 1],$$

and extend V to \mathbb{R} in a periodic way. The constant S serves as the scaling parameter to increase or decrease the effect of the potential energy V . For this case, $c_0 = 1$ and $M_1 - m_1 = 1$.

The Lax–Friedrichs based big-T method is used to compute this effective Hamiltonian. The p domain $[-1, 1]$ is sampled by 41 mesh points. The computational x domain $[0, 1]$ is discretized by 401 mesh points. The initial condition for the big-T method is taken to be $\cos(2\pi x)$. The results are shown in Fig. 8.

3.2 A double-well type Hamiltonian

Let $n = 2$. We consider a prototypical example

$$H(p) = \min \{|p - e_1|, |p + e_1|\} \quad \text{for } p = (p_1, p_2) \in \mathbb{R}^2,$$

where $e_1 = (1, 0)$. The shape of \overline{H} is more sensitive to the structure of the potential V instead of just the oscillation.

3.2.1 A unstable potential

We consider the following situation

$$\begin{cases} H(p) = \min \{|p - e_1|, |p + e_1|\} & \text{for } p = (p_1, p_2) \in \mathbb{R}^2, \\ V(x) = S \cdot (1 + \sin 2\pi x_1)(1 + \sin 2\pi x_2) & \text{for } x = (x_1, x_2) \in \mathbb{T}^2. \end{cases} \quad (3.2)$$

The constant $S > 0$ serves as a scaling parameter to adjust the oscillation of the potential V . Note that V attains its minimum along lines $x_1 = -1/4 + \mathbb{Z}$ and $x_2 = -1/4 + \mathbb{Z}$, which is clearly not a stable situation.

For this kind of nonconvex Hamiltonian H and V , complete quasi-convexification does not occur. However, we still see that \overline{H} eventually becomes a “less nonconvex” function. Let $\overline{H}(p, S)$ be the effective Hamiltonian corresponding to $H(p) - V(x)$. We have that

Theorem 3.2 *Assume that (3.2) holds. Then*

$$\lim_{S \rightarrow \infty} \overline{H}(p, S) = \max \{|p_2|, \min \{|p_1 - 1|, |p_1 + 1|\}\} \text{ for } p = (p_1, p_2) \in \mathbb{R}^2.$$

Proof We first show that for any $S > 0$,

$$\overline{H}(p, S) \geq \max \{|p_2|, \min \{|p_1 - 1|, |p_1 + 1|\}\}. \quad (3.3)$$

Let v be a viscosity solution to

$$H(p + Dv) - V(x) = \overline{H}(p, S) \text{ in } \mathbb{T}^2. \quad (3.4)$$

Without loss of generality, suppose that v is semi-convex and differentiable at $(x_1, -1/4)$ for a.e. $x_1 \in \mathbb{R}$. Otherwise, we may use super-convolution to get a subsolution and look at a nearby line by Fubini’s theorem. More precisely, for each $\varepsilon > 0$, set

$$v^\varepsilon(x) = \sup_{y \in \mathbb{R}^n} \left(v(y) - \frac{|x - y|^2}{\varepsilon} \right).$$

Then, v^ε is semi-convex and is a subsolution to an approximate equation of (3.4).

Accordingly, for a.e. $x_1 \in \mathbb{R}$,

$$\overline{H}(p, S) \geq \min \{|p_1 + v_{x_1}(x_1, -1/4) - 1|, |p_1 + v_{x_1}(x_1, -1/4) + 1|\}.$$

Assume that $x_1 \mapsto v(x_1, -1/4)$ attains its maximum at $x_0 \in \mathbb{R}$. Then $v_{x_1}(x_0, -1/4) = 0$ due to the semi-convexity of v . Hence

$$\overline{H}(p, S) \geq \min \{|p_1 - 1|, |p_1 + 1|\}.$$

Now, similarly, we can show that

$$\overline{H}(p, S) \geq |p_2 + v_{x_2}(-1/4, x_2)| \text{ for a.e. } x_2 \in \mathbb{T}.$$

Integrating both sides on $[0, 1]$ and using Jensen’s inequality, we derive $\overline{H}(p, S) \geq |p_2|$. Thus, (3.3) holds.

Next we show that

$$\lim_{S \rightarrow +\infty} \overline{H}(p, S) \leq \max \{|p_2|, \min \{|p_1 - 1|, |p_1 + 1|\}\}. \quad (3.5)$$

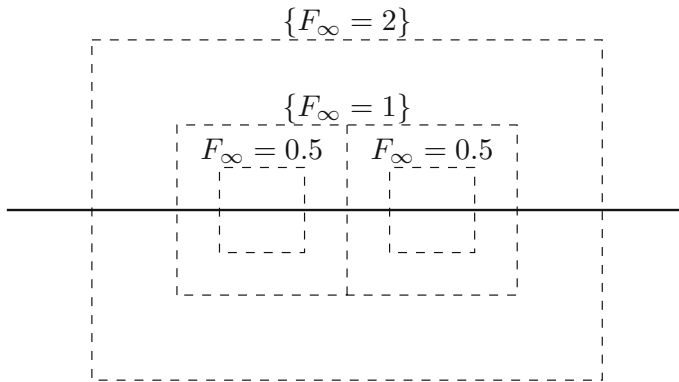


Fig. 9 Level curves of F_∞

In fact, for any $\delta > 0$, it is not hard to construct $\phi \in C^1(\mathbb{T}^2)$ such that

$$H(p + D\phi) \leq \max\{|p_2|, \min\{|p_1 - 1|, |p_1 + 1|\}\} + \delta$$

$$\text{on } (\mathbb{T} \times \{-1/4\}) \bigcup ((-1/4) \times \mathbb{T}).$$

Clearly,

$$\overline{H}(p, S) \leq \max_{x \in \mathbb{T}^2} (H(p + D\phi(x)) - S(1 + \sin 2\pi x_1)(1 + \sin 2\pi x_2)).$$

Sending $S \rightarrow \infty$ and then $\delta \rightarrow 0$, we obtain (3.5). \square

Remark 5 Write $F_\infty(p) = \max\{|p_2|, \min\{|p_1 - 1|, |p_1 + 1|\}\}$. Simple computations show that

- For $r \in [0, 1)$, $\{F_\infty = r\}$ consists of two disjoint squares centered at $(\pm 1, 0)$ respectively.
- For $r = 1$, $\{F_\infty = 1\}$ consists of two adjacent squares centered at $(\pm 1, 0)$ respectively.
- For $r > 1$, $\{F_\infty = r\}$ is a rectangle centered at the origin.

See Fig. 9. We can say that $F_\infty(p)$ looks more convex than the original Hamiltonian $H(p) = \min\{|p - e_1|, |p + e_1|\}$.

3.2.2 A stable potential

We consider the following

$$\begin{cases} H(p) = \min\{|p - e_1|, |p + e_1|\} & \text{for } p = (p_1, p_2) \in \mathbb{R}^2, \\ V(x) = S \cdot (\sin^2(2\pi x_1) + \sin^2(2\pi x_2)) & \text{for } x = (x_1, x_2) \in \mathbb{T}^2. \end{cases} \quad (3.6)$$

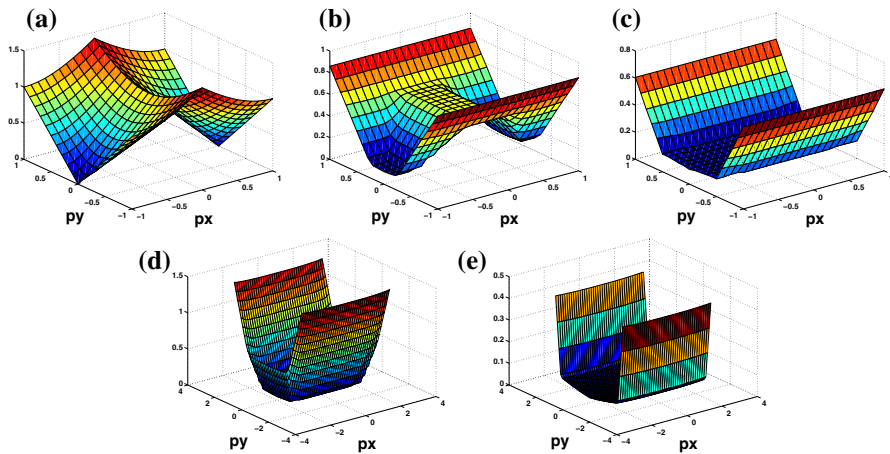


Fig. 10 Surface plots. **a** The original Hamiltonian: $S = 0$, **b** $S = 0.50$, **c** $S = 1.0$, **d** $S = 2.0$, **e** $S = 4.0$

The constant $S > 0$ serves as a scaling parameter to adjust the oscillation of the potential V . Note that V attains its minimum at points (x_1, x_2) such that $2(x_1, x_2) \in \mathbb{Z}^2$, which is a stable situation.

For this kind of potential, it is easy to show that

$$\lim_{S \rightarrow \infty} \overline{H}(p, S) = 0 \quad \text{locally uniformly in } \mathbb{R}^2.$$

More interestingly, numerical computations (Figs. 10, 11) suggest that \overline{H} becomes quasiconvex at least when $S \geq 1$.

Question 2 Assume that (3.6) holds. Does there exist L such that when $S > L$, \overline{H} is quasiconvex?

Numerical Example 3 We consider (3.6). The constant S serves as the scaling parameter to increase or decrease the effect of the potential.

We use the Lax–Friedrichs based big-T method to compute the effective Hamiltonian. The computation for time-dependent Hamilton–Jacobi equations is done by using the LxF-WENO 3rd-order scheme. The initial condition for the big-T method is taken to be $\cos(2\pi x_1) \sin(2\pi x_2)$. See Figs. 10 and 11 for results.

Numerical Example 4 We consider

$$\begin{cases} H(p) = \min \{|p - e_1|, |p + e_1|\} & \text{for } p = (p_1, p_2) \in \mathbb{R}^2, \\ V(x) = S \cdot (3 + \sin(2\pi x_1) + \sin(4\pi x_1) + \sin(2\pi x_2)) & \text{for } x = (x_1, x_2) \in \mathbb{T}^2. \end{cases}$$

This is the case that V is not even. The constant S serves as the scaling parameter to increase or decrease the effect of the potential. See Fig. 12 below. Clearly, \overline{H} is not even when $S = 0.125$, $S = 0.25$, $S = 0.5$, $S = 0.95$ and $S = 1.0$. Loss of evenness for all S implies that \overline{H} can not have a decomposition formula like (2.18) regardless of the oscillation of the V .

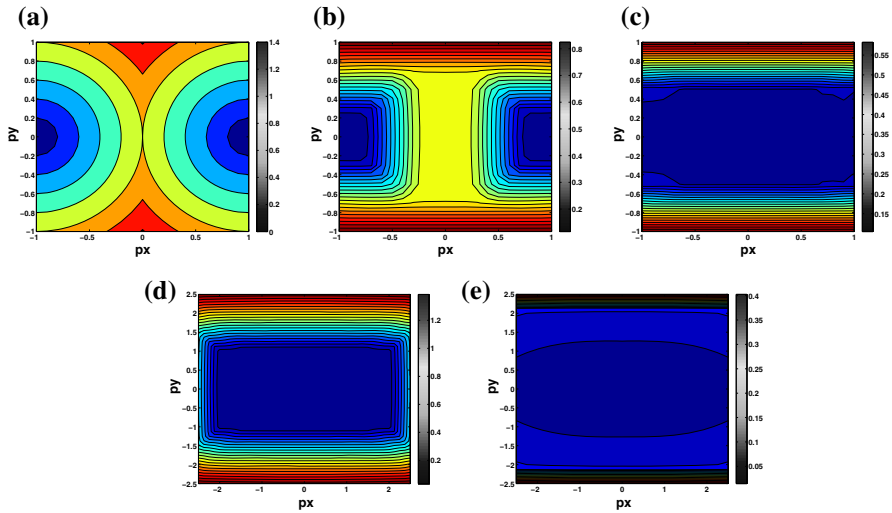


Fig. 11 Contour plots. **a** The original Hamiltonian: $S = 0$, **b** $S = 0.50$, **c** $S = 1.0$, **d** $S = 2.0$, **e** $S = 4.0$

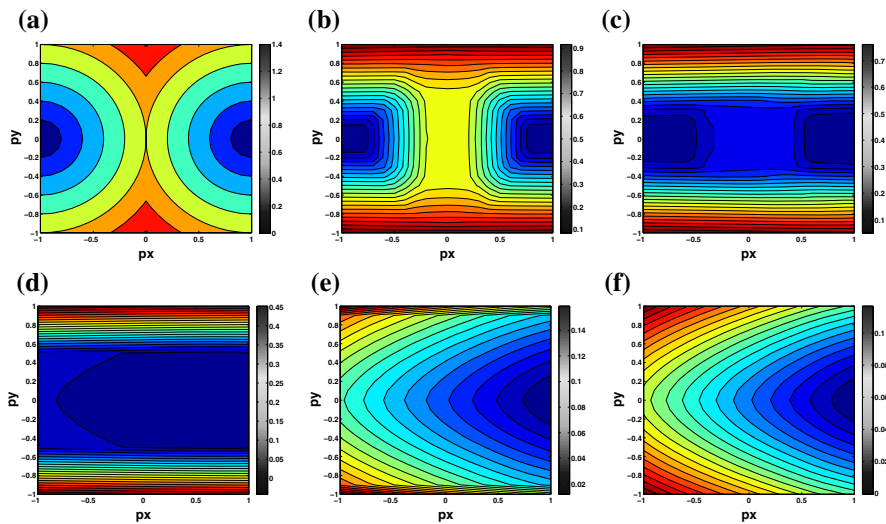


Fig. 12 **a** The original Hamiltonian: $S = 0$, **b** $S = 0.125$, **c** $S = 0.25$, **d** $S = 0.5$, **e** $S = 0.95$, **f** $S = 1.0$

4 Appendix: Some application in random homogenization

As a by-product, we show that all Hamiltonians in Sect. 2 are actually regularly homogenizable in the stationary ergodic setting. Let us first give a brief overview of stochastic homogenization.

4.1 Brief overview of stochastic homogenization

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose that $\{\tau_y\}_{y \in \mathbb{R}^n}$ is a measure-preserving translation group action of \mathbb{R}^n on Ω which satisfies that

(1) **(Semi-group property)**

$$\tau_x \circ \tau_y = \tau_{x+y} \quad \text{for all } x, y \in \mathbb{R}^n.$$

(2) **(Ergodicity)** For any $E \in \mathcal{F}$,

$$\tau_x(E) = E \quad \text{for all } x \in \mathbb{R}^n \quad \Rightarrow \quad \mathbb{P}(E) = 0 \quad \text{or} \quad \mathbb{P}(E) = 1.$$

The potential $V(x, \omega) : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$ is assumed to be stationary, bounded and uniformly continuous. More precisely, $V(x + y, \omega) = V(x, \tau_y \omega)$ for all $x, y \in \mathbb{R}^n$ and $\omega \in \Omega$, $\text{ess sup}_\Omega |V(0, \omega)| < +\infty$ and

$$|V(x, \omega) - V(y, \omega)| \leq c(|x - y|) \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } \omega \in \Omega,$$

for some function $c : [0, \infty) \rightarrow [0, \infty)$ satisfying $\lim_{r \rightarrow 0} c(r) = 0$.

For $\varepsilon > 0$, denote $u^\varepsilon(x, t, \omega)$ as the unique viscosity solution to

$$\begin{cases} u_t^\varepsilon + H(Du^\varepsilon) - V\left(\frac{x}{\varepsilon}, \omega\right) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u^\varepsilon(x, 0, \omega) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \quad (4.1)$$

Here $H \in C(\mathbb{R}^n)$ is coercive. A basic question is whether u^ε , as $\varepsilon \rightarrow 0$, converges to the solution to an effective deterministic equation (1.2) almost surely as in the periodic setting.

The stochastic homogenization of Hamilton–Jacobi equations has received much attention in the last 17 years. The first results were due to Rezakhanlou and Tarver [39] and Souganidis [41], who independently proved convergence results for general convex, first-order Hamilton–Jacobi equations in stationary-ergodic setting. These results were extended to the viscous case with convex Hamiltonians by Kosygina, Rezakhanlou and Varadhan [27] and, independently, by Lions and Souganidis [30]. New proofs of these results based on the notion of intrinsic distance functions (maximal subsolutions) appeared later in Armstrong and Souganidis [4] for the first-order case and in Armstrong and Tran [5] for the viscous case. See Davini, Siconolfi [16], Armstrong, Souganidis [4] for homogenization of quasiconvex, first-order Hamilton–Jacobi equations. See also [3, 28, 31].

One of the prominent open problems in the field is to prove/disprove homogenization in the genuinely nonconvex setting. In [6], Armstrong et al. showed that, for $H(p) = (|p|^2 - 1)^2$, (4.1) homogenizes in all space dimensions $n \geq 1$. In the next paper [7], Armstrong, Tran and Yu proved that, for $n = 1$, (4.1) homogenizes for general coercive H . Gao [22] generalized the result in [7] to the general non separable Hamiltonians $H(p, x, \omega)$ in one space dimension. A common strategy in papers [6, 7, 22] is to identify the shape of \overline{H} in the periodic setting first and then recover it in

the stationary ergodic setting. In particular, in contrast to previous works, our strategy does not depend on finding some master ergodic quantities suitable for subadditive ergodic theorems. Such kind of ergodic quantities may not exist at all for genuinely nonconvex H .

Ziliotto [42] gave a counterexample to homogenization of (4.1) in case $n = 2$. See also the paper by Feldman and Souganidis [21]. Basically, [21, 42] show that, if H has a strict saddle point, then there exists a potential energy V such that $H - V$ is not homogenizable.

Based on min–max formulas established in Sect. 2, we prove that, for the Hamiltonians H appear in Theorem 2.1, Corollary 2.4, Lemma 2.5, and Theorem 2.6, $H - V$ is always regularly homogenizable in all space dimensions $n \geq 1$. See the precise statements in Theorems 4.1, 4.3, and Corollary 4.2 in Sect. 4.2. Theorem 4.1 includes the result in [6] as a special case. Also, the result of Corollary 4.2 implies that, in some specific cases, even if H has strict saddle points, $H - V$ is still regularly homogenizable for every V with large oscillation. See the comments after its statement and some comparison between this result and the counterexamples in [21, 42].

The authors tend to believe that a prior identification of the shape of \bar{H} might be necessary in order to tackle homogenization in the general stationary ergodic setting. In certain special random environment like finite range dependence (i.i.d), the homogenization was established for a class of Hamiltonians in interesting works of Armstrong and Cardaliaguet [2], Feldman and Souganidis [21]. Their proofs are based on completely different philosophy and, in particular, rely on specific properties of the random media.

In the viscous case (i.e., adding $-\varepsilon \Delta u^\varepsilon$ to Eq. (4.1)), the stochastic homogenization problem for nonconvex Hamiltonians is more formidable. For example, the homogenization has not even been proved or disproved for simple cases like $H(p, x) = (|p|^2 - 1)^2 - V(x)$ in one dimension. Min–max formulas in the inviscid case are in general not available here due to the nonlocal effect (or regularity) from the viscous term. Nevertheless, see a preliminary result in one dimensional space by Davini and Kosygina [15].

The following definition was first introduced in [7].

Definition 1 We say that $H - V$ is regularly homogenizable if for every $p \in \mathbb{R}^n$, there exists a unique constant $\bar{H}(p) \in \mathbb{R}$ such that, for every $R > 0$ and for a.s. $\omega \in \Omega$,

$$\limsup_{\lambda \rightarrow 0} \max_{|x| \leq \frac{R}{\lambda}} |\lambda v_\lambda(x, p, \omega) + \bar{H}(p)| = 0. \quad (4.2)$$

Here, for $\lambda > 0$, $v_\lambda(\cdot, p, \omega) \in W^{1,\infty}(\mathbb{R}^n)$ is the unique bounded viscosity solution to

$$\lambda v_\lambda + H(p + Dv_\lambda) - V(x, \omega) = 0 \quad \text{in } \mathbb{R}^n.$$

According to Lemma 5.1 in [4], (4.2) is equivalent to saying that for a.s. $\omega \in \Omega$,

$$\lim_{\lambda \rightarrow 0} |\lambda v_\lambda(0, p, \omega) + \bar{H}(p)| = 0.$$

Clearly, if $H - V$ is regularly homogenizable, then random homogenization holds, that is, solution u^ε of (4.1) converges to u , the solution to (1.2) with \overline{H} defined by (4.2), as $\varepsilon \rightarrow 0$.

4.2 Stochastic homogenization results

The main claim is that $H(p) - V(x, \omega)$ is regularly homogenizable provided that $H(p)$ is of a form in Theorem 2.1, Corollary 2.4, Lemma 2.5, and Theorem 2.6. The proof is basically a repetition of arguments in the proofs of the aforementioned results except that the cell problem in the periodic setting is replaced by the discount ergodic problem in the random environment. This is because of the fact that the cell problem in the random environment might not have sublinear solutions at all. Here are the precise statements of the results.

Theorem 4.1 *Assume that $H \in C(\mathbb{R}^n)$ satisfies (H1)–(H3). Assume further that $\text{ess inf}_\Omega V(0, \omega) = 0$. Then $H - V$ is regularly homogenizable and*

$$\overline{H} = \max \{ \overline{H}_1, \overline{H}_2, 0 \}.$$

As mentioned, this theorem includes the result in [6] as a special case. An important corollary of this theorem is the following:

Corollary 4.2 *Let $H \in C(\mathbb{R}^n)$ be a coercive Hamiltonian satisfying (H1)–(H3), except that we do not require H_2 to be quasiconcave. Assume that $\text{ess inf}_\Omega V(0, \omega) = 0$, $\text{ess sup}_\Omega V(0, \omega) = \overline{m}$, and*

$$\overline{m} > \max_U H = \max_{\mathbb{R}^n} H_2.$$

Then $H - V$ is regularly homogenizable and

$$\overline{H} = \max \{ \overline{H}_1, 0 \}.$$

In particular, \overline{H} is quasiconvex in this situation.

It is worth emphasizing that we do not require any structure of H in U except that $H > 0$ there. In particular, H is allowed to have strict saddle points in U . Therefore, Corollary 4.2 implies that, in some specific cases, even if H has strict saddle points, $H - V$ is still regularly homogenizable provided that the oscillation of V is large enough. In a way, this is a situation when the potential energy V has much power to overcome the depths of all the wells created by the kinetic energy H and it “irons out” all the nonconvex pieces to make \overline{H} quasiconvex. This also confirms that the counterexamples in [21, 42] are only for the case that V has small oscillation, in which case V only sees the local structure of H around its strict saddle points, but not its global structure.

Let us now state the most general result in this stochastic homogenization context that we have.

Theorem 4.3 Assume that (H6) holds for some $m \in \mathbb{N}$. Assume further that $\text{ess inf}_{\Omega} V(0, \omega) = 0$, $\text{ess sup}_{\Omega} V(0, \omega) = \bar{m}$. Then $\varphi(|p|) - V(x, \omega)$ and $k_{m-1}(|p|) - V(x, \omega)$ are regularly homogenizable. Moreover, (2.16) and (2.17) hold in this random setting as well:

$$\overline{H}_m = \max \{ \overline{K}_{m-1}, \overline{\Phi}_{2m}, \varphi(s_{2m}) \},$$

and

$$\overline{K}_{m-1} = \min \{ \overline{H}_{m-1}, \overline{\Phi}_{2m-1}, \varphi(s_{2m-1}) - \bar{m} \}.$$

In particular, \overline{H}_m and \overline{K}_{m-1} are both even. Here we use the same notations as in Theorem 2.6.

We also have the following conjecture which was proven to be true in one dimension [7].

Conjecture 2 Assume that $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is continuous and coercive. Set $H(p, x, \omega) = \varphi(|p|) - V(x, \omega)$ for $(p, x, \omega) \in \mathbb{R}^n \times \mathbb{R}^n \times \Omega$. Then H is regularly homogenizable.

We believe that Conjecture 1 should play a significant role in proving the above conjecture as in the one dimensional case. An initial step might be to obtain random homogenization for the specific φ satisfying (H8). Below is closely related elementary question:

Question 3 Let w be periodic semi-concave (or semi-convex) function. Denote \mathcal{D} as the collection of all regular gradients, i.e.,

$$\mathcal{D} = \{ Dw(x) : w \text{ is differentiable at } x \}.$$

Is \mathcal{D} a connected set?

The periodicity assumption is essential. Otherwise, it is obviously false, e.g. $w(x) = -|x|$. When $n = 1$, the connectedness of \mathcal{D} follows easily from the periodicity and a simple mean value property (Lemma 2.6 in [7]).

As for the double-well type Hamiltonian $H(p) = \min\{|p - e_1|, |p + e_1|\}$ in the two dimensional space, the following question is closely related to Question 2 and counterexamples in [21, 42].

Question 4 Assume that $n = 2$ and $H(p) = \min\{|p - e_1|, |p + e_1|\}$ for all $p \in \mathbb{R}^2$, where $e_1 = (1, 0)$. Does there exist $L > 0$ such that, if

$$\text{osc}_{\mathbb{R}^2 \times \Omega} V = \text{ess sup}_{\Omega} V(0, \omega) - \text{ess inf}_{\Omega} V(0, \omega) > L,$$

then $H - V$ is regularly homogenizable?

4.3 Proof of Theorem 4.1

As a demonstration, we only provide the proof of Theorem 4.1 in details here. The extension to Theorem 4.3 is clear as it is a combination of the proof of Theorem 4.1 and induction by using the ideas of Lemma 2.5 and Theorem 2.6. Compared with the proof for the special case in [6], the following proof is much clearer and simpler.

We need the following comparison result.

Lemma 4.4 Fix $\lambda \in (0, 1)$, $R > 0$. Suppose that u, v are respectively a viscosity subsolution and a viscosity supersolution to

$$\lambda w + H(p + Dw) - V = 0 \quad \text{in } B(0, R/\lambda). \quad (4.3)$$

Assume further that there exists $C > 0$ such that

$$\begin{cases} \lambda(|u| + |v|) \leq C & \text{on } \overline{B}(0, R/\lambda), \\ |H(p) - H(q)| \leq C|p - q| & \text{for all } p, q \in \mathbb{R}^n. \end{cases}$$

Then

$$\lambda(u - v) \leq \frac{C(|x|^2 + 1)^{1/2}}{R} + \frac{C^2}{R} \quad \text{in } B(0, R/\lambda).$$

Proof Let

$$\tilde{v}(x) = v(x) + \frac{C(|x|^2 + 1)^{1/2}}{R} + \frac{C^2}{R\lambda} \quad \text{for } x \in B(0, R/\lambda).$$

Then, \tilde{v} is still a viscosity supersolution to (4.3) and furthermore, $\tilde{v} \geq u$ on $\partial B(0, R/\lambda)$. Hence, the comparison principle yields $\tilde{v} \geq u$ in $B(0, R/\lambda)$. \square

Proof of Theorem 4.1 Fix $p \in \mathbb{R}^n$. For $\lambda > 0$, let $v_\lambda(y, p)$ be the unique bounded continuous viscosity solution to

$$\lambda v_\lambda + H(p + Dv_\lambda) - V(y) = 0 \quad \text{in } \mathbb{R}^n. \quad (4.4)$$

In order to prove Theorem 4.1, it is enough to show that

$$\mathbb{P} \left[\lim_{\lambda \rightarrow 0} |\lambda v_\lambda(0, p) + \overline{H}(p)| = 0 \right] = 1. \quad (4.5)$$

Let us note first that, as H_1 is quasiconvex and H_2 is quasiconcave, $H_1 - V$ and $H_2 - V$ are regularly homogenizable (see [4, 16]). It is clear that

$$\max \{ \overline{H}_1, \overline{H}_2 \} \leq H. \quad (4.6)$$

Once again, we divide our proof into few steps.

Step 1. Assume that $\overline{H}_1(p) \geq \max \{\overline{H}_2(p), 0\}$. We proceed to show that

$$\mathbb{P} \left[\lim_{\lambda \rightarrow 0} |\lambda v_\lambda(0, p) + \overline{H}_1(p)| = 0 \right] = 1. \quad (4.7)$$

Since $H \geq H_1$, by the usual comparison principle, it is clear that

$$\mathbb{P} \left[\liminf_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p) \geq \overline{H}_1(p) \right] = 1. \quad (4.8)$$

It suffices to show that

$$\mathbb{P} \left[\limsup_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p) \leq \overline{H}_1(p) \right] = 1. \quad (4.9)$$

Let $v_{1\lambda}(y, -p)$ be the viscosity solution to

$$\lambda v_{1\lambda} + H_1(-p + Dv_{1\lambda}) - V = 0 \quad \text{in } \mathbb{R}^n. \quad (4.10)$$

Since $H_1 - V$ is regularly homogenizable, we get that, for any $R > 0$,

$$\mathbb{P} \left[\lim_{\lambda \rightarrow 0} \max_{y \in B(0, R/\lambda)} |\lambda v_{1\lambda}(y, -p) + \overline{H}_1(-p)| = 0 \right] = 1. \quad (4.11)$$

Fix $R > 0$. Pick $\omega \in \Omega$ such that (4.11) holds. For each $\varepsilon > 0$ sufficiently small, there exists $\lambda(R, \omega, \varepsilon) > 0$ such that, for $\lambda < \lambda(R, \omega, \varepsilon)$,

$$\max_{y \in B(0, R/\lambda)} |\lambda v_{1\lambda}(y, -p, \omega) + \overline{H}_1(-p)| \leq \varepsilon.$$

Note that by inf-sup representation formula and the even property of H_1 , we also have that \overline{H}_1 is also even, i.e., $\overline{H}_1(-p) = \overline{H}_1(p)$. In particular,

$$-\lambda v_{1\lambda}(y, -p, \omega) \geq \overline{H}_1(p) - \varepsilon \geq -\varepsilon \quad \text{for } y \in B(0, R/\lambda).$$

Due to the quasiconvexity of H_1 , this implies that, for $q \in D^- v_{1\lambda}(y, -p, \omega)$ for some $y \in B(0, R/\lambda)$,

$$H_1(-p + q) = -\lambda v_{1\lambda}(y, -p, \omega) + V(y) \geq -\varepsilon. \quad (4.12)$$

In particular, $|H_1(-p + q) - H(-p + q)| \leq \delta_\varepsilon$, where $\delta_\varepsilon = \max\{H(p) - H_1(p) : H_1(p) \geq -\varepsilon\}$.

Denote by $w = -v_{1\lambda}(y, -p, \omega) - \frac{2\overline{H}_1(p)}{\lambda}$. Then w is a viscosity subsolution to

$$\lambda w + H(p + Dw) - V = 2\varepsilon + \delta_\varepsilon \quad \text{in } B(0, R/\lambda).$$

Hence $w - \frac{2\varepsilon + \delta_\varepsilon}{\lambda}$ is a subsolution to (4.3). By Lemma 4.4, we get

$$\lambda w(0) - \lambda v_\lambda(0, p, \omega) \leq \frac{C}{R} + 2\varepsilon + \delta_\varepsilon.$$

Hence, (4.9) holds. Compare this to Step 2 in the proof of Theorem 2.1 for similarity.

Step 2. Assume that $\overline{H}_2(p) \geq \max \{ \overline{H}_1(p), 0 \}$. We proceed in the same way as in Step 1 (except that we use $v_{2\lambda}(y, p)$ instead of $v_{2\lambda}(y, -p)$ because of the quasiconcavity of H_2) to get that

$$\mathbb{P} \left[\lim_{\lambda \rightarrow 0} |\lambda v_\lambda(0, p) + \overline{H}_2(p)| = 0 \right] = 1. \quad (4.13)$$

Compare this to Step 3 in the proof of Theorem 2.1 for similarity.

Step 3. We now consider the case $\max \{ \overline{H}_1(p), \overline{H}_2(p) \} < 0$. Our goal is to show

$$\mathbb{P} \left[\lim_{\lambda \rightarrow 0} |\lambda v_\lambda(0, p)| = 0 \right] = 1. \quad (4.14)$$

This step basically shares the same philosophy as Step 4 in the proof of Theorem 2.1. Let us still present a proof here.

Thanks to the assumption that $\text{ess inf}_\Omega V(0, \omega) = 0$ and the fact that $H \geq 0$,

$$\mathbb{P} \left[\liminf_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p) \geq 0 \right] = 1. \quad (4.15)$$

We therefore only need to show

$$\mathbb{P} \left[\limsup_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p) \leq 0 \right] = 1. \quad (4.16)$$

For $\sigma \in [0, 1]$, $H_i - \sigma V$ are still regularly homogenizable for $i = 1, 2$. Let \overline{H}_i^σ be the effective Hamiltonian corresponding to $H_i - \sigma V$. By repeating Steps 1 and 2 above, we get that:

$$\begin{cases} \text{For } p \in \mathbb{R}^n \text{ and } \sigma \in [0, 1], \text{ if } \max \{ \overline{H}_1^\sigma(p), \overline{H}_2^\sigma(p) \} = 0, \\ \text{then } H - \sigma V \text{ is regularly homogenizable at } p \text{ and } \overline{H}^\sigma(p) = 0, \end{cases} \quad (4.17)$$

where $\overline{H}^\sigma(p)$ is its corresponding effective Hamiltonian. Take $p \in \mathbb{R}^n$ so that

$$\max \{ \overline{H}_1^\sigma(p), \overline{H}_2^\sigma(p) \} = 0. \quad (4.18)$$

for some $\sigma \in [0, 1]$. For $\lambda > 0$, let v_λ^σ be the viscosity solution to

$$\lambda v_\lambda^\sigma + H(p + Dv_\lambda^\sigma) - \sigma V = 0 \quad \text{in } \mathbb{R}^n,$$

then by (4.17),

$$\mathbb{P} \left[\lim_{\lambda \rightarrow 0} |\lambda v_{\lambda}^{\sigma}(0, p)| = 0 \right] = 1.$$

Since $V \geq 0$, the usual comparison principle gives us that $v_{\lambda}^{\sigma} \leq v_{\lambda}$. Hence, (4.16) holds true.

It remains to show that, if $\max \{ \overline{H}_1(p), \overline{H}_2(p) \} < 0$, then (4.18) holds for some $\sigma \in [0, 1]$. As $\overline{H}^0(p) = H(p) = 0$ for $p \in \partial U$, we only need to consider the case $p \notin \partial U$. There are two cases, either $p \in U$ or $p \in \mathbb{R}^n \setminus \overline{U}$. Again, it is enough to consider the case that $p \in U$. For this p , we have that

$$\overline{H}_2^0(p) = H_2(p) > 0 \quad \text{and} \quad \overline{H}_2^1(p) = \overline{H}_2(p) < 0.$$

By the continuity of $\sigma \mapsto \overline{H}_2^{\sigma}(p)$, there exists $\sigma \in [0, 1]$ such that $\overline{H}_2^{\sigma}(p) = 0$. This, together with the fact that $\overline{H}_1^{\sigma}(p) \leq H_1(p) < 0$, leads to (4.18). \square

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