1. Given an arbitrary matrix $A \in \mathbb{C}^{m \times n}$, you have constructed the QR decomposition by using the following three different procedures:

(a) the classical Gram-Schmidt method;
(b) the modified Gram-Schmidt method;
(c) the Householder transform based method.

Each method should return $Q$ and $R$ matrices in a suitable format, where $Q \in \mathbb{C}^{m \times n}$ is a matrix with orthonormal columns, and $R \in \mathbb{C}^{n \times n}$ is an upper triangular matrix.

In this project we are going to use these three factorizations to solve least-squares problems.

2. Solve the following three systems by the three different QR decomposition methods.

(a) Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \\ 4 & 2 & 3 \\ 4 & 2 & 2 \end{pmatrix}$$

Solve the system $Ax = b$, where $b$ is taken such that the true solution is $x = (1, 1, \cdots, 1)^T$.

(b) Let

$$A = \begin{pmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{pmatrix}$$

Solve the system $Ax = b$, where $b$ is taken such that the true solution is $x = (1, 1)^T$.

(c) Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 9 \end{pmatrix}$$

Solve the system $Ax = b$, where $b = (6, 15)^T$. Notice that this is an under-determined system.
3. Numerical discretization for inverse problems usually results in ill-conditioned linear systems. To solve such linear systems we use the Tikhonov regularization method. Then we may apply the QR method to solve the regularized system.

Consider the following over-determined linear system:

$$Ax = b,$$  \hspace{1cm} (0.1)

where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$.

The Tikhonov regularization can be formulated as a minimization problem:

$$\min_{x \in \mathbb{R}^n} \|b - Ax\|_2^2 + \alpha^2 \|x\|_2^2$$

where $\alpha \geq 0$ is a non-negative number.

**Questions:**

(a) prove that the necessary condition to minimize the above function is

$$(A^T A + \alpha^2 I)x = A^T b;$$

(b) show that the above system can be rewritten as

$$\begin{pmatrix} A \\ \alpha I \end{pmatrix} x = \begin{pmatrix} b \\ 0 \end{pmatrix};$$

(c) apply the QR method with the above regularization to solve the Hilbert system $Ax = b$, where $A$ is the n-th order Hilbert matrix (use the matlab command `hilb` to check it out), $b$ is taken such that the true solution is $x = (1, 1, \cdots, 1)^T$;

(d) to observe the performance of the above method, we may apply the QR method to solve the regularized system by taking

i. $n = 10$, and $\alpha = 0, 0.00000001, 0.0001, 0.001, 0.1$, respectively;

ii. $n = 20$, and $\alpha = 0, 0.00000001, 0.0001, 0.001, 0.1$, respectively;

iii. $n = 30$, and $\alpha = 0, 0.00000001, 0.0001, 0.001, 0.1$, respectively.

(e) can you choose an $\alpha$ which you think is ideal so that you get a best solution in the sense that it is closest to the true solution in the 2-norm sense?

4. What to turn in: you should write a short description (README file) of how to run the codes that you have written and email a tar ball of all the files to Justin Droba at drobajus@msu.edu. To make your email indicate that it is an Math850 project, please put “850 Homework” in the subject line.

**Due date: Wednesday, Oct. 14, 2009.**