1. (20 points.) Let $C$ be the unit circle oriented counterclockwise. Evaluate
\[ \int_C \frac{e^z + 1}{\sin z} \, dz. \]

**Solution.** The function $\sin z$ has zeros at $n\pi$, $n \in \mathbb{Z}$. The only zero in the interior of $C$ is at the origin. By the residue theorem,
\[ \int_C \frac{e^z + 1}{\sin z} \, dz = 2\pi i \text{res}_0 \frac{e^z + 1}{\sin z}. \]
We have
\[ \text{res}_0 \frac{e^z + 1}{\sin z} = \frac{e^0 + 1}{\cos 0} = 2. \]
Thus
\[ \int_C \frac{e^z + 1}{\sin z} \, dz = 4\pi i. \]

2. (22 points) Find
\[ \int_0^\infty \frac{\sin x}{x(1 + x^2)} \, dx. \]

**Solution.** Let $\Gamma$ denote the contour that consists of the upper semi-circle $S_r$ of radius $r$ oriented counterclockwise, the intervals $[\epsilon, r]$ and $[-r, -\epsilon]$ and $-S_\epsilon$, where $S_\epsilon$ is the upper semi-circle of radius $\epsilon$ oriented counterclockwise. Put
\[ f(\zeta) = \frac{e^{i\zeta}}{\zeta(1 + \zeta^2)}. \]
Clearly, if $r > 1$ and $\epsilon < 1$, then $f$ has one pole of order 1 in int $\Gamma$ at the point $i$.
We have
\[ \text{res}_i f = \frac{e^{-1}}{1 + 3i^2} = -\frac{1}{2e}. \]
Thus by the residue theorem,
\[ \int_\Gamma f(z) \, dz = 2\pi i \text{res}_i f = \frac{\pi i}{e}. \]
We have
\[ \left| \int_{S_r} f(z) \, dz \right| \leq \frac{2\pi r}{r(r^2 - 1)} \to 0 \quad \text{as} \quad r \to \infty. \]
Since $f$ has a simple pole at 0, we have
\[ \lim_{\epsilon \to 0} \int_{S_\epsilon} f(z) \, dz = \pi i \text{res}_0 f = \pi i \frac{e^0}{1 + 3 \cdot 0^2} = \pi i. \]
Thus
\[ \int_{-r}^{-\epsilon} \frac{e^{ix}}{x(1 + x^2)} \, dx + \int_{\epsilon}^{r} \frac{e^{ix}}{x(1 + x^2)} \, dx \to \pi i \left( 1 - \frac{1}{e} \right). \]
On the other hand,
\[
\int_{-\varepsilon}^{\varepsilon} \frac{e^{ix}}{x(1 + x^2)} dx + \int_{\varepsilon}^{r} \frac{e^{ix}}{x(1 + x^2)} dx = \int_{\varepsilon}^{r} \frac{e^{ix}}{x(1 + x^2)} dx - \int_{\varepsilon}^{r} \frac{e^{-ix}}{x(1 + x^2)} dx
\]
\[
= 2i \int_{\varepsilon}^{r} \frac{\sin x}{x(1 + x^2)} dx \to 2i \int_{0}^{\infty} \frac{\sin x}{x(1 + x^2)} dx.
\]

Thus
\[
\int_{0}^{\infty} \frac{\sin x}{x(1 + x^2)} dx = \frac{\pi}{2} \left( 1 - \frac{1}{e} \right).
\]

3. (15 points). State the Riemann conformal mapping theorem.

4. (22 points). Let \( p \) and \( q \) be polynomials of degrees \( \deg p \) and \( \deg q \) such that \( \deg q > \deg p + 1 \). Prove that the sum of the residues of the rational function \( \frac{p}{q} \) over all its poles is 0.

**Solution.** Let \( r \) be a positive number such that all zeros of \( q \) lie inside the disk of radius \( r \) centered at the origin. Let \( C_r \) be the circle of radius \( r \) centered at the origin with counterclockwise orientation. By the residue theorem,
\[
\sum_{\zeta \text{ is a zero of } q} \text{res}_\zeta \frac{p}{q} = \int_{C_r} \frac{p(\zeta)}{q(\zeta)} d\zeta,
\]
and the right-hand side does not depend on \( r \). We have
\[
\left| \int_{C_r} \frac{p(\zeta)}{q(\zeta)} d\zeta \right| \leq 2\pi r \max_{|\zeta|=r} \left| \frac{p(\zeta)}{q(\zeta)} \right| \to 0 \quad \text{as} \quad r \to \infty,
\]
since \( \deg q > \deg p + 1 \). Thus
\[
\sum_{\zeta \text{ is a zero of } q} \text{res}_\zeta \frac{p}{q} = 0.
\]

5. (21 points). Let \( f(z) = \sin z - 9z^3 + z^9 \). Find the number of zeros of \( f \) in the unit disk \( \{ \zeta \in \mathbb{C} : |\zeta| < 1 \} \).

**Solution.** Let us show that
\[
|\sin z + z^9| < 9 = 9|z|^2 \quad \text{if} \quad |z| = 1.
\]
We have
\[
|\sin z| = \frac{1}{2} |e^{iz} - e^{-iz}| \leq \frac{1}{2} \left( |e^{iz}| + |e^{-iz}| \right) = \frac{1}{2} \left( e^{Re iz} + e^{-Re iz} \right) < 3, \quad |z| = 1.
\]
Thus
\[
|\sin z + z^9| \leq |\sin z| + 1 < 4, \quad |z| = 1.
\]
By Rouché’s theorem, inside the unit disk \( f \) has the same number of zeros as the function \( 9z^3 \) has. Hence, \( f \) has 3 zeros inside the unit disk.