Geometry Primer

1 Connections and Curvature

This section presents the basics of calculus on vector bundles. It begins with the basic abstract definitions, then gives some concrete geometric examples.

Let E be a (real or complex) vector bundle over a manifold M. There are three levels of geometric structures on E:

- Metrics
- Covariant derivatives
- Second covariant derivatives. These decompose into

  (i) the covariant Hessian (the symmetric part), and

  (ii) the curvature (the skew-symmetric part).

Definition A *metric* on a vector bundle E is a smooth choice of a hermitian inner product on the fibers of E, that is, an $h \in \Gamma(E^* \otimes E^*)$ such that

$$\text{(i)}\ \ h(\alpha,\beta) = \overline{h(\beta,\alpha)} \qquad \forall \alpha,\beta \in \Gamma(E),$$

$$\text{(ii)}\ \ h(\alpha,\alpha) \geq 0 \qquad \forall \alpha \in \Gamma(E) \text{ and } h(\alpha,\alpha)=0 \text{ iff } \alpha \equiv 0.$$

We will take our hermitian metrics to be conjugate linear in the second variable. When E is a real vector bundle, (i) simply means that h is symmetric.

A metric on the tangent bundle TM is called a *Riemannian metric* on M.

In a local coordinate system $\{x^i\}$ on $U \subset M$ the vector fields $\frac{\partial}{\partial x^i}$ give a basis of the vector space T_xM at each $x \in U$ and the Riemannian metric is given by the symmetric matrix

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

by the formula $g = \sum_i g_{ij}(x)\, dx^i \otimes dx^j$.

Similarly, a *local frame* of E over $U \subset M$ is a set $\{\sigma_\alpha\}$ of sections of E over U such that the vectors $\{\sigma_\alpha(x)\}$ form a basis of the fiber $\pi^{-1}(x)$ at each $x \in U$. Write $\{\sigma^\alpha\} \in \Gamma(E^*)$ for the dual framing (so $\sum_\alpha \sigma^\alpha \cdot \sigma_\beta = \delta^\alpha_\beta$). In such a framing the metric on E is given by the hermitian matrix

$$h_{\alpha\overline{\beta}} = h(\sigma_\alpha, \sigma_\beta)$$

by the formula $h = \sum_\alpha h_{\alpha\beta}\, \sigma^\alpha \otimes \overline{\sigma^\beta}$, and for $\phi = \sum \phi^\alpha \sigma_\alpha$ we have $h(\phi, \phi) = \sum h_{\alpha\overline{\beta}} \phi^\alpha \overline{\phi^\beta}$.

A frame is *orthogonal* or *unitary* if $\{\sigma_1, \dots, \sigma_\ell\}$ is an orthonormal basis for E_x at each x. Local unitary frames always exist (start with any frame and apply the Gram-Schmidt process). In a unitary frame, the metric is simply $h = \sum \sigma^\alpha \otimes \overline{\sigma^\alpha}$, so the coefficients $h_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$ are constant.

An inner product on a vector space V induces inner products on V^*, on the exterior algebra $\Lambda^*(V)$, and on tensor products of these vector spaces. Applying this on each fiber shows that a metric on E induces metrics on E^*, $\Lambda^*(E)$ and on tensor product bundles. Simple examples:

- A metric h on E gives a metric on $E \otimes E$ by the formula
$$h(\alpha \otimes \beta, \alpha \otimes \beta) = h(\alpha, \alpha)\ h(\beta, \beta) \qquad\qquad \text{for } \alpha, \beta \in \Gamma(E)$$
and one on $\Lambda^2(E)$ by (using the convention $\alpha \wedge \beta = \frac{1}{\sqrt{2}}(\alpha \otimes \beta - \beta \otimes \alpha)$)
$$h(\alpha \wedge \beta, \alpha \wedge \beta) = h(\alpha, \alpha)\ h(\beta, \beta) - [h(\alpha, \beta)]^2.$$

- A metric h on E gives an identification of E with E^*, and hence gives a metric on E^*. When $E = TM$ this identification is given in local coordinates by
$$\frac{\partial}{\partial x^i} \mapsto \sum g_{ij} dx^i \quad \text{ and } \quad dx^i \mapsto \sum (g^{-1})^{ij} \frac{\partial}{\partial x^j}.$$
The ij component of the induced metric g^* on T^*M is
$$g^*(dx^i, dx^j) \;=\; \sum g\left((g^{-1})^{ik} \frac{\partial}{\partial x^k},\ (g^{-1})^{j\ell} \frac{\partial}{\partial x^\ell} \right) \;=\; \sum g_{k\ell} (g^{-1})^{ik} (g^{-1})^{j\ell} \;=\; (g^{-1})^{ij}.$$

A useful and standard convention is to write g_{ij} for the metric and g^{ij} for the components of its inverse, and to omit all summation signs, agreeing that repeated indices are summed. If one uses upper indices on the coordinate 1-forms dx^i and thinks of the coordinate vector fields $\partial/\partial x^i$ as having lower indices, then all formulas are consistent in the sense that all sums are over one upper and one lower index.

Connections

We would next like to define the "directional derivative" of a section $\phi \in \Gamma(E)$. To specify the direction we choose a vector field X; the dirctional derivative should compare the value of ϕ at $x \in M$ with the value at nearby points $x_t = \exp_x(tX)$. But the naive definition
$$\partial_X \phi(x) = \lim_{t \to 0} \frac{\phi(x_t) - \phi(x)}{t}$$
makes no sense because $\phi(x)$ and $\phi(x_t)$ are in different fibers of E and cannot be subtracted. Thus to define a derivative we need an additional geometric structure on E: an isomorphism between nearby fibers. Actually, we need this only infinitesimally. This is what a "connection" does.

There are many definitions of connections. We will start by defining a connection as an operator on sections with the properties expected of a directional derivative.

Definition 1.1 *A covariant derivative (or connection) on E is a bilinear map*

$$\nabla : \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E)$$

that assigns to each vector field X and each $\phi \in \Gamma(E)$ a "covariant directional derivative" $\nabla_X \phi$ satisfying, for each $f \in C^\infty(M)$,

(i) $\nabla_{fX}\phi = f\nabla_X \phi$

(ii) $\nabla_X(f\phi) = (X \cdot f)\phi + f\nabla_X\phi$ *(product rule).*

Given connections on vector bundles E and F we get one on $E \otimes F$ by the product rule:

$$\nabla_X^{E\otimes F}(\phi \otimes \psi) = \nabla_X^E \phi \otimes \psi + \phi \otimes \nabla_X^F \psi, \qquad \phi \in \Gamma(E), \psi \in \Gamma(F).$$

Similarly, a connection on E induces one on E^*: for $\phi \in \Gamma(E), \alpha \in \Gamma(E^*)$, the derivative of the function $\alpha(\phi)$ is, according to the product rule, $X \cdot \alpha(\phi) = (\nabla^{E^*}\alpha)\phi + \alpha(\nabla^E \phi)$, so ∇^{E^*} is defined by

$$(\nabla^{E^*}\alpha)\phi \;=\; X \cdot \alpha(\phi) \;-\; \alpha(\nabla^E\phi).$$

In particular, the metric h can be considered a section of the bundle $E^* \otimes E^*$. Then for $\phi, \psi \in \Gamma(E)$, $h(\phi,\psi)$ is the trace of a section of $E^* \otimes E^* \otimes E \otimes E$ so, again applying the product rule, for any vector field X

$$X \cdot h(\phi,\psi) = (\nabla_X h)(\phi,\psi) + h(\nabla_X\phi,\psi) + h(\phi,\nabla_X\psi). \tag{1.1}$$

Definition 1.2 *A connection ∇ is compatible with the metric h on E if $\nabla h = 0$.*

Each vector bundle with metric admits a compatible connection (see below). The difference of two connections is an $\mathrm{End}(E)$-values 1-form (from the definition $(\nabla - \nabla')_X\phi$ is $C^\infty(M)$-linear in X and ϕ). Conversely, given a compatible connection ∇ and $A \in \Gamma(T^*M \otimes \mathrm{End}(E))$,

$$\nabla' = \nabla + A$$

is a connection, which is compatible iff $A \in \Gamma(T^*M \otimes \mathrm{SkewEnd}(E))$. Thus the space of all compatible connections is an infinite-dimensional affine space.

Henceforth we will always assume that the connection is compatible with the metric, and will write the metric $h(\alpha,\beta)$ as $\langle\alpha,\beta\rangle$. Then (1.1) becomes

$$X \cdot \langle\alpha,\beta\rangle = \langle\nabla_X\alpha,\beta\rangle + \langle\alpha,\nabla_X\beta\rangle.$$

In a local framing $\{\sigma_\alpha\}$ over a coordinate patch $\{x^i\}$, the covariant derivative determines connection forms $\omega^\alpha_{\beta i}$ by

$$\nabla_{\frac{\partial}{\partial x^i}}\phi^\alpha = \sum \omega^\alpha_{\beta i}\,\phi^\beta.$$

For a general section $\phi = \sum \phi^\alpha \sigma_\alpha$ and vector field $X = \sum X^i \frac{\partial}{\partial x^i}$ we then have

$$\nabla_X \phi = \sum X^i \nabla_{\frac{\partial}{\partial x^i}} (\phi^\alpha \sigma_\alpha) = \sum X^i \left(\frac{\partial \phi^\alpha}{\partial x^i} + \omega^\alpha_{\beta i} \phi^\beta \right) \sigma_\alpha. \tag{1.2}$$

Thus the connection forms give the difference between the covariant derivative and the ordinary derivative in the framing. Note that it is the covariant derivative that is intrinsic; when we change framings the operators $\frac{\partial}{\partial x^i}$ and the connection forms both change.

We can now prove existence. Let $\{U_\gamma, \rho_\gamma\}$ be a partition of unity where each U_γ is a local coordinate chart over which E is trivialized by a local frame $\{\sigma_\alpha\}$. For vector fields $X = \sum X^i \frac{\partial}{\partial x^i}$ supported in one U_γ set

$$\nabla_X \phi = \begin{cases} \sum X^i \dfrac{\partial \phi}{\partial x^i} & \text{on } U_\gamma \\ 0 \quad \text{outside } U_\gamma \end{cases}$$

and for general vector fields set $\nabla_X \phi = \sum \nabla_{\rho_\gamma} \phi$. It is easily verified that this defines a connection. If the frame $\{\sigma_\alpha\}$ is unitary, then the coefficients of the metric H are constant on each U_γ. Consequently $\nabla h = 0$, so the connections is compatible with h.

In the special case where E is the tangent bundle we can impose an additional requirement on the connection. A connection ∇ on TM is called *torsion-free* or *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y] \qquad \text{for all } X, Y \in \Gamma(TM).$$

The following fact, often called the Fundamental Lemma of Riemannian Geometry, shows that these two conditions determine a connection.

Lemma 1.3 *On a manifold with Riemannian metric g, there is a unique connection ∇ on TM, the "Levi-Civita connection", that is (a) compatible with the metric, and (b) torsion free.*

Proof. For any three vector fields X, Y, Z, condition (a) requires that

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Computing $X \cdot g(Y, Z) + Y \cdot g(Z, X) - Z \cdot g(X, Y)$ using this formula and condition (b) yields

$$\begin{aligned} 2g(\nabla_X Y, Z) \;&=\; X \cdot g(Y, Z) + Y \cdot g(Z, X) - Z \cdot g(X, Y) \\ &\qquad - g(X, [Y, Z]) - g(Y, [X, Z]) - g(Z, [X, Y]). \end{aligned} \tag{1.3}$$

Both sides are linear in Z and g is non-degenerate. Uniqueness follows because the righthand side depends only on g. Conversely, requiring that this hold for all Z defines $\nabla_X Y$. One checks directly that this defines a torsion free connection with $\nabla g = 0$. $\square$

In local coordinates on a Riemannian manifold we can write the metric as $\{g_{ij}\}$. Taking coordinate vector fields $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$, we have $[X, Y] = 0$ and, from (1.3)

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \;=\; \sum_l \Gamma^\ell_{ij} \frac{\partial}{\partial x^\ell}$$

where Γ^l_{ij} are the Christoffel symbols

$$\Gamma^l_{ij} = \sum \frac{1}{2} g^{lk} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right).$$

For general vector fields $X = \sum X^i \frac{\partial}{\partial x^i}$ and $Y = \sum Y^j \frac{\partial}{\partial x^j}$ we have, as in (1.2),

$$\nabla_X Y = \sum X^i \left(\frac{\partial Y^j}{\partial x^i} + Y^k \Gamma^j_{ik} \right) \frac{\partial}{\partial x^j}.$$

Again, the Christoffel symbols and the operators $\frac{\partial}{\partial x^i}$ depend on the coordinates, but the covariant derivative does not.

A connection (on any vector bundle) gives a way of parallel transporting sections along curves. Fix a smooth curve $\gamma : [a, b] \to M$ from $x = \gamma_a$ to $y = \gamma_b$ and a vector ξ in the fiber E_x at x. We can then solve the initial value problem

$$\nabla_T \xi_t = 0 \qquad \text{with} \qquad \xi_a = \xi \tag{1.4}$$

where $T = \dot{\gamma}$ is the tangent vector to $\gamma(t)$. Evaluating the solution at $t = b$ yields a vector $\xi_b \in E_y$. This process defines a linear map

$$P_\gamma : E_x \to E_y$$

called the *parallel transport of* ξ *along* γ.

Remark 1.4 To show the existence and uniqueness of solutions of (1.4), cover γ with finitely many coordinate patches $\{U_i\}$ on which E is trivialized. In the trivialization on U_i the above equation has the form

$$\sum T^i \left(\frac{\partial \xi^\alpha}{\partial x^i} + \xi^\beta \omega^\alpha_{\beta i} \right) = 0. \tag{1.5}$$

Hence in each patch we can begin at $\gamma_c \in U_{i-1} \cap U_i$ and, by the fundamental theorem of ODEs, find a unique solution for $t \in [c, d]$ where $\gamma_d \in U_i \cap U_{i+1}$.

Having integrated, we can differentiate again and see that the connection is infinitestimal parallel transport

$$(\nabla_X \xi)_p \;=\; \lim_{t \to 0} \frac{P_{-t}\xi(p_t) - \xi(p)}{t} \tag{1.6}$$

where P_{-t} denotes parallel transport along the path $x_t = \exp(tX)$ from p_t back to p.

Proof. Along $\gamma(t) = \exp(tX)$ the solution to the parallel transport equation (1.4) can be written in local frame around $p \in M$ as $\xi = \sum \xi^\alpha(t)\sigma_\alpha$. The Taylor series of the coefficients is

$$\xi^\alpha(t) \;=\; \xi^\alpha(0) + tX^i \frac{\partial \xi^\alpha}{\partial x^i} + O(t^2)$$

and, since ξ satisfies the parallel transport equation (1.5), we have

$$P_t(\eta^\alpha) \;=\; \eta^\alpha(0) - tX^i \omega^\alpha_\beta \eta^\beta + O(t^2).$$

Replacing t by $-t$ and η by $\xi^\alpha(t)$, we see that the RHS of (1.6) is

$$\lim_{t\to 0}\ \frac{1}{t}\left(\xi^\alpha(0)+tX^i\frac{\partial\xi^\alpha}{\partial x^i}+tX^i\omega^\alpha_\beta\xi^\beta-\xi^\alpha(0)\right)\ =\ X^i\left(\frac{\partial\xi^\alpha}{\partial x^i}+\omega^\alpha_\beta\xi^\beta\right)\sigma_\alpha\ =\ (\nabla_X\xi)_p\,.$$

□

Caution While the limit (1.6) looks very similar to the limit defining the Lie derivative $\mathcal{L}_XY$, the two are unrelated. In particular, parallel transport is dependent on the choice a Riemannian metric, while the Lie derivative is defined solely in terms of the vector fields X and Y.

The definition of compatibility has the following two important consequences.

Lemma 1.5 *When the connection is compatible with the metric,*

1. *Parallel transport is an isometry, and*
2. *We have the pointwise inequality*
$$|d|\xi||\leq|\nabla\xi|.$$

Proof. (1) Given a path $\gamma(t)$ and vectors ξ_0,η_0 in the fiber of E at $\gamma(0)$, extend ξ_0,η_0 to vector fields ξ_t,η_t that are parallel along γ. Then for all t we have

$$\frac{d}{dt}\langle\xi_t,\eta_t\rangle\ =\ T\cdot\langle\xi_t,\eta_t\rangle\ =\ \langle\nabla_T\xi_t,\psi_t\rangle+\langle\xi_t,\nabla_T\eta_t\rangle\ =\ 0.$$

Thus inner products are preserved by parallel transport.

(2) For a quick proof, note that the equation $df^2=2f\,df$ gives $d|\xi|^2=2|\xi|d|\xi|$, while compatibility with the metric gives $|d|\xi|^2|=|2\langle\xi,\nabla\xi\rangle|\leq 2|\xi||\nabla\xi|$. Combining these gives the inequality in (2).

For a more enlightening proof, use polar coordinates in the fiber: on the set Ω where $\phi\neq 0$, set $\phi=\frac{\xi}{|\xi|}$. Then $\xi=|\xi|\,\phi$ and differentiating the equation $|\phi|^2=1$ shows that $2\langle\phi,\nabla\phi\rangle=0$. Hence

$$\begin{aligned}|\nabla\xi|^2\ =\ |\nabla(|\xi|\,\phi)|^2\ =\ |d|\xi|\,\phi+|\xi|\cdot\nabla\phi\ |^2\ &=\ |d|\xi||^2\,|\phi|^2+2|\xi|\,d|\xi|\,\langle\phi,\nabla\phi\rangle+|\xi|^2|\nabla\phi|^2\\ &=\ |d|\xi||^2+|\xi|^2|\nabla\phi|^2\\ &\geq\ |d|\xi||^2,\end{aligned}$$

so (2) holds on Ω and hence everywhere. □

Covariant Second Derivatives

A connection on E

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E)$$

together with the Levi-Civita connection on T^*M gives a connection on $T^*M \otimes E$. The composition

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M \otimes E)$$

is the *covariant second derivative*. Since

$$\begin{aligned} \nabla_X(\nabla_Y \xi) \;=\; \nabla_X(\nabla\xi(Y))) &= (\nabla_X \nabla\xi)(Y) + \nabla\xi(\nabla_X Y) \\ &= (\nabla^2\xi)(X,Y) + (\nabla\xi)(\nabla_X Y) \end{aligned}$$

the covariant second derivative is given by

$$(\nabla^2\xi)(X,Y) = \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi \qquad \text{for } X, Y \in \Gamma(TM),\ \xi \in \Gamma(E).$$

This expression is $C^\infty(M)$–bilinear in both X and Y.

Taking minus the trace of the covariant second derivative (in analogy with $d^*d = -\sum \partial_i\partial_i$ in euclidean space) gives a second order operator

$$-\mathrm{tr}\,\nabla^2 : \Gamma(E) \to \Gamma(E)$$

called the *trace Laplacian*. It is the same as the composition of ∇ with its adjoint ∇^* (exercise), and is given in a local orthonormal frame $\{e_i\}$ by

$$-\mathrm{tr}\,\nabla^2\xi = -\sum \left(\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_i e_i}\right)\xi.$$

Unlike second derivatives in euclidean space, covariant second derivatives do not commute. The expression that measures the failure to commute

$$\begin{aligned} (\nabla^2\xi)(X,Y) - (\nabla^2\xi)(Y,X) &= \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi - \nabla_Y\nabla_X\xi + \nabla_{\nabla_Y X}\xi \\ &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi. \end{aligned}$$

$C^\infty(M)$– linear in X, Y *and* ξ. This last fact, which is easily verified, means that the difference of these second order operators is a zeroth order operator, i.e. a tensor.

Definition The curvature of a connection ∇ is the tensor $F \in \Gamma(T^*M \otimes T^*M \otimes \mathrm{End}(E))$ given, for $X, Y \in \Gamma(TM)$, by

$$F(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \tag{1.7}$$

When ∇ is the Levi-Civita connection of a Riemannian metric g, the curvature is denoted $R(X,Y)$ and is called the *Riemannian curvature* of (M, g).

Proposition 1.6 (Symmetries of the curvature) *Let ∇ be a connection on $E \to M$ compatible with a metric $\langle\ ,\ \rangle$. Then for all vector fields X, Y, Z and sections $\xi, \eta \in \Gamma(E)$,*

(a) $F(X,Y) = -F(Y,X)$

(b) $\langle F(X,Y)\xi, \xi\rangle = -\langle \xi, F(X,Y)\xi\rangle$

(c) $(\nabla_X F)(Y,Z) + (\nabla_Y F)(Z,X) + (\nabla_Z F)(X,Y) = 0$

When $E = TM$, the Riemannian curvature R has an additional symmetry:

(d) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

Properties (a) and (b) show that the curvature can be considered as a 2-form with values in the bundle of skew-hermitian (skew-symmetric in the real case) endomorphisms of E, that is

$$F \in \Gamma(\Lambda^2(T^*M) \otimes \mathrm{SkewEnd}(E))$$

In (c) we are using the connection on this bundle obtained from the Levi-Civita connection on T^*M and the given one on E. Properties (c) and (d) are called, respectively, the second and first Bianchi identities.

Proof. Symmetry (a) is obvious from the definition of F. For (b), note that

$$\begin{aligned}\langle \nabla_X \nabla_Y \xi, \xi\rangle &= X \cdot \langle \nabla_Y \xi, \xi\rangle - \langle \nabla_Y \xi, \nabla_X \xi\rangle \\ &= X \cdot Y \cdot \langle \xi, \xi\rangle - X \cdot \langle \xi, \nabla_Y \xi\rangle - Y \cdot \langle \xi, \nabla_X \xi\rangle + \langle \xi, \nabla_Y \nabla_X \xi\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\langle F(X,Y)\xi, \xi\rangle &= \langle (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})\xi, \xi\rangle \\ &= (X \cdot Y - Y \cdot X - [X,Y]) \cdot \langle \xi, \xi\rangle + \langle \xi, (\nabla_Y\nabla_X - \nabla_X\nabla_Y - \nabla_{[Y,X]})\xi\rangle.\end{aligned}$$

Then (b) follows after noting that $[X,Y]f = XYf - YXf$ for $f \in C^\infty(M)$.

The remaining two symmetries follow from the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \qquad \forall X, Y, Z \in \Gamma(TM).$$

(The proof is straightforward: using $[X,Y] = XY - YX$ the lefthand side expands to a sum of 12 terms, which cancel.) For (d) we expand $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y$ using the definition (1.7) of curvature and the fact that the Levi-Civita connection is torsion-free. The result is

$$\begin{aligned}(\nabla_X\nabla_Y Z - &\nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) + (\nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X) \\ &\qquad + (\nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y) \\ &= (\nabla_X([Y,Z]) - \nabla_{[Y,Z]}X) + (\nabla_Y([Z,X]) - \nabla_{[Z,Y]}Y) + (\nabla_Z([X,Y]) - \nabla_{[X,Y]}Z) \\ &= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.\end{aligned}$$

The proof of (c) is similar. $\square$

Notice that each of the equations in Proposition 1.6 is tensorial, that is, linear over $C^\infty(M)$ in each of their variables. To prove tensorial formulas, it is sufficient to fix an (arbitary) point p

and verify the formula at p for the basis vectors of some trivialization. Often, the proof can be considerably shortened by a clever choice of trivialization. As an example, here is a second proof of formula (d) of Proposition 1.6.

Proof. Fix $p \in M$ and local coodinates $\{x^i\}$ around p. It suffices to verify (d) for the basis vector fields $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ and $Z = \frac{\partial}{\partial x^k}$. For these, we have $[X,Y] = [X,Z] = [Y,Z] = 0$, so by the definition of curvature, expression (d) is

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y.$$

But the connection is torsion-free, so the fact that $[X,Y] = 0$ implies that $\nabla_X Y = \nabla_Y X$; similarly $\nabla_X Z = \nabla_Z X$ and $\nabla_Y Z = \nabla_Z Y$. Hence the 6 terms above cancel in pairs, leaving 0. $\square$

Exercises

(1.1) Use a partition of unity to prove that the set

$$\mathrm{Metric}(M) = \{\text{all Riemannian metrics on the manifold } M\}$$

is a non-empty convex cone (without vertex) in the vector space $\Gamma(Sym^2(T^*M))$.

(1.2) Let ∇ and ∇' be connections compatible with a metric $\langle\ ,\ \rangle$ on a vector bundle E. Prove:

(a) For any $f \in C^\infty(M)$, $\nabla'' = f\nabla + (1-f)\nabla'$ is a connection compatible with the metric.

(b) $\nabla - \nabla' = A$ is an $\mathrm{End}(E)$-valued 1-form (i.e., an element of $\Gamma(T^*M \otimes \mathrm{End}(E))$ that is skew-hermitian when E is complex and skew-symmetric when E is real.

(c) Conversely, with ∇ and A as in (b), show that $\nabla' = \nabla + A$ is a connection compatible with the metric.

Note that (b) and (c) show that

$$\mathcal{A} = \{\text{all compatible connections on } E\}$$

is an infinite-dimensional affine space modeled on $\Gamma(T^*M \otimes \mathrm{SkewEnd}(E))$ where $\mathrm{SkewEnd}(E)$ is the bundle of skew-hermitian endomorphisms of E.

Hint: For (b), use the fact that any $C^\infty(M)$-linear map $\Phi : \Gamma(E) \to \Gamma(F)$ arises in this way from a bundle map $\phi : E \to F$ by composition: $\Phi(f\xi) = f\Phi(\xi)\ \ \forall f \in C^\infty(M)$.

(1.3) Let ∇ be the Levi-Civita connection of a Riemannian manifold (M,g). In a local coordinate system $\{x^i\}$, we write the metric as

$$g = \sum g_{ij}\ dx^i \otimes dx^j$$

and define the Christoffel symbols by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum\ \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

(a) Show that $\nabla_i = \partial_i + \Gamma^k_{ij}$, i.e. for vector fields $X = \sum X^i \frac{\partial}{\partial x^i}$ and $Y = \sum Y^j \frac{\partial}{\partial x^j}$

$$\nabla_X Y = \sum X^i \left(\frac{\partial}{\partial x^i} + \Gamma^k_{ij} Y^j \right) \frac{\partial}{\partial x^k}.$$

(b) Show that the torsion-free condition implies that $\Gamma^k_{ij} = \Gamma^k_{ji}$.

The components of the Riemannian curvature tensor R are defined by

$$\sum R^i_{jk\ell} \frac{\partial}{\partial x^i} = R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right) \frac{\partial}{\partial x^j}$$

(c) Derive the classical expression $R^i_{jkl} = \sum (\partial_k \Gamma^i_{\ell j} - \partial_\ell \Gamma^i_{kj}) + (\Gamma^m_{\ell j}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{\ell m})$

(1.4) Let ∇ and ∇' be two connections on a vector bundle $E \to M$. Write $\nabla' = \nabla + A$ where A is an End(E)-valued 1-form. Show that the curvatures of ∇ and ∇' are related by

$$F^{\nabla'} = F^{\nabla} + d^{\nabla} A + [A, A]$$

where $d^{\nabla} : \Gamma(T^*M) \otimes End(E) \to \Gamma(\Lambda^2 T^*M \otimes End(E))$ is the covariant exterior derivative defined by

$$d^{\nabla} A(X, Y) = (\nabla_X A)(Y) - (\nabla_Y A)(X),$$

and $[A, A]$ is the End(E)-valued 2-form given by $[A, A](X, Y) = A(X)A(Y) - A(Y)A(X)$.

(1.5) Prove the second Bianchi identity: the curvature satisfies (c) of Proposition 1.6.

2 The Basic Differential Operators on Bundles

Much of geometric analysis involves working with first and second order differential operators on vector bundles. A homogeneous first order linear differential operator $D : \Gamma(E) \to \Gamma(F)$ between vector bundles is a composition $D = \sigma \circ \nabla$, or more explicitly

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\sigma} \Gamma(F)$$

where ∇ is a connection on E and σ is a bundle map called the *symbol* of D. In this section we will introduce several basic first order operators, giving their constructions and local formulas. These examples will be put into a general setting in the next section.

Example 1 The exterior derivative d is a first order operator

$$d : \Omega^p_M \to \Omega^{p+1}_M$$

where Ω^p_M denotes the set of p-forms, i.e. the smooth sections of the bundle $\Lambda^{p+1}T^*M$). We will give three descriptions of d: one global, one in a local frame of TM, and two in local coordinates.

On a Riemannian manifold (M, g) we have the Levi-Civita connection ∇ on $\Lambda^P T^*M$. Consider the composition

$$\Gamma(\Lambda^p T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes (\Lambda^p T^*M) \xrightarrow{\sigma} \Gamma(\Lambda^{p+1}T^*M) \tag{2.8}$$

where σ is the bundle map defined by exterior multiplication: $\sigma(\alpha \otimes \omega) = \alpha \wedge \omega$. In a local frame $\{e_i\}$ on TM over $U \subset M$ and dual frame $\{e^i\}$ of T^*M we have $\nabla\omega = \sum e^i \otimes \nabla_{e_i}\omega$ and $d\omega = \sum e^i \wedge \nabla_{e_i}\omega$, and hence is local frames

$$d \;=\; \sum_{i=1}^n e^i \wedge \nabla_{e_i}. \tag{2.9}$$

We can similarly express d in terms of local coordinates $\{x^i\}$ on $U \subset M$. Write the elements in the frame as $e_i = \sum A^j_i \frac{\partial}{\partial x^j}$ and $e^k = \sum B^k_\ell \, dx^\ell$ where $A(x)$ and $B(x)$ are matrices satisfying

$$\delta^k_i = e^k(e_i) \;=\; \sum B^k_\ell \, dx^\ell \left(A^j_i \frac{\partial}{\partial x^i}\right) \;=\; \sum B^k_\ell A^j_i \, \delta^\ell_j \;=\; (AB)^k_i \,. \tag{2.10}$$

Therefore $B = A^{-1}$. Formula (2.9) then becomes $d = \sum (AB)^j_\ell \, dx^\ell \wedge \nabla_j = \sum \delta^j_\ell \, dx^\ell \wedge \nabla_j$, giving the local coordinate formula

$$d \;=\; \sum_{i=1}^n dx^i \wedge \nabla_i. \tag{2.11}$$

in local coordinates. Finally, substitute $\nabla_i = \frac{\partial}{\partial x^i} + \Gamma^\ell_{ik}$. Using the symmetry $\Gamma^\ell_{ik} = \Gamma^\ell_{ki}$, one can show that the term involving Γ^ℓ_{ik} vanishes (see Exercise 2.5), leaving

$$d \;=\; \sum_{i=1}^n dx^i \wedge \frac{\partial}{\partial x^i} \tag{2.12}$$

This is the usual formula for d. In particular, reading backwards, we have shown that the composition (2.20) is precisely the exterior derivative d on p-forms.

Example 2. Given a connection ∇ on E we get one on E-valued p-forms: $\boldsymbol{\nabla}(e \otimes \omega) = \nabla e \otimes \omega + e \otimes \nabla^{LC}\omega$. Tenoring (2.20) with E gives a covariant exterior derivative

$$d^\nabla : \; \Gamma(\Lambda^p T^*M \otimes E) \longrightarrow \Gamma(\Lambda^{p+1}T^*M \otimes E) \tag{2.13}$$

that is given globally by $d^\nabla = \sigma \circ \boldsymbol{\nabla}$ and locally as

$$d^\nabla \;=\; \sum_{i=1}^n e^i \wedge \boldsymbol{\nabla}_{e_i}.$$

This operator allows us to cast some previous facts in an elegant form:

Lemma 2.1 *For a p-form ω and $\xi \in \gamma(E)$, we have*

(a) $d^\nabla(\omega \otimes \xi) \;=\; d\omega \otimes \xi + (-1)^{deg\,\omega}\omega \wedge \nabla\xi.$

(b) $d^\nabla \circ d^\nabla \xi \;=\; F^\nabla \xi$.

(c) $d^\nabla F^\nabla = 0$.

(d) If $\nabla' = \nabla + A$ *then* $F^{\nabla'} = F^\nabla + d^\nabla A + [A, A]$.

Proof. It suffices to prove these locally in a dual from $\{e^i, e_i\}$. For (a) we have $d^\nabla(\omega \otimes \xi) = \sum e^i \wedge (\nabla_i \omega \otimes \xi + \omega \otimes \nabla_i \xi) = d\omega \otimes \xi + (-1)^{\deg \omega} \omega \wedge \nabla \xi$. For (b) we note that

$$\begin{aligned} d^\nabla \circ d^\nabla \xi \;&=\; \sum e^i \wedge \boldsymbol{\nabla}_{e_i}(e^j \wedge \boldsymbol{\nabla}_{e_j}\xi) \\ &=\; \sum e^i \wedge \big(\boldsymbol{\nabla}_i e^j \wedge \boldsymbol{\nabla}_j \xi + e^j \wedge \boldsymbol{\nabla}_i \boldsymbol{\nabla}_j \xi\big) \\ &=\; \sum e^i \wedge e^j \wedge \boldsymbol{\nabla}_i \boldsymbol{\nabla}_j \xi + (\nabla_i e^j, e_k)\, e^i \wedge e^k \boldsymbol{\nabla}_j \xi \end{aligned}$$

and then note that differentiating the duality pairing $(e^j, e_k) = \delta^j_k$ gives $(\nabla_i e^j, e_k) + (e^j, \nabla_i e_k) = 0$. Hence

$$d^\nabla \circ d^\nabla \xi \;=\; \sum e^i \wedge e^j \left(\boldsymbol{\nabla}_i \boldsymbol{\nabla}_j \xi - \boldsymbol{\nabla}_{\nabla_i e_j}\right) \xi$$

Then (b) follows because $(e^i \wedge e^j)(X, Y) = e^i(X) e^j(Y) - e^i(Y) e^j(X)$.

For (c), note that $d^\nabla F$ is an $\mathrm{End}(E)$-valued 3-form defined as the skew-symmetrization of ∇F. Noting that $F(X, Y)$ is already skew in X and Y, we have

$$\begin{aligned} (d^\nabla F)(X, Y, Z) \;&=\; \frac{1}{6}\left[(\nabla_X F)(Y, Z) - (\nabla_Y)(X, Z)) - (\nabla_Z F)(Y, X)\right. \\ &\qquad\qquad \left. -(\nabla_X F)(Z, Y) - (\nabla_Y)(Z, X)) - (\nabla_Z F)(X, Y)\right] \\ &=\; \frac{1}{3}\left[((\nabla_X F)(Z, Y) + (\nabla_Y)(Z, X)) + (\nabla_Z F)(X, Y)\right] \end{aligned}$$

which vanishes by the second Bianchi identity. Finally, (d) was Exercise 1.4. □

For further examples of differential operators, one can consider adjoints. Recall that the metric on an oriented Riemannian manifold (M, g) determines a volume form given in local coordinates by

$$dv_g \;=\; \sqrt{det\, g_{ij}}\;\; dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

This in turns defines a measure on M (even when M is not orientable), allowing us to integrate functions on M. We will also consider vector bundles E and F over M, both equipped with metrics and compatible connections.

Definition 2.2 *The (formal) adjoint of a linear operator* $D : \Gamma(E) \to \Gamma(F)$ *is the operator* $D : \Gamma(F) \to \Gamma(E)$ *defined by*

$$\int_M \langle D\xi, \eta \rangle \; dv_g \;=\; \int_M \langle \xi, D^* \eta \rangle \; dv_g \tag{2.14}$$

for all $\xi \in \Gamma(E), \eta \in \Gamma(F)$ *with compact support.*

Example 3. The adjoint of d on Ω^*_M is

$$d^* : \Omega^p_M \to \Omega^{p-1}_M.$$

To obtain a specific formula, recall that when M is oriented and n-dimensional, the Hodge star operator $* : \Omega^p_M \to \Omega^{n-p}_M$ is defined in terms of the inner product on $\Lambda^p(T^*M)$ and by the pointwise condition

$$\langle \xi, *\eta \rangle \;=\; \langle \xi, \eta \rangle \; dv_g. \tag{2.15}$$

It satisfies $*^2 = (-1)^{p(n-p)} Id.$ on Ω^p_M. For a quick proof, note that it suffices to check this on basis elements. In fact, after reordering, it suffices to check for $\alpha = e^1 \wedge e^2 \wedge \cdots \wedge e^p$ and $\beta = e^{p+1} \wedge \cdots \wedge e^n$. Then $|\alpha| = |\beta| = 1$, $*\alpha = \beta$ and $**\alpha = *\beta = (-1)^{p(n-p)}\alpha$ using (2.15).

Now for any compactly-supported $\xi \in \Omega^{p-1}_M$ and $\eta \in \Omega^p_M$ we have

$$0 \;=\; \int_M d(\xi \wedge *\eta) \;=\; \int_M d\xi \wedge *\eta + (-1)^{p-1}\, \xi \wedge d{*}\eta.$$

Rewriting $d{*}\eta$ in the last term as $(-1)^{(n-p+1)(p-1)} **d{*}\eta$ and using (2.15), this becomes

$$0 \;=\; \int_M \langle d\xi, \eta \rangle \; dv_g + (-1)^{(p-1)(n-p)} \int_M \langle \xi, *d{*}\eta \rangle \; dv_g.$$

Comparing with (2.14) we see that

$$d^* \;=\; (-1)^{n(p-1)+1} \; *d* \tag{2.16}$$

when M is oriented. Reversing orientation changes $*$ to $-*$, so does not change $*d*$. As a result, d^* is defined even when M is not orientable.

We can also given local formulas for d^* similar to those for d. Before doing so, however, it is useful to introduce two new ideas.

The first is an algebraic operation. Let $\lrcorner$ denote *contraction* or *interior multplication*: for $X \in Vect(M)$ and p-form ω, we define $X \lrcorner\, \omega$ to be the $(p-1)$-form

$$(X \lrcorner\, \omega)(Y_1, \ldots, Y_{p-1}) \;=\; \omega(X, Y_1, \ldots, Y_{p-1}) \qquad \forall Y_1, \ldots Y_{p-1} \in Vect(M).^1$$

We will need two facts about the contraction operator (to be proven later):

- Interior multiplication is the adjoint of wedge multiplication: for each vector X, p-form α and $(p-1)$-form β we have
$$\langle X \lrcorner\, \alpha, \beta \rangle \;=\; \langle \alpha, X^* \wedge \beta \rangle$$
where X^* denotes the 1-form dual to X by the metric, i.e. $X^* = g(X, \cdot)$.

- The Cartan identity: for each vector X, 1-form α and p-form ω we have
$$(X \lrcorner\, \alpha \wedge + \alpha \wedge X \lrcorner\,)\omega \;=\; \alpha(X)\, \omega$$

[1] A technical remark is needed here: there are two common choices for the pairing $\Lambda^*(T^*M) \otimes \Lambda^*(TM) \to \mathbb{R}$ used in the above formula. With our convention, when $\omega = \alpha_1 \wedge \cdots \wedge \alpha_p$ is the product of 1-forms, we have $X \lrcorner\, \omega = \alpha_1(X)\, a_2 \wedge \cdots \wedge \alpha_p \;-\; \alpha_2(X)\, \alpha_1 \wedge \alpha_3 \wedge \cdots \wedge \alpha_p + \cdots$ (with the alternative pairing there is a factor of $1/p$ on the righthand side).

The second new idea is a technique for choosing local frames that expedite calculations.

Lemma 2.3 (Useful frame) *At any $p \in M$, we can choose dual orthonormal frames $\{e_i\}$, $\{e^i\}$ of TM and T^*M such that $(\nabla_{e_i} e_j)_p = 0$ $[e_i, e_j]_p = 0$, and $(\nabla_{e_i} e^j)_p = 0$ for all i, j.*

Proof. Choose an orthonormal basis $\{e_i(p)\}$ of T_pM and parallel transport outward along geodesics from p to construct a local frame $\{e_i\}$. By the parallel transport equation (1.4) we have $(\nabla_{e_i} e_j)_p = 0$, and hence $[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i$ vanishes at p for all i and j. Finally, differentiating the duality relation $e^j(e_i) = \delta_i^j$ shows that $(\nabla_{e_i} e^j)_p = 0$. $\square$

The frames in the above lemma are useful for verifying tensorial identities since they eliminate the appearance of connection forms in many calculations. For example, at the point p the second covariant derivative is simply

$$(\nabla^2 \xi)(e_i, e_j) \;=\; \nabla_{e_i} \nabla_{e_j}.$$

We will do several such calculations below. But a word of caution: only first derivatives of the frame vanish at p and one cannot assume the vanishing of second derivatives such as $(\nabla_i \nabla_j e^k)_p$ (if this were true that all curvatures would vanish!).

We can now give a local formula for d^*. Given an E-valued $(p-1)$-form α and an E-valued p-form β, consider the $(n-1)$-form ω defined locally by

$$\omega \;=\; \sum_i \langle \alpha, \epsilon^i \wedge \beta \rangle \; e_i \lrcorner \, dv_g$$

where $\{e_i\}$ is a local frame of TM and $\{e^i\}$ is the dual frame. In a different frame $\hat{e}_i = A_i^j \, e_j$ and $\hat{e}^k = B_k^i e^k$ we have $B = A^{-1}$ as in (2.10), so

$$\sum_i \langle \alpha, \hat{e}^i \wedge \beta \rangle \; \hat{e}_i \lrcorner \, dv_g \;=\; \sum_i \, (BA)_j^i \langle \alpha, \epsilon^j \wedge \beta \rangle \; e_i \lrcorner \, dv_g \;=\; \omega.$$

Thus ω is independent of the frame, so is a globally-defined form. Fixing $p \in M$ and taking $\{e_i\}$ to be a useful frame at p, we have

$$d\omega \;=\; \sum e^k \wedge \nabla_k \omega \;=\; \sum \big(\langle \nabla_k \alpha, e^i \wedge \beta \rangle + \langle \alpha, e^i \nabla_k \beta \rangle \big) \; e^k \wedge e_i \lrcorner \, dv_g.$$

But by the Cartan identify $e^k \wedge e_i \lrcorner \, dv_g = (e^k \wedge e_i \lrcorner \; + e_i \lrcorner \, e^k \wedge) dv_g = \delta_i^k \, dv_g$. Thus

$$d\omega \;=\; \langle \sum e_i \lrcorner \, \nabla_i \alpha, \beta \rangle + \langle \alpha, d\beta \rangle \; dv_g.$$

Again, the operator $\sum e_i \lrcorner \, \nabla_i$ is independent of the frame, so is globally defined. Integrating, we have

$$0 \;=\; \int_M d\omega \;=\; \langle \sum e_i \lrcorner \, \nabla_i \alpha, \beta \rangle + \langle \alpha, d\beta \rangle \; dv_g,$$

which shows that $d^* = -\sum e_i \lrcorner \, \nabla_i$. Then repeating the argument that led to (2.11) and (2.12, we have three local formulas for d^* similar to those for d:

$$d^* \;=\; -\sum e_i \lrcorner \, \nabla_{e_i} \qquad \text{in a local orthonormal frame} \tag{2.17}$$

$$\;=\; -\sum g^{ij} e_i \lrcorner \, \nabla_{e_j} \qquad \text{in any local frame} \tag{2.18}$$

$$\;=\; -\sum g^{ij} \frac{\partial}{\partial x^i} \lrcorner \, \nabla_j \qquad \text{in a local coordinates.} \tag{2.19}$$

From these expressions we see that the symbol of d^* is $-\sigma^*(e^i \otimes \omega) = -e_i^* \lrcorner\, \omega$ where e_i^* is the metric dual of e_i.

Example 4. As in Example 2, a connection on E induces one on $\Lambda^* T^*M \otimes M$ and we can generalize d^* to an operator $(d^\nabla)^*$ on E-valued p-forms whose symbol is also interior multiplication:

$$\Gamma(\Lambda^p T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes (\Lambda^p T^*M) \xrightarrow{-\sigma^*\otimes 1} \Gamma(\Lambda^{p-1} T^*M). \tag{2.20}$$

The formulas are the same as for d^*:

$$\begin{aligned} (d^\nabla)^* &= (-1)^{p(n-p)} * d^\nabla * && \text{on } E\text{-valued } p\text{-forms} \\ &= -\sum g^{ij} e_i \lrcorner\, \nabla_{e_j} && \text{in any local frame.} \end{aligned}$$

Example 5. Fix a bundle E over a Riemannian manifold (M, g) and a connection ∇ on E compatible with a metric $\langle\ ,\ \rangle$ on E. The adjoint ∇^* of the covariant derivative $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ is the composition

$$\nabla^* : \Gamma(T^*M \otimes E) \xrightarrow{\boldsymbol{\nabla}} \Gamma(T^*M \otimes T^*M \otimes E) \xrightarrow{g} \Gamma(E)$$

of the connection on $T^*M \otimes E$ with the metric on the cotangent space (cf. Exercise 2.3). In local coordinates $\{x^i\}$ and in a local frame $\{e_i\}$ of TM it is given by

$$\nabla^* \;=\; -\sum_{ij} g^{ij}\, \frac{\partial}{\partial x^i} \lrcorner\, \nabla_j \qquad \text{and} \qquad \nabla^* \;=\; -\sum_{ij} g^{ij}\; e_i \lrcorner\, \nabla_{e_j}$$

and in an orthonormal frame $\{e_i\}$ the latter formula simplifies to $\nabla^* = -\sum e_i \lrcorner\, \nabla_{e_i}$.

Our final example is a second order differential operator. As in Example 5, we fix a bundle $E \to (M, g)$ with a metric and a compatible connection ∇.

Example 6. The *trace Laplacian* of ∇ is $-\mathrm{tr}_g\, \nabla^2$, the trace (using the metric g) of the second covariant derivative. It is given a local frame by $-\mathrm{tr}\, \nabla^2 = \sum g^{ij} \left(\nabla_{e_i} \nabla_{e_j} - \nabla_{\nabla_i e_j}\right)$. Alternatively, it is the composition of ∇ with ∇^*:

Lemma 2.4 $\nabla^*\nabla \;=\; -tr\nabla^2$ *as operators on* $\Gamma(E)$.

Proof. In a useful frame $\{e_i\}$ at $p \in M$ we have

$$\nabla^*\nabla\xi \;=\; -\sum_{i,j} e_i \lrcorner\, \nabla_i (e^j \otimes \nabla_j \xi) \;=\; -\sum\; e_i \lrcorner\, e^j \wedge \nabla_i \nabla_j \xi \;=\; -\sum \nabla_i \nabla_i \xi \;=\; -\mathrm{tr}\, \nabla^2 \xi.$$

Because both $\nabla^*\nabla$ and $\mathrm{tr}\, \nabla^2$ are independent of the frame, so $\nabla^*\nabla = -\mathrm{tr}\, \nabla^2$ at all $p \in M$. $\square$

Underlying some of these calculations is the idea of a derivation. Recall that an algebra A over a ring R is *graded* if $A = \bigoplus A^k = A^0 \oplus A^1 \oplus A^2 \oplus \cdots$ and multiplication in A respects the grading in the sense that $x \in A^k, y \in A^\ell \implies xy \in A^{k+\ell}$. For example,

- For any vector space V, $\mathrm{Sym}^*(V^*) = \bigoplus \mathrm{Sym}^k(V^*)$ and $\Lambda^*(V^*) = \bigoplus \Lambda^k(V^*)$ are graded $\mathbb{R}$-algebras.
- For any vector bundle $E \to M$, $\mathrm{Sym}^*(E^*)$ and $\Lambda^*(E^*)$ are graded $C^\infty(M)$-algebras.

Definition 2.5 *If A is a graded algebra, a linear map $D : A \to A$ is a* even derivation *if*

$$D(\alpha \cdot \beta) = D\alpha \cdot \beta + \alpha \cdot D\beta \qquad \forall \alpha, \beta \in A$$

and a odd derivation *if*

$$D(\alpha \cdot \beta) = D\alpha \cdot \beta + (-1)^{deg\,\alpha}\, \alpha \cdot D\beta \qquad \forall \alpha, \beta \in A.$$

Lemma 2.6 *(a) If D and D' are derivations, then $[D, D']$ is a derivation whose parity is the product of the parities of D and D'.*

Furthermore, if A is a graded R-algebra that is generated by $R = A^0$ and A^1 (as is the case for the above examples), then

(b) A derivation D on A is uniquely determined by its values on A^0 and A^1.

(c) Each linear map $l : A^1 \to A^1$ extends uniquely to an odd derivation $\mathcal{L} : A \to A$ with $\mathcal{L} \equiv 0$ on A^0.

Proof. The proof of (a) is straightforward. For (b) and (c), note that each element of A can be written as a sum of elements of the form $\xi = f\ e^1 \cdot e^2 \cdots e^k$ for $f \in A^0$ and $e^1, \ldots, e^k \in A^1$. By the derivation property, we can express $D\xi$ of such elements as a sum of terms involving only Df and De^i. $\square$

Example 2.7 (a) Given a vector space V, contraction $v \lrcorner$ by $v \in V$ is an odd derivation of $A = \Lambda^*(V^*)$ and of $B = \mathrm{Sym}^*(V^*)$.

Proof. One way to prove this is to *define* $v \lrcorner$ to be the unique odd derivation that vanishes on A^0 and satisfies $v \lrcorner \alpha = \alpha(v)$ for $\alpha \in A^1 = V^*$. Then for $\alpha = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p$ we have

$$v \lrcorner \alpha \;=\; \alpha_1(v)\, a_2 \wedge \cdots \wedge \alpha_p \;-\; \alpha_2(v)\, \alpha_1 \wedge \alpha_3 \wedge \cdots \wedge \alpha_p + \cdots$$

and hence $(v \lrcorner \alpha)(w_1, w_2, \ldots) = \alpha(v, w_1, \ldots) - \alpha(w_1, v, w_2, \ldots) + \cdots = \alpha(v, w_1, w_2, \ldots)$, which agrees with our previous definition of contraction. $\square$

(b) Essentially the same proof shows that each contraction $v \lrcorner$ by $v \in V$ is an even derivation of $A = \mathrm{Sym}^*(V^*)$.

(c) Similarly, contraction $X \lrcorner$ by $X \in Vect(M)$ is an odd derivation of Ω^*_M and an even derivation of $\mathrm{Sym}^*(T^*M)$.

(d) $d : \Omega^*_M \to \Omega^*_M$ is an odd derivation.

(e) For each $X \in Vect(M)$, ∇_X is an even derivation on $\mathrm{Sym}^*(E^*)$ and $\Lambda^*(E^*)$.

(f) For each $X, Y \in Vect(M)$ the Riemannian curvature $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is an even derivation of Ω^*_M.

(g) By Lemma 2.6, any bundle endomorphism $A : T^*M \to T^*M$ extends uniquely to an even derivation $\mathcal{A} : \Omega^*_M \to \Omega^*_M$ with $\mathcal{A}f = 0$ for all $f \in C^\infty(M)$.

The following notation is extremely helpful for doing local calculations involving endomorphisms of exterior algebras.

Let V be a vector space with dual space V^*. Each $v \in V$ and $\alpha \in V^*$ determine endomorphims

$$a^*_\alpha : \Lambda^p(V^*) \to \Lambda^{p+1}(V^*)$$
$$a_v : \Lambda^p(V^*) \to \Lambda^{p-1}(V^*)$$

of $\Lambda^*(V^*)$ by exterior and interior multiplication. Because $\alpha\wedge\beta = -\beta\wedge\alpha$, we have $a^*_\alpha a^*_\beta = -a^*_\beta a^*_\alpha$. Similarly, a_v anti-communtes with a_w. Thus we have

$$a_v a_w \;=\; -a_w a_v, \qquad a^*_\alpha a^*_\beta \;=\; -a^*_\beta a^*_\alpha, \qquad a_v a^*_\alpha + a^*_\alpha a_v \;=\; \alpha(v)\cdot \mathrm{Id}. \tag{2.21}$$

where the last is the Cartan Identity. This notation is used in physics, where V^* is the space of states of a fermion, V the states of the corresponding antiparticle, and $\Lambda^p(V^*)$ is the p-particle state space. The operators a^*_α and a_v are fermion creation and annihalation operators, and (2.21) are the "fermion anticommutation relations".

Products of the a^*_α and a_v provide a convenient notation for describing operators on $\Lambda^*(V^*)$.

Example 2.8 Suppose V is finite-dimensional. Choose a basis $\{e_i\}$ of V with dual basis $\{e^i\}$ and write a_{e_i} as a_i and $a^*_{e^i}$ as a^{*i}. These are a basis of $\mathrm{End}(\Lambda^*(V^*))$, and we have the following important expressions (proved in Exercise 2.4 below).

(a) The fermion number operator F, defined as $p\cdot\mathrm{Id}$ on $\Lambda^p(V^*)$, is $\sum a^{*i}a_i$.

(b) The parity operator $(-1)^F$, which is Id. on the forms of even degree and $-$Id. on those of odd degree, is
$$(-1)^F = \prod(1-2a^{*i}a_i) = \prod(a_i a^{*i} - a^{*i}a_i).$$

(c) The extension of a matrix $A \in \mathrm{End}(V^*)$ to an even derivation of $\Lambda^*(V^*)$ vanishing on $\Lambda^0(V^*)$ is $\mathcal{A} = \sum A^i_j a^{*j} a_i$ where A^i_j is the matrix of A defined by $A(e^i) = \sum A^i_j e^j$.

Now consider a Riemannian manifold (M,g). A orthonormal frame $\{e_i\}$ on TM and dual frame $\{e^j\}$ on a open set $U \subset M$ defines, as above, operators $a_i, a^{*i} \in \Gamma(\mathrm{End}(\Lambda^*(T^*M))$ by contracting with e_i and wedging with e^i).

Now by Example 2.7e, the Levi-Civita connection extends as an even derivation to a connection on $\Lambda^*(T^*M)$. By Example 2.7f, its curvature extends an even derivation that vanishes on Ω^0_M which, by Lemma 2.6b, is equal to the curvature of the connection on $\Lambda^*(T^*M)$. In coordinates, the components of the curvature acting on T^*M are defined by

$$R(e_k,e_\ell)e^i \;=\; \sum_j R^i_{jk\ell}e^j$$

for each i,k,ℓ. Accordingly, as in Example 2.8c, the curvature $R(e_k,e_\ell)$ acts on $\Lambda^*(T^*M)$ by

$$R(e_k,e_\ell) \;=\; -\sum_{i,j} R^i_{jk\ell}a^{*j}a_i \tag{2.22}$$

Theorem 2.9 (Bochner Formula) *The Laplace-Beltrami operator is*

$$\Delta \;=\; dd^* + d^*d \;=\; \nabla^*\nabla + \mathcal{R}$$

where $\mathcal{R}$ is the curvature endomorphism given locally by $\mathcal{R} = \sum R^i_{jk\ell}\; a^{*k} a_\ell a^{*j} a_i$.

Proof. In a useful frame at $p \in U$,

$$d = \sum a^{*k}\nabla_k \qquad \text{and} \qquad d^* = -\sum a_k \nabla_k.$$

Calculating at p using (2.21),

$$\begin{aligned} dd^* + d^*d \;&=\; -\sum a^{*k}a_\ell \nabla_k\nabla_\ell + a_\ell a^{*k}\nabla_\ell\nabla_k \\ &=\; -\sum (a^{*k}a_\ell + a_\ell a^{*k})\nabla_k\nabla_\ell - a_\ell a^{*k}(\nabla_k\nabla_\ell - \nabla_\ell\nabla_k) \\ &=\; \sum -\nabla_k\nabla_k + a_\ell a^{*k} R(e_k, e_\ell) \\ &=\; \nabla^*\nabla + \mathcal{R}. \end{aligned}$$

where in the last step we used (2.22) and the fact that $R(e_k, e_\ell) = -R(e_k, e_\ell)$. The expression (1.4) is tensorial, so independent of the choice of frame. $\square$

Remark Instead of using a local orthonormal frame, one can use local coordinates, defining A^{*i} to be $dx^i\wedge$ and a_i to be $\frac{\partial}{\partial x^i} \lrcorner$. Then $\mathcal{R}$ is given by

$$\mathcal{R} = \sum \left(g^{\ell m} R^i_{jk\ell}\right)\; a^{*k} a_m a^{*j} a_i \tag{2.23}$$

and this formula is independent of the coordinate system.

Exercises

(2.1) Re-prove the second Bianchi identify as stated in Proposition 0.6 this time using a useful frame (take X, Y, Z to be basis vectors in the frame and choose ξ with $(\nabla\xi)_p = 0$).

(2.2) Prove the formula $\nabla^* = -\sum_{ij} g^{ij}\; e_i \lrcorner \nabla_{e_j}$ by first showing that for any $\xi \in \Gamma(E)$ and $\eta \in \Gamma(T^*M \otimes E)$ the $(n-1)$-form $\omega = \sum \langle \eta, e^i \otimes \xi\rangle\; e_i \lrcorner\, dv_g$ is well-defined (independent of the frame), then computing $d\omega$ in a useful frame, and integrating.

(2.3) Let ∇ be a connection on a bundle E that is compatible with the metric $\langle\;,\;\rangle$ on E. Show that $\langle \nabla^*\nabla\phi, \phi\rangle + \langle \phi, \nabla^*\nabla\phi\rangle = 2|\nabla\phi|^2 + d^*d|\phi|^2$ for all $\phi \in \Gamma(E)$, and consequently

$$\int_M \langle \phi, \nabla^*\nabla\psi\rangle \;=\; \int_M \langle \nabla\phi, \nabla\psi\rangle$$

for all smooth sections $\phi, \psi \in \Gamma(E)$ that are compactly supported in the interior of M.

(2.4) Using the notation of Example 2.8 show that, for each i, $a^{*i}a_i$ is an even derivation of $\Lambda^8(V^*)$ that vanishes on $\Lambda^0(V^*)$. Then use Lemma 2.6b and equations (2.21) to prove the formulas in parts (a), (b) and (c) of Example 2.8.

3 Elliptic Operators

A second order linear differential operator (LDO) $\mathbb{R}^n$ has the form

$$D \;=\; \sum a^{ij}(x)\,\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} \;+\; b^i(x)\,\frac{\partial}{\partial x^i} \;+\; c(x) \tag{3.24}$$

(and applies to smooth functions in the obvious way). The coefficients depend on the coordinate system: after a diffeomorphism of $\mathbb{R}^n$ the coefficients will look quite different. Below, we will give a description of LDO's that does not depend on coordinates at all, and hence carries over to vector bundles on manifolds.

The set $C^\infty(M)$ of all smooth functions on a manifold M is a ring. For each $p \in M$, evaluation at p is a ring homomorphism $C^\infty(M) \to \mathbb{R}$ whose kernel is the ideal

$$I_p \;=\; \{f \in C^\infty(M) \mid f(p) = 0\}.$$

The *k-jet* of functions at p is the vector space

$$J^k_p \;=\; C^\infty(M)\Big/ I_p^{k+1}.$$

The projection $C^\infty(M) \to J^k_p$ takes a function f to its k-jet $[f]_k$; two functions have the same k-jet if they agree through order k. In coordinates, the k-jet $[f]_k$ is uniquely represented by the degree k Taylor polynomial of f at p.

Similarly, the set $\Gamma(E)$ of smooth sections of a vector bundle E over M is a module over $C^\infty(M)$, and the k-jets of sections of E is

$$J^k(E)_p \;=\; \Gamma(E)\Big/ I_p^{k+1}\cdot\Gamma(E).$$

In particular, $J^0(E)_p$ is the fiber E_p. In coordinates $\{x^i\}$ near p and a basis $\{\sigma_\alpha\}$ of E_p, the k-jet of a section ξ is uniquely represented by its degree k Taylor polynomial

$$[\xi]_k \;=\; \sum_\alpha \left(a_0^\alpha + \sum_i a_i^\alpha\,(x-p)^i + \cdots + \sum a^\alpha_{i_1 i_2 \cdots k}\,(x-p)^{i_1 i_2 \cdots k} \right) \sigma_\alpha.$$

The coefficients give one choice of basis of the finite-dimensional vector space $J^k(E)_p$.

A linear differential operator is a linear map from $J^k(E)_p$ to the fiber F_p of F that depends smoothly on p. Equivalently, it is a linear map from $\Gamma(E)$ to F_p that vanishes on sections that vanish to order $k+1$ at p. Noting that I_p^{k+1} is generated by functions of the form f^{k+1} for $f \in I_p$, we have:

Definition 3.1 *Fix vector bundles E and F over M. A k^{th} order linear differential operator from E to F is a linear map*

$$D : \Gamma(E) \to \Gamma(F)$$

such that $D(f^{k+1}\xi)_p = 0$ for all $f \in I_p, \xi \in \Gamma(E)$ (and this holds for no $k' < k$).

Lemma 3.2 *A 0^{th}-order LDO is a vector bundle map $E \to F$.*

Proof. Recall that a map $\mathcal{L} : \Gamma(E) \to \Gamma(F)$ is linear over $C^\infty(M)$ if and only if there is a vector bundle map $L : E \to F$ so that $\mathcal{L}(\xi) = L \circ \xi$ for all $\xi \in \Gamma(E)$[2]. Thus if suffices to show that any 0^{th}-order LDO D is $C^\infty(M)$-linear.

Given $f \in C^\infty(M)$ and $p \in M$, we can write $f = c + g$ where $g \in I_p$ and c is the constant $f(p)$. Then

$$D(f\xi)(p) \;=\; D(c\xi + g\xi)_p \;=\; cD(\xi)_p + \cancel{D(g\xi)_p} \;=\; f(p)D(\xi)_p.$$

Thus $D(f\xi) = fD\xi$ for all $\xi \in \Gamma(E)$, as required. $\square$

Examples (1) A connection, regarded as map $\Gamma(E) \to \Gamma(T^*M \otimes E)$, is a first order LDO: for $f \in I_p$,

$$\nabla(f\xi)(p) \;=\; (df \otimes \xi + \cancel{f\nabla\xi})_p \;=\; (df \otimes \xi)_p$$

is generally non-zero, while $\nabla(f^2\xi)(p) = \big(2fdf \otimes \xi + f^2\nabla\xi\big)_p = 0$.

(2) On a Riemannian manifold, a trace Laplacian $\nabla^*\nabla : \Gamma(E) \to \Gamma(F)$ satisfies

$$\begin{aligned} \nabla^*\nabla(\phi\xi) &= \nabla^*(d\phi \otimes \xi + \phi\nabla\xi) \\ &= d^*d\phi \otimes \xi - 2\langle d\phi, \nabla\xi\rangle + \phi\,\nabla^*\nabla\xi \end{aligned} \tag{3.25}$$

where $\langle d\phi, \nabla\xi\rangle$ is the inner product on 1-forms. Taking $\phi = f^3$ for $f \in I_p$, we have that f, df and d^*df all vanish at p. Thus $\nabla^*\nabla$ is a second order LDO.

We noted that the coefficients of the operator (3.24) change under diffeomorphisms. But an important observation is that *the leading coefficient of an LDO is intrinsically defined.* This leading coefficient is called the symbol of the operator and can be defined without reference to coordinates as follows.

Definition 3.3 *The* symbol *of a k^{th} order LDO $D : \Gamma(E) \to \Gamma(F)$ is the vector bundle map*

$$\sigma_D : Sym^k(T^*M) \to Hom(E, F)$$

whose value at $p \in M$ is defined by

$$\sigma_D(\omega_1 \otimes \cdots \otimes \omega_k)_p(\xi) \;=\; \frac{1}{k!}\, D(f_1 f_2 \cdots f_k \xi)(p) \tag{3.26}$$

where the f_i are functions satisfying $df_i(p) = \omega_i$.

Notice that (3.26)

- Is clearly symmetric in the ω_i.
- Is independent of the choice of $\{f_i\}$ because if $df_i(p) = \omega_i = dg_i(p)$ then $f_i - g_i \in I_p^2$, so $f_1 \cdots (f_i - g_i) \cdots f_k \in I_p^{k+1}$ and hence the righthand side of (3.26) is unchanged when f_i is replaced by g_i.

[2] Lee, *Introduction to smooth manifolds*, Proposition 5.16

- Depends only on the value of ξ at p because if sections ξ and ξ' of E are equal at p then $\xi' = \xi + \sum f_\alpha \eta^\alpha$ for some $f_\alpha \in I_p$, so $f_1 \cdots f_k f_\alpha \in I_p^{k+1}$ and hence $\sum D(f_1 \cdots f_k f_\alpha \eta_\alpha)(p) = 0$.

We can reformulate the definition of symbol using a standard algebraic fact. Recall that a symmetric bilinear function $B(x,y)$ on a vector space V has an associated quadratic function $Q(x) = B(x,x)$, and given Q we can recover B by the *polarization formula*

$$B(x,y) \;=\; \frac{1}{2}[Q(x+y) - Q(x) - Q(y)].$$

Similarly, a symmetric multilinear function $B : \mathrm{Sym}^k(V) \to \mathbb{R}$ has an associated homogeneous polynomial $P(x) = B(x \otimes x \otimes \cdots \otimes x)$ on V homogeneous of degree k, and we can recover B by the polarization formula

$$B(x_1, \ldots x_k) \;=\; \frac{1}{k!}\,\frac{d^k}{ds_1 \cdots ds_k}\,P(s_1 x_1 + \cdots + s_k x_k).$$

Applying this to the symbol (3.26) gives, at each $p \in M$, a polynomial on T_p^*M homogeneous of degree k with values in $\mathrm{Hom}(E_p, F_p)$. As p varies, this defines a map from the pullback π^*E of E by $\pi : T^*M \to M$ to the pullback π^*F. That is our second way to think of the symbol.

Definition 3.4 *The* symbol *of a k^{th} order LDO $D : \Gamma(E) \to \Gamma(F)$ is the section*

$$\sigma_D \in \Gamma\left(Hom(\pi^*E, \pi^*F)\right)$$

defined by $\sigma_D(p,\omega) = \sigma(\omega \otimes \cdots \omega)_p$. It is homogeneous of degree k on each fiber of T^*M.

Geometric analysis is especially studies the class of elliptic operators. Ellipticity is a property of the symbol:

Definition 3.5 *A LDO D is* elliptic *if its symbol map σ_D is an isomorphism at each point $(p,\omega) \in T^*M$ with $\omega \neq 0$.*

To make Definition 3.5 more concrete, consider the second order LDO (3.24) acting on functions on $\mathbb{R}^n$. For $\phi \in C^\infty(\mathbb{R}^n)$ and $\omega = \sum w_k\, dx^k$ we have

$$\begin{aligned}
\sigma_D(0,\omega)\phi \;=\; \sigma_D(\omega \otimes \omega)\phi\big|_{x=0} \;&=\; \sum \omega_k \omega_\ell\, \sigma_D(dx^k \otimes dx^\ell)\phi\Big|_0 \\
&=\; \frac{1}{2}\sum \omega_k \omega_\ell\, D(x^k x^\ell \phi)_0 \\
&=\; \frac{1}{2}\sum \omega_k \omega_\ell \left(a^{ij}\partial_i \partial_j (x^k x^\ell \phi) + b^i \partial_i (x^k x^\ell \phi) + x^k x^\ell \phi\right)_0 \\
&=\; \sum a^{ij}(0)\,\omega_i \omega_j.
\end{aligned}$$

Similarly, the symbol of D at a general point $x \in \mathbb{R}^n$ is $\sigma_D(x,\omega) = \sum a^{ij}(x)\omega_i\omega_j$. In fact, in general, *the symbol is the homogeneous function of ω associated with the leading order coefficients of the operator.*

Now at any point p, $a^{ij}(p)$ is a symmetric matrix, so can be diagonalized by a choice of basis for $T_p^*\mathbb{R}^n$. Thus there is a basis in which $\sigma_D(p,\omega) = \sum \lambda^i \omega_i^2$. Ellipticity is then the condition that this function have no zeros for $\omega \neq 0$. By the Intermediate Value Theorem, ellipticity implies that either $a^{ij}(p)$ or $-a^{ij}(p)$ is positive definite for all p (assuming that M is connected).

This conclusion applies on manifolds. Any second order LDO acting on functions on a manifold M has the form (3.24) in local coordinates, and the above calculation shows that the operator is elliptic if and only if the symbol (or its negative) is defines a Riemannian metric on M.

We can now go down our list of globally-defined operators and determine which are elliptic.

Examples. First note that Definition 3.5 implies that *an operator $D : \Gamma(E) \to \Gamma(F)$ cannot be elliptic unless E and F are bundles of the same rank.*

(a) A connection $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ is not elliptic if $\dim M > 1$ because the ranks aren't equal.

(b) $d : \Omega^*_M \to \Omega^*_M$ has $\sigma_d(\omega) = \omega \wedge \xi$ and is not elliptic, again if $\dim M > 1$, because for each ω, any ξ that can written as $\omega \wedge \alpha$ for some α lies in the kernel of $\sigma_d(\omega)$.

(c) Similarly, $d^* : \Omega^*_M \to \Omega^*_M$ has $\sigma_{d^*}(\omega) = -\omega^* \lrcorner\, \xi$ and is not elliptic (take $\xi = 1 \in \Omega^0_M$).

(d) For a trace Laplacian we fix $(p,\omega) \in T^*M$ and f with $df(p) = \omega$ and calculate as in (3.25):

$$\sigma_{\nabla^*\nabla}(p,\omega)\xi \;=\; \frac{1}{2}\,\nabla^*\nabla(f^2\xi)_p \;=\; \frac{1}{2}\,d^*\delta(f^2)\xi_p \;=\; \frac{1}{2}\,d^*(2f\omega)\xi_p \;=\; -|\omega|^2\xi_p$$

Thus $\sigma_{\nabla^*\nabla}(p,\omega) = -|\omega|^2 \cdot \mathrm{Id.}$ is elliptic.

(e) By the Bochner formula, the Hodge Laplacian $\Delta = dd^* + d^*d$ has the same leading order term – and hence the same symbol – as $\nabla^*\nabla$. Thus Δ is elliptic.

(f) $D = d + d^* : \Omega^{\mathrm{even}}_M \to \Omega^{\mathrm{odd}}_M$ is elliptic because its symbol

$$\sigma_D(\omega)\xi \;=\; \omega \wedge \xi - \omega^* \lrcorner\, \xi$$

satisfies $[\sigma_D(\omega)]^2 = -\left[\omega \wedge \omega^* \lrcorner\, + \omega^* \lrcorner\, \omega\wedge\right] = -|\omega|^2 \cdot \mathrm{Id.}$ by the Cartan identity.

Examples (b) and (c) illustrate a general fact about how symbols behave under adjoints. If E and F are bundles with metrics on a Riemannian manifold (M,g), and $D : \Gamma(E) \to \Gamma(F)$ is a LDO, we can define the adjoint of the symbol $\sigma_D(\omega)$ to be the section

$$(\sigma_D)^* \in \Gamma\left(\mathrm{Hom}(\pi^*F, \pi^*E)\right)$$

defined by

$$\langle(\sigma_D)^*(\omega)\eta, \xi\rangle_E \;=\; \langle \eta, \sigma_D(\omega)\xi\rangle_F \qquad \forall \omega \in T^*M, \xi \in \Gamma(\pi^*E), \eta \in \Gamma(\pi^*F).$$

Exercise 3.2 shows that $\sigma_D^* = \pm\sigma_{D^*}$, that is, the symbol of the adjoint is the adjoint of the symbol up to sign.

We conclude this section by focusing on the most important type of first order and second order elliptic operators. First,

Proposition 3.6 *Given a connection ∇ on E, any first order LDO $D : \Gamma(E) \to \Gamma(F)$ can be written*

$$D \;=\; \sigma_D \circ \nabla + C \tag{3.27}$$

and any second order LDO can be written

$$D \;=\; \sigma_D \circ \nabla^2 + B \circ \nabla + C \tag{3.28}$$

$B : T^*M \otimes E \to F$ and $C : E \to F$ are bundle maps.

Proof. For each $p \in M$ and each $f \in I_p$ we have

$$\left[(D - \sigma_d \circ \nabla)\,(f\xi)\right]_p \;=\; \sigma_D(df)\xi\big|_p - \left[\sigma_D(df \otimes \xi + \cancel{f\nabla\xi})\right]_p \;=\; 0.$$

Thus $C = D - \sigma_D \circ \nabla$ is a LDO of order 0, so is a bundle map. The proof of (3.28) is similar. $\square$

Definition 3.7 *A* Dirac-type Operator *is any first order LDO*

$$D : \Gamma(E) \to \Gamma(F)$$

whose symbol satisfies $\sigma_D^*(\omega)\sigma_D(\omega) = -|\omega|^2 \cdot Id$.

This condition on the symbol implies that D is elliptic. It also implies that the symbol of D^*D, which by Exercise 3.2 is $\sigma_{D^*D}(\omega) = \sigma_D^*\sigma_D(\omega)$, is equal to the symbol of $\nabla^*\nabla$. It follows that $D^*D - \nabla^*\nabla$ is a first order differential operator, so D^*D has a Bochner-type formula

$$D^*D \;=\; \nabla^*\nabla + \text{lower order terms.} \tag{3.29}$$

Of course, the Bochner formula itself is an example. Specifically, taking

$$D = d + d^* : \Omega_M^{\text{even}} \to \Omega_M^{\text{odd}},$$

as in Example f above, we have $D^* = d + d^* : \Omega_M^{\text{odd}} \to \Omega_M^{\text{even}}$. Then $D^*D = dd^* + d^*d$ is the Hodge Laplacian and the standard Bochner formula has the form (3.29).

Exercises

(3.1) Complete the proof of Proposition 3.6 by proving that any second order LDO $D : \Gamma(E) \to \Gamma(F)$ has the form (3.28). Then state and sketch the proof of the corresponding formula for k^{th} order operators.

(3.2) Let $D : \Gamma(E) \to \Gamma(F)$ and $\tilde{D} : \Gamma(F) \to \Gamma(G)$ be linear differential operators with symbols σ_D and $\sigma_{\tilde{D}}$ and orders k and $\tilde{k}$.

(a) For $f \in C^\infty(M)$ let M_f denote multiplication by f. Show that $[D, M_f]$ is an LDO of order $k-1$.

(b) Show that $\sigma_{D \circ \tilde{D}} = \sigma_D \circ \sigma_{\tilde{D}}$.

(c) Show that $\sigma_{D^*} = (-1)^k(\sigma_D)^*$.

(3.3) Let ∇ be a connection on a (real) bundle E that is compatible with the metric $\langle\ ,\ \rangle$ on E. Show that $\langle \phi, \nabla^*\nabla\phi\rangle = |\nabla\phi|^2 + \frac{1}{2}d^*d|\phi|^2$ for all $\phi \in \Gamma(E)$, and consequently

$$\int_M \langle \phi, \nabla^*\nabla\psi\rangle \;=\; \int_M \langle \nabla\phi, \nabla\psi\rangle$$

for all smooth compactly suported sections $\phi, \psi \in \Gamma(E)$.

(3.4) Complete the proof of Proposition 3.6 (in the lecture notes) by proving that any second order LDO $D : \Gamma(E) \to \Gamma(F)$any second order LDO can be written

$$D \;=\; \sigma_D \circ \nabla^2 + B \circ \nabla + C$$

where $B : T^*M \otimes E \to F$ and $C : E \to F$ are bundle maps. Then state and sketch the proof of the corresponding formula for k^{th} order operators.

(3.5) Let $D : \Gamma(E) \to \Gamma(F)$ and $\tilde{D} : \Gamma(F) \to \Gamma(G)$ be linear differential operators with symbols σ_D and $\sigma_{\tilde{D}}$ and orders k and $\tilde{k}$.

(a) Let M_f denote multiplication by $f \in C^\infty(M)$. Show that $[D, M_f]$ is an LDO of order $k - 1$.

(b) Show that $\sigma_{D\circ\tilde{D}} = \sigma_D \circ \sigma_{\tilde{D}}$.

(c) Show that $\sigma_{D^*} = (-1)^k(\sigma_D)^*$.

4 The Spectral Theorem and the Index

We will continue to consider a first order simple elliptic operator $D : \Gamma(V) \to \Gamma(W)$ on a compact n-manifold (X, g). We extend the analysis of section 3 to the second major theme of elliptic theory: variational problems. This leads to the Spectral Theorem for D, to the proof of the Hodge Theorem, and to the problem of computing the index.

We start by regarding D as a operator between Sobolev spaces. Writing $D = \sigma \circ \nabla + K$, we have $|D\phi| \leq a|\nabla\phi| + b|\phi|$ pointwise. Integrating over the compact manifold X gives the inequality $\| D\phi \|_{0,2} \leq C \| \phi \|_{1,2}$. Thus D extends to a bounded, linear, and hence smooth map

$$D : L^{1,2}(V) \to L^2(W). \tag{4.1}$$

We will prove the Spectral Theorem using variational methods. The key step is to show that the Lagrangian

$$L(\phi) = \int_X |D\phi|^2.$$

can be minimized over closed subspaces of $L^{1,2}$. Note that L is a smooth function on $L^{1,2}$ (it is the composition of (4.1) with the square of the norm), and that the elliptic regularity estimate (**??**) gives

$$\| \phi \|_{1,2}^2 \leq \; C \left(L(\phi) + \| \phi \|_{0,2}^2 \right). \tag{4.2}$$

Recall that in a Hilbert space H a sequence ϕ_n *converges weakly* to ϕ_0, written $\phi_n \rightharpoonup \phi_0$, if $\langle \phi_n, \psi \rangle \to \langle \phi_0, \psi \rangle$ for all $\psi \in H$. It is easy to see that weak limits are unique and that if $\phi_n \rightharpoonup \phi_0$ and $L : H \to H'$ is bounded then $L\phi_n \rightharpoonup L\phi_0$.

Lemma 4.1 (Minimization) *Let S be the unit sphere in L^2 and let V be a closed linear subspace of L^2. Then if $V \cap L^{1,2}$ is non-empty, there is a $\phi_0 \in S(V) = S \cap V \cap L^{1,2}$ that minimizes $L(\phi)$ on $S(V)$.*

Proof. If $V \cap L^{1,2}$ is non-empty, we can choose a sequence $\{\phi_n\} \in S(V)$ minimizing L, i.e. with $L(\phi_n) \to L_0 = \inf\{L(\phi) \,|\, \phi \in S(V)\}$. This sequence is bounded in $L^{1,2}$ by (4.2). We can then choose a subsequence, still denoted $\{\phi_n\}$, that converges weakly in $L^{1,2}$ to some $\phi_0 \in L^{1,2}$ (the unit ball in a Hilbert space is weakly compact), and a further subsequence $\phi_n \to \phi_0$ in L^2 (the embedding $L^{1,2} \subset L^2$ is compact). Hence $\phi_0 \in S(V)$ and, since (4.1) is bounded, $D\phi_n \rightharpoonup D\phi_0$ in L^2. We then have

$$\begin{aligned} L(\phi_0) - L(\phi_n) \;&=\; \int 2\langle D\phi_0, D(\phi_0 - \phi_n)\rangle - |D(\phi_0 - \phi_n)|^2 \\ &\leq\; 2\int \langle D\phi_0, D(\phi_0 - \phi_n)\rangle \end{aligned}$$

where the righthand side $\to 0$ by weak convergence. Thus $L_0 \leq L(\phi_0) \leq \liminf L(\phi_n) = L_0$, so L achieves its minimum at ϕ_0. $\square$

Theorem 4.2 (Spectral Theorem for D^*D) *Let D be a first order self-adjoint simple elliptic operator on a compact Riemannian manifold. Then there is an L^2 orthogonal decomposition*

$$L^2(V) = \bigoplus E_\lambda$$

where each E_λ is a finite-dimensional space of smooth solutions of $D^*D\phi = \lambda\phi$ and the spectrum $\{\lambda\}$ is real, non-negative, and without accumulation points.

Proof. Applying the Minimization Lemma to $V_1 = L^2(E)$ gives an element $\phi_1 \in S(V_1)$ that minimizes $L(\phi)$ over $S(V_1)$. Let V_2 be the subspace of V_1 L^2-perpendicular to ϕ_1. The Minimization Lemma then produces a minimum ϕ_2 of $L(\phi)$ on $S(V_2)$; let V_3 be the L^2-perpendicular to ϕ_2 in V_2. Continuing, we obtain an L^2 orthonormal sequence $\{\phi_n\} \in S(V_n)$ with $L(\phi_1) \leq L(\phi_2) \leq \ldots$.

The numbers $\lambda_n = L(\phi_n)$ are real and non-negative. They are also unbounded: otherwise we have $\| \phi_n \|_{1,2} < C(L(\phi_n) + \| \phi_n \|_2) < C'$ for infinitely many n, and the compact embedding $L^{1,2} \to L^2$ would yield an L^2 convergent subsequence, contradicting the orthogonality of the $\{\phi_n\}$. This argument also implies that there are only finitely many ϕ_n with $\lambda_n < C$. Hence the sequence $\{\lambda_n\}$ has no accumulation points and the spaces

$$E_\lambda = \ L^2 \text{ span } \{\phi_n \mid L(\phi_n) = \lambda \ \}$$

are finite dimensional.

To show that the ϕ_n span $L^2(E)$, let W be the $L^{1,2}$ space of the ϕ_n, let $W^\perp$ be its $L^{1,2}$ orthogonal complement, and let V be the L^2 closure of $W^\perp$. If $V \cap L^{1,2} = 0$ then $W = L^{1,2}$, so W is dense in L^2 and therefore the ϕ_n span. Otherwise, the Minimization Lemma yields $\psi \in S(V)$ minimizes $L(\phi)$ over $S(V)$. Since $\{\lambda_n\}$ is unbounded there is a ϕ_n with $L(\phi_n) > L(\psi)$. But this ϕ_n minimizes E over $(\text{span}\{\phi_1 \ldots \phi_{n-1}\})^\perp \cap L^{1,2} \cap S(V)$ and hence over $V \cap L^{1,2} \cap S(V)$; thus $L(\phi_n) \leq L(\psi)$, contradicting the choice of ϕ_n.

Finally, we compute the variational equations of our minimization problem. Fix n and $m > n$. Then

$$\phi_t = \frac{\phi_n + t\phi_m}{\sqrt{1+t^2}}$$

is a path in $S(V_n)$. Hence

$$L(\phi_n) \ \leq \ L(\phi_t) \ = \ L(\phi_n) + 2t \int \langle D\phi_n, D\phi_m \rangle + O(t^2),$$

so $0 \leq \int \langle D\phi_n, D\phi_m \rangle$. Replacing ϕ_m by $-\phi_m$ gives the opposite inequality. The argument holds with n and m interchanged, so

$$0 = \int \langle D\phi_n, D\phi_m \rangle \qquad \forall m \neq n.$$

Since the $\{\phi_n\}$ span L^2

$$0 = \int \langle D\phi_n, D\psi\rangle = \int \langle D^* D\phi_n, \psi\rangle \qquad \forall \psi \perp \phi_n.$$

Thus $D^*D\phi_n = c\phi_n$ where this Lagrange multiplier is

$$c = c\int |\phi_n|^2 = \int \langle D^*D\phi_n, \phi_n\rangle = \int |D\phi_n|^2 = \lambda_n.$$

To complete the proof we need only show that the eigenfunctions ϕ_n are smooth; this is shown in the course of proving Theorem 4.4 below. $\square$

By the spectral theorem, each $\phi \in L^2(V)$ has an L^2-orthogonal "Fourier series" expansion

$$\phi = \sum a_\lambda \phi_\lambda \tag{4.3}$$

where the ϕ_λ are eigenvectors of D^*D. The L^2 norm of ϕ is then

$$\| \phi \|^2 = \sum |a_\lambda|^2 < \infty. \tag{4.4}$$

Intuitively, we can think of $L^2(V)$ as a vector space with basis $\{\phi_\lambda\}$ and D^*D as a big diagonal matrix with the eigenvalues $\{\lambda\}$ along the diagonal. This simple linear algebra picture describes D^*D except for issues of convergence. The following proof is an application of this viewpoint.

Corollary 4.3 *There is a L^2-orthogonal decomposition*

$$L^2(V) = \ker\, D \,\oplus\, D^* L^{1,2}(W).$$

Proof. Given $\phi \in L^2(V)$ with expansion (4.3), set $\psi_n = \sum \lambda^{-1} a_\lambda\, D\phi_\lambda$, where the sum is over all λ with $0 < \lambda \leq n$. Then each ψ_n is L^2 perpendicular to ker D^*. Hence by the Poincaré inequality (**??**) we have, for all $n < m$,

$$\| \psi_m - \psi_n \|_{1,2}^2 \leq\; C\; \| D^*(\psi_m - \psi_n) \|_{0,2}^2 \leq\; C\; \| \sum_{n\leq\lambda\leq m} a_\lambda \phi_\lambda \|_{0,2}^2 \leq\; C \sum_{n}^{\infty} |a_\lambda|^2.$$

Thus by (4.4) the sequence $\{\psi_n\}$ is Cauchy in $L^{1,2}$, and the limit ψ satisfies $D^*\psi - \phi = \sum_{\lambda=0} a_\lambda \phi_\lambda \in$ ker D. $\square$

The Hodge Theorem follows easily from Corollary 4.3 (exercise 4-A).

We can also state a spectral theorem for *first order* operators. Of course, it makes no sense to speak of eigenvectors of $D : \Gamma(V) \to \Gamma(W)$ when $V \neq W$. Instead, we consider

$$D' = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix} : L^{1,2}(V \oplus W) \to L^2(V \oplus W). \tag{4.5}$$

This operator is still simple and elliptic, and is now self-adjoint. The eigenvalues of such an operator, which we henceforth call D, are exactly the square roots of the eigenvalues of D^*D.

Theorem 4.4 (Spectral Theorem for D) *Let D be a first order self-adjoint simple elliptic operator on a compact Riemannian manifold. Then there is an L^2 orthogonal decomposition*

$$L^2(V) = \bigoplus E_\mu \oplus E_{-\mu}$$

where each $E_{\pm\mu}$ is a finite-dimensional space of smooth solutions of $D'\phi = \pm\mu\,\phi$ and the spectrum $\{\pm\mu\}$ is real, without accumulation points.

Proof. Clearly ker $D \subset$ ker D^*D, while if $D^*D\phi = 0$ then

$$0 = \int_X \langle \phi, D^*D\phi\rangle = \int_X \langle |D\phi|^2.$$

Thus the zero eigenspace of D is that same as the zero eigenspace of D^*D. On the other hand, if $D^*D\phi = \lambda\phi$, wet $\mu = \sqrt{\lambda}$. Then for with $\lambda \neq 0$, $\psi_\pm = (\pm\phi,\, \mu^{-1}D\phi)$ satisfies

$$D\psi_\pm \;=\; \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}\begin{pmatrix} \pm\phi \\ \mu^{-1}D\phi \end{pmatrix} \;=\; \begin{pmatrix} \mu^{-1}\lambda\phi \\ \pm D\phi \end{pmatrix} \;=\; \pm\mu\,\psi.$$

Thus the λ-eigenspace of D^*D decomposes into eigenspaces $E_\mu \oplus E_{-\mu}$ of D. The spectral decomposition and properties of the spectrum then follow from Theorem 4.2. Finally, applying Example **??** to (4.6) shows that ψ_+ and ψ_- are smooth, and hence so is the eigenfunction ϕ of D^*D. $\square$

We end this section by mentioning the index problem for first order elliptic operators.

A bounded linear map $L : X \to Y$ between Hilbert spaces is *Fredholm* if it has finite-dimensional kernel, and its range is closed and has finite codimension. Then coker $L = Y/\text{image } L$ is finite-dimensional and is naturally identified with ker L^*. In our case, the Spectral Theorem implies that

$$D : L^{1,2}(V) \to L^2(W)$$

is Fredholm. Hence D has a well-defined index

$$\begin{aligned} \text{index } D &= \text{dim ker } D - \text{dim coker } D \\ &= \text{dim ker } D - \text{dim ker } D^*. \end{aligned}$$

Example 4.5 *By the Hodge Theorem, the index of $d + d^* : \Omega^{ev} \to \Omega^{odd}$ is the euler class of X.*

The index has two basic properties (see [Lg]). First, it is a continuous function on the space Fred(X, Y) of Fredholm maps from X to Y (with the operator norm topology). Second, it is stable under compact perturbations: if L is Fredholm and K is a compact operator, then $L + K$ is Fredholm with index $(L + K) =$ index $(L) +$ index (K).

Now, writing $D = \sigma \circ \nabla + K$, we make three observations.

(1) Index D is independent of K.

Proof. K is a smooth tensor, so induces a bounded linear map $K : L^2(V) \to L^2(W)$. Precomposing with the compact inclusion of $L^{1,2}$ into L^2 this becomes a compact perturbation of $D : L^{1,2}(V) \to L^2(W)$. $\square$

(2) Index D is independent of the connection.

Proof. Changing ∇ to $\nabla + A$ changes D to $D + K'$ where $K' = \sigma \circ A$ is smooth. Then (1) applies. $\square$

(3) Index D depends only on the homotopy class of σ.

Proof. Given a continuous path of symbol maps σ_t, it suffices to one can construct a continuous path of Fredholm operators D_t with symbols σ_t. This can be done using pseudo-differential operators, see [AS] (first order differential operators have linear symbols; to make the argument we must move to a class of operators with continuous symbols).

These observations raise the index problem.

Index Problem Find an effective way of computing the index of D from its symbol map $\sigma : T^*X \otimes V \to W$.

The solution is given by the Atiyah-Singer Index Theorem ([AS], [Sh]). In the case when the symbol is *universal* (i.e. arising functorially from a vector space map as, for example, $d + d^*$ arises from interior and exterior multiplication) the answer is expressed in terms of the characteristic classes of the manifold X and the bundles V and W. We will give such a formula in the next section.

Exercises

(4.1) Let X be the unit circle. Explicitly describe the eigenspaces of

$$\begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \qquad \text{acting on} \qquad \Omega^0_X \oplus \Omega^1_X.$$

(4.2) Use the expansion (4.3) to prove the following version of the Poincaré inequality: if $\phi \in L^2$ is L^2-perpendicular to the space of all eigenspaces with eigenvalues $\lambda \leq \Lambda$ then

$$\| \phi \|_{1,2} \leq C \ \| D\phi \|_{0,2} \qquad \text{where} \qquad C^2 = 1/\Lambda.$$

(4.3) Fix $k \geq 0$. Define a norm on $\phi \in \Gamma(V)$ by expanding ϕ as in (4.3) and setting

$$\| \phi \|^2 = \sum_\lambda (1 + \lambda)^k \, |a_\lambda|^2.$$

(a) Prove that this norm is equivalent to the $L^{k,2}$ norm.
(b) Define Sobolev spaces of functions $W^{s,2}(M)$ for real numbers $s > 0$.

(4.4) Read the last page above and answer these questions:

Let $D_t : \Gamma(V) \to \Gamma(W)$, $t \in [0,1]$, be a path of first order simple elliptic operators.

(a) Show that the non-zero spectrum of $D_t^* D_t$ is the same as that of $D_t D_t^*$.

(b) Assuming that the eigenvalues depend continuously on t (they do), use part (a) to show that the index is independent of t.