Math 930  Problem Set 6
Due Monday, October 27

Problem (6.1) Prove the second Bianchi identity
\[ \nabla_X R(Y, Z) + \nabla_Y R(Z, X) + \nabla_Z R(X, Y) = 0 \]
for all vector fields \( X, Y, Z \).

Problem (6.2) As in Problem 5.1, let \( \{e_i\} \) be a local orthonormal frame of \( TM \) with dual frame \( \{e^i\} \) of \( T^*M \) and a connection 1-form \( \omega^i_j \) defined by \( \nabla_X e^i = \sum \omega^i_j(X) e^j \). Show that in this frame, the curvature is the \( \mathfrak{so}(n) \)-valued 2-form
\[ \Omega^i_j = d\omega^i_j + \omega^k_i \wedge \omega^j_k \]
with \( \Omega^i_j = \sum -R^i_{jkl} e^k \wedge e^l \) (note minus!). Use the conventions \( \alpha \wedge \beta(X, Y) = \alpha(X) \beta(Y) - \alpha(Y) \beta(X) \) and \( R^i_{jkl} = \langle R(e_k, e_l)e^i, e^j \rangle \). (Note: here we are using the connection and curvature on \( T^*M \)).

Problem (6.3) A locally symmetric space is a Riemannian manifold \((M^n, g)\) whose curvature tensor \( R \) satisfies \( \nabla R = 0 \).
(a) Prove that if \( M \) has constant sectional curvature then \( M \) is a locally symmetric space.
(b) Let \( \gamma : [0, b] \to M \) be a geodesic in a locally symmetric space, and let \( X, Y, Z \) be parallel vector fields along \( \gamma \). Prove that \( R(X, Y)Z \) is parallel along \( \gamma \).
(c) Prove that a connected, 2-manifold is locally symmetric if and only if it has constant sectional curvature.

Problem (6.4) Let \( \gamma : [0, \infty) \to M \) be a geodesic in a locally symmetric space with tangent \( T = \dot{\gamma} \). Let \( v = T(0) \) be the tangent at \( p = \gamma(0) \). Define a linear transformation \( K_v : T_pM \to T_pM \) by
\[ K_v(w) = -R(v, w) v \quad \text{for } w \in T_pM. \]
(a) Show that \( K_v \) is self-adjoint.
(b) Choose an orthonormal basis \( \{e_i\} \) of \( T_pM \) that diagonalizes \( K_v \), so \( K_v(e_i) = \lambda_i e_i \) for \( i = 1, \ldots, n \). Extend these to vector fields \( e_i(t) \) along \( \gamma \) by parallel transport. Prove that
\[ K_T(e_i(t)) = \lambda_i e_i(t) \]
for all \( i \) and all \( t \geq 0 \), where \( \lambda_i \) is independent of \( t \).
(c) Show that \( X(t) = \sum_i x_i(t)e_i(t) \) is a Jacobi field along \( \gamma \) if and only if
\[ \frac{d^2x_i}{dt^2} + \lambda_i x_i = 0 \quad i = 1, \ldots, n \]
The conjugate points of \( p \) along \( \gamma \) are those points \( q = \gamma(t) \) for which there exists a Jacobi field along \( \gamma \) that vanishes at \( p \) and at \( q \).
(d) Show that the conjugate points of \( p \) along \( \gamma \) are \( \gamma(\pi k/\sqrt{\lambda_i}) \), where \( k \) is an integer and \( \lambda_i \) is a positive eigenvalue of \( K_v \).