Math 930  Problem Set 4  
Due Monday, September 29

Problem 4.1  Many interesting manifolds arise as *Riemannian submersions*. To understand this idea, do Problem 3.8 on page 45 of the textbook.

Problem 4.2  A smooth function $\phi : M \to \mathbb{R}$ determines a 1-form $d\phi$. If there is an affine connection on $M$ then the covariant Hessian of $\phi$ is defined by

$$(\nabla^2 \phi)(X,Y) = \nabla_X(\delta\phi)(Y).$$

(which differs slightly from the equation at the bottom of page 54 of the textbook).

(a) Show that $(\nabla^2 \phi)(X,Y) = X \cdot Y \phi - (\nabla_X Y) \phi$.

(b) Prove that $\nabla^2 \phi$ is a $(2,0)$ tensor.

(c) Show that a connection $\nabla$ is torsion-free if and only if the covariant Hessian $\nabla^2 \phi$ is a symmetric tensor for every $\phi \in C^\infty(M)$.

Problem 4.3  Do Problem 4.4 on page 63 of the textbook.

Problem 4.4  Do Exercise 5.7 on page 82 of the textbook.

Problem 4.5  Lobatchevski’s non-euclidean geometry is the metric

$$g = \frac{1}{y^2} \left( dx \otimes dx + dy \otimes dy \right)$$

on the upper half-plane $H = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ (thus $g_{11} = g_{22} = \frac{1}{y^2}$ and $g_{12} = 0$).

(a) Show that the Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma^1_{11} = \Gamma^2_{22} = \Gamma^2_{12} = 0, \quad \Gamma^2_{11} = \frac{1}{y}, \quad \Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}.$$  

(b) Fix the vector $X_0 = (0,1) \in T_{(0,1)}H$ and let $X_t$ be its parallel transport along the line $\gamma(t) = (t,1)$. Show that $X_t$ is the unit vector that makes an angle $t$ with the positive $x$-axis, measured *clockwise*.

*Hint:* Since $X_t$ is a unit vector and $y = 1$, we can write $X_t = (\cos \theta(t), \sin \theta(t))$ for some function $\theta(t)$. Write down the equation of parallel transport (using the above $\Gamma^k_{ij}$), and show that $\frac{d\theta}{dt} = -1$. 